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WEAK PERIODIC SOLUTIONS OF SOME QUASILINEAR PARABOLIC EQUATIONS WITH DATA MEASURES

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Abstract

The goal of this paper is to study the existence of weak periodic solutions for some quasilinear parabolic equations with data measures and critical growth nonlinearity with respect to the gradient. The classical techniques based on C^{α}-estimates for the solution or its gradient cannot be applied because of the lack of regularity and a new approach must be considered. Various necessary conditions are obtained on the data for existence. The existence of at least one weak periodic solution is proved under the assumption that a weak periodic super solution is known.The results are applied to reaction-diffusion systems arising from chemical kinetics.

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Contents

1. Introduction

Periodic behavior of solutions of parabolic boundary value problems arises from many biological, chemical, and physical systems, and various methods have been proposed for the study of the existence and qualitative property of periodic solutions. Most of the work in the earlier literature is devoted to scalar semilinear parabolic equations under either Dirichlet or Neumann boundary conditions (cf. [4], [5], [14], [15], [18], [19], [20], [23], [24], [25]) all these works examine the classical solutions. In recent years attention has been given to weak solutions of parabolic equations under linear boundary conditions, and different methods for the existence problem have been used (cf [1], [2], [3], [6], [7], [9], [8], [10], [11], [16], [21], [22], etc.).

In this work we are concerned with the periodic parabolic problem

$$(1.1)\qquad \begin{cases} u_t - \Delta u = J\left(t,x,u,\nabla u\right) + \lambda f & \text{in } Q_T \\ u\left(t,x\right) = 0 & \text{on } \sum_T \\ u\left(0,x\right) = u(T,x) & \text{in } \Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$, $Q_T =]0,T[\times\Omega$, $\sum_T =]0,T[\times\partial\Omega$, $T>0$, λ are given numbers, $-\Delta$ denotes the Laplacian operator on L^1 with Dirichlet boundary conditions, the perturbation $J : Q_T \times \mathbb{R} \times \mathbb{R}^N \to [0,+\infty[$ is measurable and continuous with respect to u and ∇u, and f is a given nonnegative measure on Q_T.

The work by Amann [4] is concerned with the problem (1.1) under the hypothesis that f is regular enough and the growth of the nonlinearities J with respect to the gradient is sub-quadratic, namely

$$J(t,x,u,\nabla u) \leq c(|u|)\left(|\nabla u|^2 + 1\right).$$

He obtained the existence of maximal and minimal solutions in $C^1(\overline{\Omega})$ by using the method of sub- and super-solutions and Schauder's fixed point theorem in a suitable Banach space (see also [5], [12]).

In this work we are interested in situations where f is irregular and where the growth of J with respect to ∇u is arbitrary and, in particular, larger than $|\nabla u|^2$ for large $|\nabla u|$. The fact that f is not regular requires that one deals with "weak" solutions for which ∇u, u_t and even u itself are not bounded. As a consequence, the classical theory using C^α-a priori estimates to prove existence fails. Let us make this more precise on a model problem like

$$\text{(1.2)} \qquad \begin{cases} u_t - \Delta u + au = |\nabla u|^p + \lambda f & \text{in} \quad Q_T \\ u(t,x) = 0 & \text{on} \quad \sum_T \\ u(0,x) = u(T,x) & \text{in} \quad \Omega, \end{cases}$$

where $|\cdot|$ denotes the $\mathbb{R}^N$-euclidean norm, $a \geq 0$ and $p \geq 1$.

If $p \leq 2$, the method of sub- and super-solutions can be applied to prove existence in (1.2) if f is regular enough. For instance if $a > 0$ and $f \in C^\alpha(Q_T)$, then (1.2) has a solution since $\underline{w} \equiv 0$ is a sub-solution and $\overline{w}(t,x) = v(x)$, where v is a solution of the elliptic problem

$$\begin{cases} av - \Delta v = |\nabla v|^p + \lambda \|f\|_\infty & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

is a super-solution of (1.2) (see Amann [4]). The situation is quite different if $p > 2$: for instance a size condition is necessary on λf to have existence in (1.2) even f is very regular, indeed we prove in Section 2.1 that there exists $\lambda^* < +\infty$ such that (1.2) does not have any periodic solution for $\lambda > \lambda^*$. On the other hand we obtain another critical value $p^* = 1 + \frac{2}{N}$ of the problem, indeed as proved in Section 2.2, existence in (1.2) with $p > p^*$ requires that f be regular enough.

We prove in Section 3, that existence of a nonnegative weak periodic super solution implies existence of nonnegative weak periodic solution in the case of sub quadratic growth. Obviously, the classical approach fails to provide existence since f is not regular enough and new techniques must be applied. We describe some of them here. Finally in Section 4, the results are applied to reaction-diffusion systems arising from chemical kinetics.

To finish this paragraph, we recall the following notations and definitions:

Notations:
$\mathcal{C}_0^\infty(Q_T) = \{\varphi : Q_T \to \mathbb{R},$ indefinitely derivable with compact support in $Q_T\}$
$\mathcal{C}_b(\Omega) = \{\varphi : \Omega \to \mathbb{R},$ continuous and bounded in $\Omega\}$
$\mathcal{M}_b(Q_T) = \{\mu$ bounded Radon measure in $Q_T\}$
$\mathcal{M}_b^+(Q_T) = \{\mu$ bounded nonnegative Radon measure in $Q_T\}$.

Definition 1.1. *Let $u \in C(]0,T[\,; L^1(\Omega))$, we say that $u(0) = u(T)$ in $\mathcal{M}_b(\Omega)$ if for all $\varphi \in \mathcal{C}_b(\Omega)$,*

$$\lim_{t\to 0} \int_\Omega (u(t,x) - u(T-t,x))\varphi dx = 0.$$

2. Necessary Conditions for Existence

Throughout this section we are given

(2.1) f a nonnegative finite measure on $]0,T[\times\Omega$
and $J:[0,T]\times\Omega\times\mathbb{R}^N\to[0,+\infty[$ is such that

(2.2) J is measurable, almost everywhere (t,x), $r\longmapsto J(t,x,r)$
is continuous, convex.

(2.3) $\forall r\in\mathbb{R}^N, J(\cdot,\cdot,r)$ is integrable on $]0,T[\times\Omega$.

(2.4) $J(t,x,0)=\min\left\{J(t,x,r),r\in\mathbb{R}^N\right\}=0.$

For $\lambda\in\mathbb{R}$, we consider the problem

$$(2.5)\quad \begin{cases} u\in L^1(0,T;W_0^{1,1}(\Omega))\cap C(]0,T[;L^1(\Omega)),\ u\geq 0 & \text{in } Q_T \\ J(t,x,\nabla u)\in L^1_{loc}(Q_T), & \\ u_t-\Delta u\geq J(t,x,\nabla u)+\lambda F & \text{in } \mathfrak{D}'(Q_T) \\ u(0)=u(T) & \text{in } \mathcal{M}_b(\Omega). \end{cases}$$

2.1. No Existence in Superquadratic Case

We prove in this section, if $J(\cdot,\cdot,r)$ is superquadratic at infinity, then there exists $\lambda^*<+\infty$ such that (2.5) does not have any periodic solution for $\lambda>\lambda^*$. The techniques used here are similar to those in [1] for the parabolic problem with initial data measure. A rather sharp superquadratic condition on J is given next where the (t,x)-dependence is taken into account. We assume

(2.6) There exists $]\varepsilon,\tau[$ open in $]0,T[\,, p>2,$ and a constant $c_0>0$

(2.7) such that $J(t,x,r)\geq c_0\,|r|^p$ almost every where $(t,x)\in\,]\varepsilon,\tau[\,\times\Omega$

(2.8) $$\int_{]\varepsilon,\tau[\times\Omega} f>0.$$

Theorem 2.1. *Assume that (2.1) – (2.4), (2.6) – (2.8) hold. Then there exists $\lambda^*<+\infty$ such that (2.5) does not have any solution for $\lambda>\lambda^*$.*

Proof. Assume u is a solution of (2.5). By (2.6) and (2.7), we have

$$u_t-\Delta u\geq c_0\,|\nabla u|^p+\lambda f \text{ in } \mathfrak{D}'\left(]\varepsilon,\tau[\,\times\Omega\right). \tag{2.9}$$

Let $\varphi\in C_0^\infty\left(]0,T[\,\times\Omega\right),\ \varphi\geq 0$ and $\varphi(\varepsilon)=\varphi(\tau)=0$. Multiply (2.9) by φ and integrate to obtain

$$\lambda\int_\varepsilon^\tau\int_\Omega f\varphi\leq\int_\varepsilon^\tau\int_\Omega\nabla u\nabla\varphi-c_0\,|\nabla u|^p\,\varphi-u\varphi_t. \tag{2.10}$$

Taking into account the equality

$$\varphi_t=-\Delta\left(G\varphi_t\right)=-\Delta\left(G\varphi\right)_t,$$

where G is the Green Kernel on Ω. We obtain from (2.10)

$$\lambda\int_\varepsilon^\tau\int_\Omega f\varphi\leq\int_\varepsilon^\tau\int_\Omega\nabla u\nabla\varphi-c_0\,|\nabla u|^p\,\varphi-\nabla u\nabla\left(G\varphi\right)_t$$

this can be extended for all $\varphi \in C^1([0,T];\ L^\infty(\Omega)) \cap L^\infty\left(0,T;W_0^{1,\infty}(\Omega)\right)$, $\varphi \geq 0$ and $\varphi(\varepsilon) = \varphi(\tau) = 0$. We obtain

$$\lambda \int_\varepsilon^\tau \int_\Omega f\varphi \leq \int_\varepsilon^\tau \int_\Omega \varphi \left[|\nabla u| \frac{|\nabla \varphi - \nabla (G\varphi)_t|}{\varphi} - c_0 |\nabla u|^p \right] dxdt \tag{2.11}$$

if we recall Young's inequality $\forall s \in \mathbb{R}\ sr \leq c_0 |r|^p + c|s|^q$, $\frac{1}{p} + \frac{1}{q} = 1$. We see that (2.11) implies

$$\begin{cases} \lambda \displaystyle\int_\varepsilon^\tau \int_\Omega f\varphi \leq c \int_\varepsilon^\tau \int_\Omega \frac{|\nabla \varphi - \nabla (G\varphi)_t|^q}{\varphi^{q-1}} dxdt \\ \forall \varphi \in C^1([0,T];L^\infty(\Omega)) \cap L^\infty\left(0,T;W_0^{1,\infty}(\Omega)\right) \\ \varphi \geq 0 \text{ and } \varphi(\varepsilon) = \varphi(\tau) = 0. \end{cases} \tag{2.12}$$

Let us prove that this implies that λ is finite (hence the existence of λ^*). We choose $\varphi(t,x) = (t-\varepsilon)^q (\tau - t)^q \Phi(x)$, Φ is a solution of

$$\begin{cases} -\Delta\Phi(x) = \lambda_1 \Phi(x), \Phi > 0 & \text{in } \Omega \\ \Phi(x) = 0 & \text{in } \partial\Omega, \end{cases}$$

where λ_1 is the first eigenvalue of $-\Delta$ in Ω. We then have from (2.12)

$$\lambda \int_\varepsilon^\tau \int_\Omega (t-\varepsilon)^q (\tau - t)^q \Phi(x) f$$

$$\leq c\int_{\varepsilon}^{\tau}\int_{\Omega}\Bigg[\Bigg|(t-\varepsilon)^{q}(\tau-t)^{q}\nabla\Phi(x)-\frac{q}{\lambda_{1}}(t-\varepsilon)^{q-1}(\tau-t)^{q-1}$$

$$\times(\tau+\varepsilon-2t)\nabla\Phi(x)\Bigg|^{q}\Bigg]\Bigg/\left[(t-\varepsilon)^{q(q-1)}(\tau-t)^{q(q-1)}\Phi(x)^{q-1}\right]dxdt$$

$$\leq c_{1}\int_{\varepsilon}^{\tau}\int_{\Omega}\frac{(t-\varepsilon)^{q}(\tau-t)^{q}\left|\nabla\Phi(x)\right|^{q}}{\Phi(x)^{q-1}}dxdt+c_{2}\int_{\varepsilon}^{\tau}\int_{\Omega}\frac{\left|\nabla\Phi(x)\right|^{q}}{\Phi(x)^{q-1}}dxdt$$

it provides

$$\lambda\int_{\varepsilon}^{\tau}\int_{\Omega}(t-\varepsilon)^{q}(\tau-t)^{q}\Phi(x)f\leq c_{3}\int_{\Omega}\frac{\left|\nabla\Phi(x)\right|^{q}}{\Phi(x)^{q-1}}dx.$$

By the definition of Φ we have $\Phi\in W_{0}^{1,\infty}(\Omega)$ and $\frac{1}{\Phi^{\alpha}(x)}\in L^{1}(\Omega)$ for all $\alpha<1$. Since $p>2$ then $\alpha=q-1<1$, therefore $\int_{\Omega}\frac{|\nabla\Phi(x)|^{q}}{\Phi(x)^{q-1}}dx<\infty$. This completes the proof. $\square$

2.2. Regularity Condition on the Data f

We consider the following problem

$$\text{(2.13)}\qquad\begin{cases}u\in L^{1}\left(0,T;W_{0}^{1,1}(\Omega)\right)\cap C(]0,T[\,;L^{1}(\Omega)),\\[2ex] J(.,u,\nabla u)\in L_{loc}^{1}(Q_{T})\\[2ex] u_{t}-\Delta u\geq J(t,x,u,\nabla u)+\lambda f\quad\text{in }\mathfrak{D}'(Q_{T})\\[2ex] u(0)=u(T)\qquad\text{in }\ \mathcal{M}_{b}(\Omega)\end{cases}$$

where f, J satisfy (2.1) – (2.4) and

$$\text{there exists } p > 1\ , c_1,\ c_2 > 0,\ J\left(t, x, s, r\right) \geq c_1 \left|r\right|^p - c_2. \tag{2.14}$$

Theorem 2.2. *Assume that (2.1) – (2.4), (2.14) hold. Assume (2.13) has a solution for some $\lambda > 0$. Then the measure f does not charge the set of $W_q^{2,1}$-capacity zero $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$.*

Remark 2.1. *We recall that a compact set K in Q_T is of $W_q^{2,1}$-capacity zero if there exists a sequence of $C_0^\infty\left(\Omega\right)$-functions φ_n greater than 1 on K and converging to zero in $W_q^{2,1}$. The above statement means that*

$$\left(K \text{ compact, } W_q^{2,1}\text{-capacity}\left(K\right) = 0\right) \Rightarrow \int_K f = 0 \tag{2.15}$$

Obviously, this is not true for any measure f as soon as $q < \frac{N}{2} + 1$ or $p > 1 + \frac{2}{N}$, (see, e.g. [7] and the references therein for more details.)

Remark 2.2. *The natural question is now the following. Let $1 \leq p < 1 + \frac{2}{N}$ and $f \in \mathcal{M}_b^+\left(Q_T\right)$, does there exist u solution of (2.13) and if this solution exists is it unique? It will make the object of a next work.*

Proof of Theorem 2.2. From (2.13), (2.14), we get the following inequality

$$u_t - \Delta u \geq c_1 \left|\nabla u\right|^p - c_2 + \lambda f \quad \text{in } \mathfrak{D}'\left(Q_T\right). \tag{2.16}$$

Let K be a compact set of $W_q^{2,1}$-capacity zero and φ_n a sequence of $C_0^\infty\left(Q_T\right)$-functions such that

$$\varphi_n \geq 1 \text{ on } K,\ \varphi_n \to 0 \text{ in } W_q^{2,1} \text{ and a.e in } Q_T,\ 0 \leq \varphi_n \leq 1. \tag{2.17}$$

Multiplying (2.16) by $\chi_n = \varphi_n^q$ leads to

$$\lambda \int_0^T \int_\Omega \chi_n f + c_1 \int_0^T \int_\Omega \chi_n |\nabla u|^p \leq c_2 \int_0^T \int_\Omega \chi_n - u \frac{\partial \chi_n}{\partial t} + \int_0^T \int_\Omega \nabla \chi_n \nabla u. \tag{2.18}$$

We use $\nabla \chi_n = q\varphi_n^{q-1} \nabla \varphi_n$, and Young's inequality to treat last integral above:

$$\int_0^T \int_\Omega \nabla \chi_n \nabla u \leq \varepsilon \int_0^T \int_\Omega \chi_n |\nabla u|^p + c_\varepsilon \int_0^T \int_\Omega |\nabla \varphi_n|^q \tag{2.19}$$

Due to (2.17), passing to the limit in (2.18), (2.19) with ε small enough easily leads to

$$\lambda \int_K f = 0. \tag{2.20}$$

$\square$

Remark 2.3. *The result obtained here is valid if one replaces in (2.13) the operator $-\Delta$ by Δ that is to say for the equation*

$$\begin{cases} u \in L^1\left(0,T; W_0^{1,1}(\Omega)\right) \cap C(]0,T[\,; L^1(\Omega)), \\ J(t,x,u,\nabla u) \in L^1_{loc}(Q_T) \\ u_t + \Delta u \geq J(t,x,u,\nabla u) + \lambda f \quad \text{in } \mathfrak{D}'(Q_T) \\ u(0) = u(T) \qquad \text{in } \mathcal{M}_b(\Omega) \end{cases}$$

or also for the equation

$$\begin{cases} u_t - \Delta u + |\nabla u|^p = \lambda f & \text{in } Q_T \\ u(t,x) = 0 & \text{on } \sum_T \\ u(0,x) = u(T,x) & \text{in } \Omega. \end{cases}$$

3. An Existence Result for Subquadratic Growth

3.1. Statement of the Result

3.1.1. Assumption

First, we clarify in which sense we want to solve problem (1.1).

Definition 3.1. *A function u is called a weak periodic solution of (1.1) if*

$$\left\{ \begin{array}{l} u \in L^2\left(0,T,H_0^1\left(\Omega\right)\right) \cap C\left(\left[0,T\right],L^2\left(\Omega\right)\right), \\ J\left(t,x,u,\nabla u\right) \in L^1\left(Q_T\right) \\ u_t - \Delta u = J\left(t,x,u,\nabla u\right) + f \quad \textit{in } \mathfrak{D}'\left(Q_T\right) \\ u\left(0\right) = u(T) \in L^2\left(\Omega\right), \end{array} \right. \tag{3.1}$$

where f is a nonnegative, integrable function and

(3.2) $J : Q_T \times \mathbb{R} \times \mathbb{R}^N \to [0,+\infty[$ *is a Caratheodory function, that means:*

$$\begin{array}{l} (t,x) \longmapsto J\left(t,x,s,r\right) \textit{ is measurable} \\ (s,r) \longmapsto J\left(t,x,s,r\right) \textit{ is continuous for almost every } (t,x) \end{array}$$

(3.3) *J is nondecreasing with respect to s and convex with respect to r.*

(3.4) $J\left(t,x,s,0\right) = \min\left\{J\left(t,x,s,r\right), r \in \mathbb{R}^N\right\} = 0$

(3.5) $J\left(t,x,s,r\right) \leq c\left(|s|\right)\left(|r|^2 + H\left(t,x\right)\right),$

where $c : [0,+\infty[\to [0,+\infty[$ is nondecreasing and $H \in L^1\left(Q_T\right)$.

Definition 3.2. *We call weak periodic sub-solution (resp. super-solution) of (1.1) a function u satisfying (4.1) with " $=$ " replaced by " $\leq$ " (resp. $\geq$).*

3.1.2. The Main Result

We state now the main result of this section

Theorem 3.1. *Suppose that hypotheses (3.2) – (3.5) hold and problem (1.1) has a nonnegative weak super-solution w. Then (1.1) has a weak periodic solution u such that: $0 \leq u \leq w$.*

3.2. Proof of the Main Result

3.2.1. Approximating Problem

Let $n \geq 1$ and $\hat{j}_n(t,x,s,\cdot)$ be the Yoshida's approximation of the function $J(t,x,s,\cdot)$ which increases almost every where to $J(t,x,s,\cdot)$ as n tends to infinity and satisfies the following properties

$$\hat{j}_n \leq J, \qquad \text{and} \qquad \|\hat{j}_{n,r}(t,x,s,r)\| \leq n.$$

Let

$$J_n(t,x,s,r) = \hat{j}_n(t,x,s,r)\, 1_{[w \leq n]}(t,x,s,r),$$

where w is a super-solution of (1.1).

It is easily seen that J_n satisfies hypotheses (3.2) – (3.5).

Moreover

$$J_n \leq J 1_{[w \leq n]} \qquad \text{and} \qquad J_n \leq J_{n+1}. \tag{3.6}$$

On the other hand, since $f \in L^1(Q_T)$, we can construct a sequence f_n in $L^\infty(Q_T)$ such that

$$f_n \leq f_{n+1}, \qquad \|f_n\|_{L^1(Q_T)} \leq \|f\|_{L^1(Q_T)}$$

and f_n converge to f in $\mathfrak{D}'(Q_T)$ as n tends to infinity.
Let

$$F_n = f_n 1_{[w \leq n]}, \qquad w_n = \min(w, n),$$

and consider the sequence (u_n) defined by: $u_0 = w_0 = 0$,

$$(3.7) \qquad \begin{cases} u_n \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \\ u_{n_t} - \Delta u_n = J_n(t,x,u_{n-1},\nabla u_n) + F_n \text{ in } \mathfrak{D}'(Q_T) \\ u_n(0) = u_n(T) \quad \in L^2(\Omega). \end{cases}$$

We will show by induction that (3.7) has a solution such that

$$(3.8) \qquad 0 \leq u_{n-1} \leq u_n \leq w_n.$$

To do this, we first consider the linear periodic problem

$$(3.9) \qquad \begin{cases} u \in L^2(0,T,H_0^1(\Omega)) \cap L^\infty(0,T,L^2(\Omega)), \; u \geq 0 \text{ in } \overline{Q_T} \\ u_t - \Delta u = F_1 \;\text{ in }\; \mathfrak{D}'(Q_T) \\ u(0) = u(T) \in L^2(\Omega). \end{cases}$$

This problem has a solution u_1 (see [17, Theorem 6.1, p. 483]). We remark that w_1 is a supersolution of (3.9) and thanks to the maximum principle, we have $w_1 - u_1 \geq 0$ on Q_T, hence there exists u_1 such that

$$0 \leq u_0 \leq u_1 \leq w_1.$$

Suppose that (3.8) is satisfied for $n-1$.
Then from (3.6), u_{n-1} is a weak sub-solution of (3.7). Let us prove that w_n is

a weak super-solution of (3.7). Indeed, by the definition of w_n and the monotonicity of J_n we have

$$\begin{cases} w_n \in L^2\left(0,T,H_0^1\left(\Omega\right)\right) \cap L^\infty\left(0,T,L^2\left(\Omega\right)\right) \\ w_{n_t} - \Delta w_n \geq J_n\left(t,x,u_{n-1},\nabla w_n\right) + F_n \text{ in } \mathfrak{D}'\left(Q_T\right) \\ w_n\left(0\right) = w_n\left(T\right) \in L^2\left(\Omega\right). \end{cases}$$

Hence (3.7) has a solution u_n such that $u_{n-1} \leq u_n \leq w_n$ (see [11]), which proves (3.8) by induction.

3.2.2. A Priori Estimates and Passing to the Limit

A Priori Estimate In this section, we are going to give several technical results as lemmas that will be very useful for the proof of the main result.

Lemma 3.2. *Let $u, v \in L^2(0,T;H_0^1(\Omega))$, such that*

$$\begin{cases} 0 \leq u \leq v \quad \text{in } Q_T \\ u_t - \Delta u \geq 0 \ \text{ in } \mathfrak{D}'\left(Q_T\right) \\ v_t - \Delta v \geq 0 \ \text{ in } \mathfrak{D}'\left(Q_T\right) \\ u\left(0\right) = u\left(T\right) \in L^2\left(\Omega\right) \\ v\left(0\right) = v\left(T\right) \in L^2\left(\Omega\right). \end{cases} \tag{3.10}$$

Then, there exists a constant $c_2 > o$, such that

$$\int_{Q_T} \left|\nabla u\right|^2 \leq c_2 \int_{Q_T} \left|\nabla v\right|^2.$$

Lemma 3.3. *Let u_n be a solution of (3.7), then there exists a constant $c_3 > o$, such that*

$$\int_{Q_T} J_n\left(t, x, u_{n-1}, \nabla u_n\right) dxdt \leq c_3.$$

Lemma 3.4. *Let $u \in L^2\left(0, T, H_0^1\left(\Omega\right)\right)$, such that*

$$\begin{cases} u_t - \Delta u = \rho \quad \text{in } \mathfrak{D}'\left(Q_T\right) \\ \rho \in M_B^+(Q_T) \\ u\left(0\right) = u\left(T\right) \in L^2\left(\Omega\right). \end{cases}$$

Then

$$u\rho \in L^1\left(Q_T\right) \qquad \text{and} \qquad \int_{Q_T} u\rho \leq \int_{Q_T} \left|\nabla u\right|^2.$$

Lemma 3.5. *Let $u, u_n \in L^2\left(0, T, H_0^1\left(\Omega\right)\right)$, such that*

$$0 \leq u_n \leq u \text{ in } Q_T \text{ and } u\left(0\right) = u\left(T\right) \in L^2\left(\Omega\right) \tag{3.11}$$

$$u_n \rightharpoonup u \text{ weakly in } L^2\left(0, T, H_0^1\left(\Omega\right)\right) \tag{3.12}$$

$$\begin{cases} u_{n_t} - \Delta u_n = \rho_n \text{ in } \mathfrak{D}'\left(Q_T\right) \\ u_n\left(0\right) = u_n\left(T\right) \in L^\infty\left(\Omega\right) \\ \rho_n \in L^2\left(Q_T\right), \quad \rho_n \geq 0 \text{ and } \left\|\rho_n\right\|_{L^1(Q_T)} \leq c \end{cases} \tag{3.13}$$

where c is a constant independent of n. Then $u_n \to u$ strongly in $L^2\left(0, T, H_0^1\left(\Omega\right)\right)$.

Proof of Lemma 3.2. Since $u \in L^2(0, T, H_0^1(\Omega))$ and $\Delta u \in L^2(0, T, H^{-1}(\Omega))$, then

$$\int_{Q_T} |\nabla u|^2 = \langle -\Delta u, u \rangle ,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between $L^2(0, T; H_0^1(\Omega))$ and $L^2(0, T; H^{-1}(\Omega))$.

Moreover, we have $\int_{Q_T} u u_t = 0$ and $0 \leq u \leq v$, then

$$\begin{aligned} \int_{Q_T} |\nabla u|^2 &= \langle u_t - \Delta u, u \rangle \leq \langle u_t - \Delta u, v \rangle \\ &\leq -\langle \Delta u, v \rangle - \langle \Delta u, v \rangle \\ &\leq 2 \int_{Q_T} \nabla u \nabla v. \end{aligned}$$

Using Young's inequality we obtain

$$\int_{Q_T} |\nabla u|^2 \leq \frac{1}{2} \int_{Q_T} |\nabla u|^2 + c \int_{Q_T} |\nabla u|^2 ,$$

where c is a positive constant . □

Proof of Lemma 3.3. Remark that

$$\begin{aligned} \int_{Q_T} J_n(t, x, u_{n-1}, \nabla u_n)\, dxdt = \int_{Q_T \cap [u_n \leq 1]} & J_n(t, x, u_{n-1}, \nabla u_n)\, dxdt \\ &+ \int_{Q_T \cap [u_n > 1]} J_n(t, x, u_{n-1}, \nabla u_n)\, dxdt. \end{aligned}$$

We note

$$I_1 = \int_{Q_T \cap [u_n \le 1]} J_n\left(t, x, u_{n-1}, \nabla u_n\right) dxdt$$

and

$$I_2 = \int_{Q_T \cap [u_n > 1]} J_n\left(t, x, u_{n-1}, \nabla u_n\right) dxdt.$$

Hypothesis (3.5) yields

$$I_1 \le c(1) \int_{Q_T} \left(|\nabla u_n|^2 + H(t, x) \right) dxdt.$$

But $H \in L^1(Q_T)$ and $0 \le u_n \le w$, then Lemma 3.2, implies that there exists a constant c_4 such that

$$I_1 \le c_4. \tag{3.14}$$

On the other hand, we have

$$I_2 \le \int_{Q_T} u_n J_n\left(t, x, u_{n-1}, \nabla u_n\right) dxdt.$$

Multiplying the equation in (3.9) by u_n and integrating by part yields:

$$I_2 \le \int_{Q_T} |\nabla u_n|^2 .$$

Using Lemma 3.2 and inequality (3.14), we complete the proof. □

Proof of Lemma 3.4. Consider the sequence $u_m = \min(u, m)$. It is clear that $u_m \in L^2(0, T, H_0^1(\Omega))$. Moreover u_m converge to u in $L^2(0, T, H_0^1(\Omega))$ and satisfies the equation

$$\left\{ \begin{array}{l} u_m \in L^2(0, T, H_0^1(\Omega)) \\ \\ u_{m_t} - \Delta u_m \geq \rho 1_{[u<m]} \text{ in } \mathfrak{D}'(Q_T) \\ \\ u_m(0) = u_m(T) \in L^\infty(\Omega). \end{array} \right. \tag{3.15}$$

Multiply (3.15) by u_m and integrate by part on Q_T, we obtain

$$\begin{aligned} \left\langle u_m, \rho 1_{[u<m]} \right\rangle &= \langle u_m, u_{m_t} - \Delta u_m \rangle \\ &= \frac{1}{2} \int_{Q_T} u_{m_t}^2 + \int_{Q_T} |\nabla u_m|^2 \\ &= \int_{Q_T} |\nabla u_m|^2 . \end{aligned}$$

Thanks to Fatou's lemma, we deduce

$$\int_{Q_T} u\rho = \int_{Q_T} |\nabla u|^2 .$$

□

Proof of lemma 3.5. By relations (3.11) – (3.13), there exists $\rho \in \mathcal{M}_b^+(Q_T)$, such that,

$$\left\{ \begin{array}{l} u_t - \Delta u = \rho \text{ in } \mathfrak{D}'(Q_T) \\ \\ u(0) = u(T) \in L^2(\Omega). \end{array} \right. \tag{3.16}$$

However,

$$\begin{aligned}\int_{Q_T} |\nabla u - \nabla u_n|^2 &= -\int_{Q_T} (u-u_n)\,\Delta(u-u_n)\\ &= -\int_{Q_T} u\Delta(u-u_n) + \int_{Q_T} u_n\Delta(u-u_n)\\ &= \int_{Q_T} \nabla u\nabla(u-u_n) - \int_{Q_T} \nabla u_n\nabla u - \int_{Q_T} u_n\nabla u_n\\ &= \int_{Q_T} \nabla u\nabla(u-u_n) - \int_{Q_T} \nabla u_n\nabla u - \int_{Q_T} u\,(u_{n_t} - \Delta u_n)\,.\end{aligned}$$

Moreover, by Lemma 3.4, we have

$$\int_{Q_T} u\,(u_{n_t} - \Delta u_n) \le \int_{Q_T} |\nabla u|^2\,.$$

Hence

$$\lim_{n\to+\infty} \int_{Q_T} |\nabla u - \nabla u_n|^2\, dxdt = 0.$$

□

Passing to the Limit According to Lemma 3.3 and estimate (3.8), $(u_n)_n$ is bounded in $L^2(0,T;H_0^1(\Omega))$. Therefore there exists $u \in L^2(0,T;H_0^1(\Omega))$, up to a subsequence still denoted (u_n) for simplicity, such that

$$\begin{aligned}&u_n \to u \quad \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T\\ &u_n \rightharpoonup u \quad \text{weakly in } L^2(0,T;H_0^1(\Omega))\,.\end{aligned}$$

However, Lemma 3.5 implies that the last convergence is strong in $L^2\left(0,T;H_0^1\left(\Omega\right)\right)$. Then to ensure that u is a solution of problem (1.1), it suffices to prove that

$$J_n\left(\cdot,\cdot,u_{n-1},\nabla u_n\right)\to J\left(\cdot,\cdot,u,\nabla u\right)\ \text{in}\ L^1\left(Q_T\right). \tag{3.17}$$

It is obvious by Lemma 3.2 and the strong convergence of u_n in $L^2\left(0,T,H_0^1\left(\Omega\right)\right)$ that

$$J_n\left(\cdot,\cdot,u_{n-1},\nabla u_n\right)\to J\left(\cdot,\cdot,u,\nabla u\right)\ \text{a.e in}\ Q_T.$$

To conclude that u is a solution of (1.1), we have to show, in view of Vitali's theorem that $\left(J_n\right)_n$ is equi-integrable in $L^1\left(Q_T\right)$.
Let K be a measurable subset of Q_T, $\epsilon>0$ and $k>0$, we have

$$\begin{aligned}\int_K J_n\left(t,x,u_{n-1},\nabla u_n\right)dxdt &= \int_{K\cap[u_n\le k]} J_n\left(t,x,u_{n-1},\nabla u_n\right)dxdt\\ &\quad+\int_{K\cap[u_n>k]} J_n\left(t,x,u_{n-1},\nabla u_n\right)dxdt.\end{aligned}$$

We note that

$$I_1=\int_{K\cap[u_n\le k]} J_n\left(t,x,u_{n-1},\nabla u_n\right)dxdt$$

and

$$I_2=\int_{K\cap[u_n>k]} J_n\left(t,x,u_{n-1},\nabla u_n\right)dxdt.$$

To deal with the term I_2, we write

$$I_2\le\frac{1}{k}\int_K u_nJ_n\left(t,x,u_{n-1},\nabla u_n\right)dxdt$$

which yields from the equation satisfied by u_n in (3.7)

$$I_2 \leq \frac{1}{k}\int_K (u_n u_t - u_n \Delta u_n)\, dxdt$$
$$\leq \frac{1}{k}\int_K |\nabla u_n|^2\, dxdt.$$

By Lemma 3.2, there exists a constant $c_2^{'} > 0$ such that

$$I_2 \leq \frac{c_2^{'}}{k}. \tag{3.18}$$

Then, there exists $k_0 > 0$,such that, if $k > k_0$ then

$$I_2 \leq \frac{\epsilon}{3}. \tag{3.19}$$

By hypothesis (3.5), we have for all $k > k_0$

$$I_1 \leq c(k)\int_{K\cap[u_n\leq k]} \left(|\nabla u_n|^2 + H(t,x)\right) dxdt.$$

The sequence $\left(|\nabla u_n|^2\right)_n$ is equi-integrable in $L^1(Q_T)$. So there exists $\delta_1 > 0$ such that if $|K| \leq \delta_1$, then

$$c(k)\int_{K\cap[u_n\leq k]} \left(|\nabla u_n|^2\right) dxdt \leq \frac{\epsilon}{3}. \tag{3.20}$$

On the other hand $H \in L^1(Q_T)$, therefore there exists $\delta_2 > 0$, such that

$$c(k) \int_{K \cap [u_n \leq k]} H(t,x)\, dxdt \leq \frac{\epsilon}{3}, \tag{3.21}$$

whenever $|K| \leq \delta_2$.
Choose $\delta_0 = \inf(\delta_1, \delta_2)$, if $|K| \leq \delta_0$, we have

$$\int_K J_n(t, x, u_{n-1}, \nabla u_n)\, dxdt \leq \epsilon.$$

4. Application to a Class of Reaction-Diffusion Systems

We will see in this section how to apply the result established below to a class of raction-diffusion systems of the form

$$(4.1)\qquad \begin{cases} u_t - \Delta u = -J(t,x,v,\nabla u) + F(t,x) & \text{in } Q_T \\ v_t - \Delta v = J(t,x,v,\nabla u) + G(t,x) & \text{in } Q_T \\ u = v = 0 & \text{on } \sum_T \\ u(0) = u(T)\,, \;\; v(0) = v(T) & \text{in } \Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$, $Q_T =]0,T[\times\Omega$, $\sum_T =]0,T[\times\partial\Omega\ T > 0$, F, G are integrable nonnegative functions and J satisfies hypotheses $(H_1) - (H_4)$.

Definition 4.1. *A couple (u,v) is said to be a weak solution of the system (4.1) if*

$$\begin{cases} u,v \in L^2\left(0,T;H_0^1(\Omega)\right) \cap C\left([0,T];L^2(\Omega)\right) \\ u_t - \Delta u = -J(t,x,v,\nabla u) + F(t,x) \;\; \textit{in } Q_T \\ v_t - \Delta v = J(t,x,v,\nabla u) + G(t,x) \;\; \textit{in } Q_T \\ u(0) = u(T)\,, \; v(0) = v(T) \in L^2(\Omega)\,. \end{cases}$$

Theorem 4.1. *Under the hypotheses (3.2) – (3.5), and $F, G \in L^2(Q_T)$, system (4.1) has a nonnegative weak periodic solution.*

To prove this result, we introduce the function w solution of the following linear problem

$$(4.2)\qquad \begin{cases} w \in L^2\left(0,T;H_0^1(\Omega)\right) \cap C\left([0,T];L^2(\Omega)\right) \\ w_t - \Delta w = F + G \quad \text{in } \mathfrak{D}'(Q_T) \\ w(0) = w(T) \in L^2(\Omega). \end{cases}$$

It is well known that (4.2) has a unique solution, see [17].
Consider now the equation

$$(4.3)\qquad \begin{cases} v \in L^2\left(0,T,H_0^1(\Omega)\right) \cap C\left([0,T],L^2(\Omega)\right) \\ v_t - \Delta v = J(t,x,v,\nabla w - \nabla v) + G \quad \text{in } \mathfrak{D}'(Q_T) \\ v(0) = v(T) \quad \in L^2(\Omega). \end{cases}$$

It is clear that solving (4.1) is equivalent to solve (4.3) and set $u = w - v$.

Proof of Theorem 4.1. We remark that w is a supersolution of (4.3). Then by a direct application of Theorem 3.1, problem (4.3) has a solution. $\square$

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