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SOME GENERALIZED CONVOLUTION PROPERTIES ASSOCIATED WITH CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract

For functions belonging to each of the subclasses $\mathcal{M}_n^*(\alpha)$ and $\mathcal{N}_n^*(\alpha)$ of normalized analytic functions in open unit disk $\mathbb{U}$, which are introduced and investigated in this paper, the authors derive several properties involving their generalized convolution by applying certain techniques based especially upon the Cauchy-Schwarz and Hölder inequalities. A number of interesting consequences of these generalized convolution properties are also considered.

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Contents

1. Introduction and Definitions

Let $\mathcal{A}_n$ denote the class of functions $f(z)$ *normalized* in the form:

$$f(z) = z + \sum_{k=n}^{\infty} a_k z^k \qquad (n \in \mathbb{N} \setminus \{1\};\ \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We denote by $\mathcal{M}_n(\alpha)$ the subclass of $\mathcal{A}_n$ consisting of functions $f(z)$ which satisfy the inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha \qquad (\alpha > 1;\ z \in \mathbb{U}). \tag{1.2}$$

Also let $\mathcal{N}_n(\alpha)$ be the subclass of $\mathcal{A}_n$ consisting of functions $f(z)$ which satisfy the inequality:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \alpha \qquad (\alpha > 1;\ z \in \mathbb{U}). \tag{1.3}$$

For $n = 2$ and $1 < \alpha < \frac{4}{3}$, the classes $M_2(\alpha)$ and $N_2(\alpha)$ were investigated earlier by Uralegaddi *et al.* (cf. [5]; see also [4] and [6]). In fact, following these earlier works in conjunction with those by Nishiwaki and Owa [1] (see also [3]), it is easy to derive Lemma 1.1 and Lemma 1.2 below, which provide the sufficient conditions for functions $f \in \mathcal{A}_n$ to be in the classes $\mathcal{M}_n(\alpha)$ and $\mathcal{N}_n(\alpha)$, respectively.

Lemma 1.1. *If $f \in \mathcal{A}_n$ given by (1.1) satisfies the condition:*

$$\sum_{k=n}^{\infty} (k-n)\,|a_k| \leqq \alpha - 1 \qquad \left(1 < \alpha < \frac{n+1}{2}\right), \tag{1.4}$$

then $f \in \mathcal{M}_n(\alpha)$.

Lemma 1.2. *If $f \in \mathcal{A}_n$ given by (1.1) satisfies the condition:*

$$\sum_{k=n}^{\infty} k\,(k-\alpha)\,|a_k| \leqq \alpha - 1 \qquad \left(1 < \alpha < \frac{n+1}{2}\right), \tag{1.5}$$

then $f \in \mathcal{N}_n(\alpha)$.

For examples of functions in the classes $\mathcal{M}_n(\alpha)$ and $\mathcal{N}_n(\alpha)$, let us first consider the function $\varphi(z)$ defined by

$$\varphi(z) := z + \sum_{k=n}^{\infty} \left(\frac{n(\alpha-1)}{k(k+1)(k-\alpha)}\right) z^k, \tag{1.6}$$

which is of the form (1.1) with

$$a_k = \frac{n(\alpha-1)}{k(k+1)(k-\alpha)} \qquad (k = n, n+1, n+2, \ldots), \tag{1.7}$$

so that we readily have

$$\sum_{k=n}^{\infty} \left(\frac{k-\alpha}{\alpha-1}\right) |a_k| = n \sum_{k=n}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1. \tag{1.8}$$

Thus, by Lemma 1.1, $\varphi \in \mathcal{M}_n(\alpha)$. Furthermore, since

$$f(z) \in \mathcal{N}_n(\alpha) \iff zf'(z) \in \mathcal{M}_n(\alpha), \tag{1.9}$$

we observe that the function $\psi(z)$ defined by

$$\psi(z) := z + \sum_{k=n}^{\infty} \left(\frac{n(\alpha-1)}{k^2(k+1)(k-\alpha)} \right) z^k \tag{1.10}$$

belongs to the class $\mathcal{N}_n(\alpha)$.

In view of Lemma 1.1 and Lemma 1.2, we now define the subclasses

$$\mathcal{M}_n^*(\alpha) \subset \mathcal{M}_n(\alpha) \qquad \text{and} \qquad \mathcal{N}_n^*(\alpha) \subset \mathcal{N}_n(\alpha),$$

which consist of functions $f(z)$ satisfying the conditions (1.4) and (1.5), respectively.

Finally, for functions $f_j \in A_n$ $(j = 1, \ldots, m)$ given by

$$f_j(z) = z + \sum_{k=n}^{\infty} a_{k,j}\, z^k \qquad (j = 1, \ldots, m), \tag{1.11}$$

the Hadamard product (or convolution) is defined by

$$(f_1 * \cdots * f_m)(z) := z + \sum_{k=n}^{\infty} \left(\prod_{j=1}^{m} a_{k,j} \right) z^k. \tag{1.12}$$

2. Convolution Properties of Functions in the Classes $\mathcal{M}_n^*(\alpha)$ and $\mathcal{N}_n^*(\alpha)$

For the Hadamard product (or convolution) defined by (1.12), we first prove

Theorem 2.1. *If* $f_j(z) \in \mathcal{M}_n^*(\alpha_j)$ $(j = 1, \ldots, m)$, *then*

$$(f_1 * \cdots * f_m)(z) \in \mathcal{M}_n^*(\beta),$$

where

$$\beta = 1 + \frac{(n-1)\prod_{j=1}^{m}(\alpha_j - 1)}{\prod_{j=1}^{m}(n-\alpha_j) + \prod_{j=1}^{m}(\alpha_j - 1)}. \tag{2.1}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, \ldots, m)$ *given by*

$$f_j(z) = z + \left(\frac{\alpha_j - 1}{n - \alpha_j}\right) z^n \qquad (j = 1, \ldots, m). \tag{2.2}$$

Proof. Following the work of Owa [2], we use the principle of mathematical induction in our proof of Theorem 2.1. Let $f_1(z) \in \mathcal{M}_n^*(\alpha_1)$ and $f_2(z) \in \mathcal{M}_n^*(\alpha_2)$. Then the inequality:

$$\sum_{k=n}^{\infty}(k-\alpha_j)|a_{k,j}| \leqq \alpha_j - 1 \qquad (j = 1, 2)$$

implies that

$$\sum_{k=n}^{\infty}\sqrt{\frac{k-\alpha_j}{\alpha_j - 1}|a_{k,j}|} \leqq 1 \qquad (j = 1, 2). \tag{2.3}$$

Thus, by applying the Cauchy-Schwarz inequality, we have

$$\left| \sum_{k=n}^{\infty} \sqrt{\frac{(k-\alpha_1)(k-\alpha_2)}{(\alpha_1-1)(\alpha_2-1)}} |a_{k,1}| |a_{k,2}| \right|^2 \leqq \left(\sum_{k=n}^{\infty} \left(\frac{k-\alpha_1}{\alpha_1-1} \right) |a_{k,1}| \right) \left(\sum_{k=n}^{\infty} \left(\frac{k-\alpha_2}{\alpha_2-1} \right) |a_{k,2}| \right) \leqq 1.$$

Therefore, if

$$\sum_{k=n}^{\infty} \left(\frac{k-\delta}{\delta-1} \right) |a_{k,1}| |a_{k,2}| \leqq \sum_{k=n}^{\infty} \sqrt{\frac{(k-\alpha_1)(k-\alpha_2)}{(\alpha_1-1)(\alpha_2-1)}} |a_{k,1}| |a_{k,2}|,$$

that is, if

$$\sqrt{|a_{k,1}| |a_{k,2}|} \leqq \left(\frac{\delta-1}{k-\delta} \right) \sqrt{\frac{(k-\alpha_1)(k-\alpha_2)}{(\alpha_1-1)(\alpha_2-1)}} \qquad (k = n, n+1, n+2, \dots),$$

then $(f_1 * f_2)(z) \in \mathcal{M}_n^*(\delta)$.

We also note that the inequality (2.3) yields

$$\sqrt{|a_{k,j}|} \leqq \sqrt{\frac{\alpha_j - 1}{k - \alpha_j}} \qquad (j = 1, 2;\; k = n, n+1, n+2, \dots).$$

Consequently, if

$$\sqrt{\frac{(\alpha_1-1)(\alpha_2-1)}{(k-\alpha_1)(k-\alpha_2)}} \leqq \frac{\delta-1}{k-\delta} \sqrt{\frac{(k-\alpha_1)(k-\alpha_2)}{(\alpha_1-1)(\alpha_2-1)}},$$

that is, if

$$\frac{k-\delta}{\delta-1} \leqq \frac{(k-\alpha_1)(k-\alpha_2)}{(\alpha_1-1)(\alpha_2-1)} \qquad (k=n,n+1,n+2,\dots), \tag{2.4}$$

then we have $(f_1 * f_2)(z) \in \mathcal{M}_n^*(\delta)$. It follows from (2.4) that

$$\delta \geqq 1 + \frac{(k-1)(\alpha_1-1)(\alpha_2-1)}{(k-\alpha_1)(k-\alpha_2)+(\alpha_1-1)(\alpha_2-1)} =: h(k)$$
$$(k=n,n+1,n+2,\dots).$$

Since $h(k)$ is decreasing for $k \geqq n$, we have

$$\delta \geqq 1 + \frac{(n-1)(\alpha_1-1)(\alpha_2-1)}{(n-\alpha_1)(n-\alpha_2)+(\alpha_1-1)(\alpha_2-1)},$$

which shows that $(f_1 * f_2)(z) \in \mathcal{M}_n^*(\delta)$, where

$$\delta := 1 + \frac{(n-1)(\alpha_1-1)(\alpha_2-1)}{(n-\alpha_1)(n-\alpha_2)+(\alpha_1-1)(\alpha_2-1)}.$$

Next, we suppose that

$$(f_1 * \cdots * f_m)(z) \in \mathcal{M}_n^*(\gamma),$$

where

$$\gamma := 1 + \frac{(n-1)\prod_{j=1}^m(\alpha_j-1)}{\prod_{j=1}^m(n-\alpha_j)+\prod_{j=1}^m(\alpha_j-1)}.$$

Then, by means of the above technique, we can show that

$$(f_1 * \cdots * f_{m+1})(z) \in \mathcal{M}_n^*(\beta),$$

where

$$\beta := 1 + \frac{(n-1)(\gamma-1)(\alpha_{m+1}-1)}{(n-\gamma)(n-\alpha_{m+1}) + (\gamma-1)(\alpha_{m+1}-1)}. \tag{2.5}$$

Since

$$(\gamma-1)(\alpha_{m+1}-1) = \frac{(n-1)\prod_{j=1}^{m+1}(\alpha_j-1)}{\prod_{j=1}^{m}(n-\alpha_j) + \prod_{j=1}^{m}(\alpha_j-1)}$$

and

$$(n-\gamma)(n-\alpha_{m+1}) = \frac{(n-1)\prod_{j=1}^{m+1}(n-\alpha_j)}{\prod_{j=1}^{m}(n-\alpha_j) + \prod_{j=1}^{m}(\alpha_j-1)},$$

Equation (2.5) shows that

$$\beta = 1 + \frac{(n-1)\prod_{j=1}^{m+1}(\alpha_j-1)}{\prod_{j=1}^{m+1}(n-\alpha_j) + \prod_{j=1}^{m+1}(\alpha_j-1)}.$$

Finally, for the functions $f_j(z)$ $(j = 1, \ldots, m)$ given by (2.2), we have

$$(f_1 * \cdots * f_m)(z) = z + \left(\prod_{j=1}^{m}\left(\frac{\alpha_j-1}{n-\alpha_j}\right)\right) z^n = z + A_n\, z^n,$$

where

$$A_n := \prod_{j=1}^{m}\left(\frac{\alpha_j-1}{n-\alpha_j}\right).$$

It follows that

$$\sum_{k=n}^{\infty}\left(\frac{k-\beta}{\beta-1}\right)|A_k|=1.$$

This evidently completes the proof of Theorem 2.1. □

By setting $\alpha_j=\alpha\ (j=1,\dots,m)$ in Theorem 2.1, we get

Corollary 2.2. *If $f_j(z)\in\mathcal{M}_n^*(\alpha)\ (j=1,\dots,m)$, then*

$$(f_1*\cdots*f_m)(z)\in\mathcal{M}_n^*(\beta),$$

where

$$\beta=1+\frac{(n-1)(\alpha-1)^m}{(n-\alpha)^m+(\alpha-1)^m}.$$

The result is sharp for the functions $f_j(z)\ (j=1,\dots,m)$ given by

$$f_j(z)=z+\left(\frac{\alpha-1}{n-\alpha}\right)z^n\qquad(j=1,\dots,m).$$

Next, for the Hadamard product (or convolution) of functions in the class $\mathcal{N}_n^*(z)$, we derive

Theorem 2.3. *If $f_j(z)\in\mathcal{N}_n^*(\alpha_j)\ (j=1,\dots,m)$, then*

$$(f_1*\cdots*f_m)(z)\in\mathcal{N}_n^*(\beta),$$

where

$$\beta=1+\frac{(n-1)\prod_{j=1}^{m}(\alpha_j-1)}{n^{m-1}\prod_{j=1}^{m}(n-\alpha_j)+\prod_{j=1}^{m}(\alpha_j-1)}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1, \ldots, m)$ *given by*

$$f_j(z) = z + \left(\frac{\alpha_j - 1}{n(n - \alpha_j)} \right) z^n \qquad (j = 1, \ldots, m). \tag{2.6}$$

Proof. As in the proof of Theorem 2.1, for $f_1(z) \in \mathcal{N}_n^*(\alpha_1)$ and $f_2(z) \in \mathcal{N}_n^*(\alpha_2)$, the following inequality:

$$\sum_{k=n}^{\infty} \left(\frac{k(k - \delta)}{\delta - 1} \right) |a_{k,1}| |a_{k,2}| \leqq 1$$

implies that $(f_1 * f_2)(z) \in \mathcal{N}_n^*(\delta)$. Also, in the same manner as in the proof of Theorem 2.1, we obtain

$$\delta \geqq 1 + \frac{(k-1)(\alpha_1 - 1)(\alpha_2 - 1)}{k(k - \alpha_1)(k - \alpha_2) + (\alpha_1 - 1)(\alpha_2 - 1)} \qquad (k = n, n+1, n+2, \ldots). \tag{2.7}$$

The right-hand side of (2.7) takes its maximum value for $k = n$, because it is a decreasing function of $k \geqq n$. This shows that $(f_1 * f_2)(z) \in \mathcal{N}_n^*(\delta)$, where

$$\delta = 1 + \frac{(n-1)(\alpha_1 - 1)(\alpha_2 - 1)}{n(n - \alpha_1)(n - \alpha_2) + (\alpha_1 - 1)(\alpha_2 - 1)}.$$

Now, assuming that

$$(f_1 * \cdots * f_m)(z) \in \mathcal{N}_n^*(\gamma),$$

where

$$\gamma := 1 + \frac{(n-1)\prod_{j=1}^{m}(\alpha_j - 1)}{n^{m-1}\prod_{j=1}^{m}(n-\alpha_j) + \prod_{j=1}^{m}(\alpha_j - 1)},$$

we have

$$(f_1 * \cdots * f_{m+1})(z) \in \mathcal{N}_n^*(\beta),$$

where

$$\beta = 1 + \frac{(n-1)(\gamma-1)(\alpha_{m+1}-1)}{n(n-\gamma)(n-\alpha_{m+1}) + (\gamma-1)(\alpha_{m+1}-1)}$$
$$= 1 + \frac{(n-1)\prod_{j=1}^{m+1}(\alpha_j - 1)}{n^{m}\prod_{j=1}^{m+1}(n-\alpha_j) + \prod_{j=1}^{m+1}(\alpha_j - 1)}.$$

Moreover, by taking the functions $f_j(z)$ given by (2.6), we can easily verify that the result of Theorem 2.3 is sharp. □

By letting $\alpha_j = \alpha$ $(j = 1, \ldots, m)$ in Theorem 2.3, we obtain

Corollary 2.4. *If* $f_j(z) \in \mathcal{N}_n^*(\alpha)$ $(j = 1, \ldots, m)$, *then*

$$(f_1 * \cdots * f_m)(z) \in \mathcal{N}_n^*(\beta),$$

where

$$\beta = 1 + \frac{(n-1)(\alpha-1)^m}{n^{m-1}(n-\alpha)^m + (\alpha-1)^m}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1, \ldots, m)$ *given by*

$$f_j(z) = z + \left(\frac{\alpha - 1}{n(n-\alpha)}\right) z^n \qquad (j = 1, \ldots, m).$$

Now we turn to the derivation of the following lemma which will be used in our investigation.

Lemma 2.5. *If* $f(z) \in \mathcal{M}_n^*(\alpha)$ *and* $g(z) \in \mathcal{N}_n^*(\beta)$, *then* $(f * g)(z) \in \mathcal{M}_n^*(\gamma)$, *where*

$$\gamma := 1 + \frac{(n-1)(\alpha-1)(\beta-1)}{n(n-\alpha)(n-\beta)+(\alpha-1)(\beta-1)}.$$

The result is sharp for the functions $f(z)$ *and* $g(z)$ *given by*

$$f(z) = z + \left(\frac{\alpha-1}{n-\alpha}\right) z^n$$

and

$$g(z) = z + \left(\frac{\beta-1}{n(n-\beta)}\right) z^n.$$

Proof. Let

$$f(z) = z + \sum_{k=n}^{\infty} a_k\, z^k \in \mathcal{M}_n^*(\alpha)$$

and

$$g(z) = z + \sum_{k=n}^{\infty} b_k\, z^k \in \mathcal{N}_n^*(\beta).$$

Then, by virtue of Lemma 1.1, it is sufficient to show that

$$\sum_{k=n}^{\infty} \left(\frac{k-\gamma}{\gamma-1}\right) |a_k|\,|b_k| \leqq 1$$

for $(f*g)(z) \in \mathcal{M}_n^*(\gamma)$. Indeed, since

$$\sum_{k=n}^{\infty} \left(\frac{k-\alpha}{\alpha-1} \right) |a_k| \leqq 1$$

and

$$\sum_{k=n}^{\infty} \left(\frac{k(k-\beta)}{\beta-1} \right) |b_k| \leqq 1,$$

if we assume that

$$\sum_{k=n}^{\infty} \left(\frac{k-\gamma}{\gamma-1} \right) |a_k|\,|b_k| \leqq \sum_{k=n}^{\infty} \sqrt{\frac{k(k-\alpha)(k-\beta)}{(\alpha-1)(\beta-1)}}\, |a_k|\,|b_k|,$$

so that

$$\sqrt{|a_k|\,|b_k|} \leqq \left(\frac{\gamma-1}{k-\gamma} \right) \sqrt{\frac{k(k-\alpha)(k-\beta)}{(\alpha-1)(\beta-1)}} \qquad (k=n,n+1,n+2,\dots)$$

then we prove that $(f*g)(z) \in \mathcal{M}_n^*(\gamma)$. Consequently, if γ satisfies the inequality:

$$\gamma \geqq 1 + \frac{(k-1)(\alpha-1)(\beta-1)}{k(k-\alpha)(k-\beta)+(\alpha-1)(\beta-1)} \qquad (k=n,n+1,n+2,\dots),$$

then $(f*g)(z) \in \mathcal{M}_n^*(\gamma)$. Thus it is easy to see that $(f*g)(z) \in \mathcal{M}_n^*(\gamma)$ with γ given already in Lemma 2.5. □

By combining Theorem 2.1 and Theorem 2.3 with Lemma 2.5, we arrive at

Theorem 2.6. *If* $f_j(z) \in \mathcal{M}_n^*(\alpha_j)$ $(j = 1, \ldots, p)$ *and* $g_j(z) \in \mathcal{N}_n^*(\beta_j)$ $(j = 1, \ldots, q)$*, then*

$$(f_1 * \cdots * f_p * g_1 * \cdots * g_q)(z) \in \mathcal{M}_n^*(\gamma),$$

where

$$\gamma = 1 + \frac{(n-1)(\alpha-1)(\beta-1)}{n(n-\alpha)(n-\beta)+(\alpha-1)(\beta-1)},$$

$$\alpha = 1 + \frac{(n-1)\prod_{j=1}^{p}(\alpha_j-1)}{\prod_{j=1}^{p}(n-\alpha_j)+\prod_{j=1}^{p}(\alpha_j-1)}, \tag{2.8}$$

and

$$\beta = 1 + \frac{(n-1)\prod_{j=1}^{q}(\beta_j-1)}{n^{q-1}\prod_{j=1}^{q}(n-\beta_j)+\prod_{j=1}^{q}(\beta_j-1)}. \tag{2.9}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, \ldots, p)$ *and* $g_j(z)$ $(j = 1, \ldots, q)$ *given by*

$$f_j(z) = z + \left(\frac{\alpha_j - 1}{n - \alpha_j}\right) z^n \qquad (j = 1, \ldots, p) \tag{2.10}$$

and

$$g_j(z) = z + \left(\frac{\beta_j - 1}{n(n - \beta_j)}\right) z^n \qquad (j = 1, \ldots, q). \tag{2.11}$$

For $\alpha_j = \alpha\ (j = 1, \dots, p)$ *and* $\beta_j = \beta\ (j = 1, \dots, q)$, Theorem 2.6 immediately yields

Corollary 2.7. *If* $f_j(z) \in \mathcal{M}_n^*(\alpha)\ (j = 1, \dots, p)$ *and* $g_j(z) \in \mathcal{N}_n^*(\beta)$ $(j = 1, \dots, q)$*, then*

$$(f_1 * \cdots * f_p * g_1 * \cdots * g_q)(z) \in \mathcal{M}_n^*(\gamma),$$

where

$$\gamma = 1 + \frac{(n-1)(\alpha-1)^p(\beta-1)^q}{n^q(n-\alpha)^p(n-\beta)^q + (\alpha-1)^p(\beta-1)^q}.$$

The result is sharp for the functions $f_j(z)\ (j = 1, \dots, p)$ *and* $g_j(z)\ (j = 1, \dots, q)$ *given by*

$$f_j(z) = z + \left(\frac{\alpha-1}{n-\alpha}\right) z^n \qquad (j = 1, \dots, p) \tag{2.12}$$

and

$$g_j(z) = z + \left(\frac{\beta-1}{n(n-\beta)}\right) z^n \qquad (j = 1, \dots, q). \tag{2.13}$$

We also have the following results analogous to Theorem 2.6 and Corollary 2.7:

Theorem 2.8. *If* $f_j(z) \in \mathcal{M}_n^*(\alpha_j)\ (j = 1, \dots, p)$ *and* $g_j(z) \in \mathcal{N}_n^*(\beta_j)$ $(j = 1, \dots, q)$*, then*

$$(f_1 * \cdots * f_p * g_1 * \cdots * g_q)(z) \in \mathcal{N}_n^*(\gamma),$$

where

$$\gamma = 1 + \frac{(n-1)(\alpha-1)(\beta-1)}{(n-\alpha)(n-\beta)+(\alpha-1)(\beta-1)}, \tag{2.14}$$

α and β are given by (2.8) and (2.9), respectively. The result is sharp for the functions $f_j(z)$ $(j = 1, \ldots, p)$ and $g_j(z)$ $(j = 1, \ldots, q)$ given by (2.10) and (2.11), respectively.

Corollary 2.9. *If $f_j(z) \in \mathcal{M}_n^*(\alpha)$ $(j = 1, \ldots, p)$ and $g_j(z) \in \mathcal{N}_n^*(\beta)$ $(j = 1, \ldots, q)$, then*

$$(f_1 * \cdots * f_p * g_1 * \cdots * g_q)(z) \in \mathcal{N}_n^*(\gamma),$$

where

$$\gamma = 1 + \frac{(n-1)(\alpha-1)^p(\beta-1)^q}{n^{q-1}(n-\alpha)^p(n-\beta)^q+(\alpha-1)^p(\beta-1)^q}. \tag{2.15}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, \ldots, q)$ and $g_j(z)$ $(j = 1, \ldots, q)$ given by (2.12) and (2.13), respectively.

3. Generalizations of Convolution Properties

For functions $f_j(z)$ $(j = 1, \ldots, m)$ given by (1.11), the *generalized* convolution (or the *generalized* Hadamard product) is defined here by

$$(3.1) \qquad (f_1 \bullet \cdots \bullet f_m)(z) := z + \sum_{k=n}^{\infty} \left(\prod_{j=1}^{m} (a_{k,j})^{\frac{1}{p_j}} \right) z^k$$

$$\left(\sum_{j=1}^{m} \frac{1}{p_j} = 1;\ p_j > 1;\ j = 1, \ldots, m \right).$$

Our first result for the generalized convolution defined by (3.1) is contained in

Theorem 3.1. *If* $f_j(z) \in \mathcal{M}_n^*(\alpha_j)$ $(j = 1, \ldots, m)$, *then*

$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{M}_n^*(\beta),$$

where

$$(3.2) \qquad \beta = 1 + \frac{(n-1)\prod_{j=1}^{m}(\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^{m}(n - \alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^{m}(\alpha_j - 1)^{\frac{1}{p_j}}}$$

and

$$\frac{n-1}{\prod_{j=1}^{m}(\alpha_j - 1)} \left(\sum_{j=1}^{m} \left(\frac{\alpha_j}{p_j} \right) - 1 \right) \geqq 2.$$

The result is sharp for the functions $f_j(z)$ $(j = 1, \dots, m)$ *given by*

$$f_j(z) = z + \left(\frac{\alpha_j - 1}{n - \alpha_j}\right) z^n \qquad (j = 1, \dots, m). \tag{3.3}$$

Proof. We use the principle of mathematical induction once again for the proof of Theorem 3.1. Since, for $f_1(z) \in \mathcal{M}_n^*(\alpha_1)$ and $f_2(z) \in \mathcal{M}_n^*(\alpha_2)$,

$$\sum_{k=n}^{\infty} \left(\frac{k - \alpha_j}{\alpha_j - 1}\right) |a_{k,j}| \leqq 1 \qquad (j = 1, 2),$$

we have

$$\prod_{j=1}^{2} \left(\sum_{k=n}^{\infty} \left\{ \left(\frac{k - \alpha_j}{\alpha_j - 1}\right)^{\frac{1}{p_j}} |a_{k,j}|^{\frac{1}{p_j}} \right\}^{p_j} \right)^{\frac{1}{p_j}} \leqq 1. \tag{3.4}$$

Therefore, by appealing to the Hölder inequality, we find from (3.4) that

$$\sum_{k=n}^{\infty} \left\{ \prod_{j=1}^{2} \left(\frac{k - \alpha_j}{\alpha_j - 1}\right)^{\frac{1}{p_j}} |a_{k,j}|^{\frac{1}{p_j}} \right\} \leqq 1,$$

which implies that

$$\prod_{j=1}^{2} |a_{k,j}|^{\frac{1}{p_j}} \leqq \prod_{j=1}^{2} \left(\frac{\alpha_j - 1}{k - \alpha_j}\right)^{\frac{1}{p_j}} \qquad (k = n, n+1, n+2, \dots). \tag{3.5}$$

Now we need to find the smallest δ $\left(1 < \delta < \frac{n+1}{2}\right)$ which satisfies the inequality:

$$\sum_{k=n}^{\infty}\left(\frac{k-\delta}{\delta-1}\right)\left(\prod_{j=1}^{2}|a_{k,j}|^{\frac{1}{p_j}}\right) \leqq 1.$$

By virtue of the inequality (3.5), this means that we find the smallest δ $\left(1 < \delta < \frac{n+1}{2}\right)$ such that

$$\sum_{k=n}^{\infty}\left(\frac{k-\delta}{\delta-1}\right)\left(\prod_{j=1}^{2}|a_{k,j}|^{\frac{1}{p_j}}\right) \leqq \sum_{k=n}^{\infty}\left(\frac{k-\delta}{\delta-1}\right)\left(\prod_{j=1}^{2}\left(\frac{\alpha_j-1}{k-\alpha_j}\right)^{\frac{1}{p_j}}\right) \leqq 1,$$

that is, that

$$\frac{k-\delta}{\delta-1} \leqq \prod_{j=1}^{2}\left(\frac{k-\alpha_j}{\alpha_j-1}\right)^{\frac{1}{p_j}} \qquad (k = n, n+1, n+2, \dots),$$

which yields

$$\delta \geqq 1 + \frac{(k-1)\prod_{j=1}^{2}(\alpha_j-1)^{\frac{1}{p_j}}}{\prod_{j=1}^{2}(k-\alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^{2}(\alpha_j-1)^{\frac{1}{p_j}}} \qquad (k = n, n+1, n+2, \dots).$$

Let us define

$$h(k) := \frac{k-1}{\prod_{j=1}^{2}(k-\alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^{2}(\alpha_j-1)^{\frac{1}{p_j}}} \qquad (k \geqq n).$$

Then, for the numerator $N(k)$ of $h'(k)$, we have

$$\begin{aligned} N(k) &= (\alpha_1 - 1)^{\frac{1}{p_1}} (\alpha_2 - 1)^{\frac{1}{p_2}} - (k-\alpha_1)^{\frac{1}{p_1}-1} (k-\alpha_2)^{\frac{1}{p_2}-1} \\ &\qquad \cdot \left(\frac{k-1}{p_1}(k-\alpha_2) + \frac{k-1}{p_2}(k-\alpha_1) - (k-\alpha_1)(k-\alpha_2) \right) \\ &\leqq (\alpha_1 - 1)^{\frac{1}{p_1}} (\alpha_2 - 1)^{\frac{1}{p_2}} - (k-\alpha_1)^{\frac{1}{p_1}-1} (k-\alpha_2)^{\frac{1}{p_2}-1} \\ &\qquad \cdot \left(\frac{1}{p_1}(k-\alpha_2)(\alpha_1 - 1) + \frac{1}{p_2}(k-\alpha_1)(\alpha_2 - 1) \right). \end{aligned}$$

Since $k \geqq n$ and $1 < \alpha_j < \frac{n+1}{2}$, we note that $k - \alpha_j > \alpha_j - 1$ $(j = 1, 2)$. This implies that

$$\begin{aligned} N(k) &\leqq -(k-\alpha_1)^{\frac{1}{p_1}-1} (k-\alpha_2)^{\frac{1}{p_2}-1} \\ &\qquad \cdot \left(\frac{1}{p_1}(k-\alpha_2)(\alpha_1 - 1) + \frac{1}{p_2}(k-\alpha_1)(\alpha_2 - 1) - (\alpha_1 - 1)(\alpha_2 - 1) \right) \\ &\leqq -(k-\alpha_1)^{\frac{1}{p_1}-1} (k-\alpha_2)^{\frac{1}{p_2}-1} \\ &\qquad \cdot \left(\frac{1}{p_1}(n-\alpha_2)(\alpha_1 - 1) + \frac{1}{p_2}(n-\alpha_1)(\alpha_2 - 1) - (\alpha_1 - 1)(\alpha_2 - 1) \right) \\ &= -(k-\alpha_1)^{\frac{1}{p_1}-1} (k-\alpha_2)^{\frac{1}{p_2}-1} \\ &\qquad \cdot \left\{ (n-1)\left(\frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} - 1 \right) - 2(\alpha_1 - 1)(\alpha_2 - 1) \right\} \\ &\leqq 0, \end{aligned}$$

by means of the condition of Theorem 3.1. This implies that $h(k)$ is decreasing for $k \geqq n$. Consequently, we have

$$\delta = 1 + \frac{(n-1)\prod_{j=1}^{2}(\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^{2}(n-\alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^{2}(\alpha_j - 1)^{\frac{1}{p_j}}}.$$

Thus the assertion of Theorem 3.1 holds true when $m = 2$.

Next we suppose that

$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{M}_n^*(\gamma),$$

where

$$\gamma = 1 + \frac{(n-1)\prod_{j=1}^{m}(\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^{m}(n-\alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^{m}(\alpha_j - 1)^{\frac{1}{p_j}}}.$$

Then, clearly, the first half of the above proof implies that

$$(f_1 \bullet \cdots \bullet f_{m+1})(z) \in \mathcal{M}_n^*(\beta)$$

with

$$\beta = 1 + \frac{(n-1)(\gamma-1)^{1-\frac{1}{p_{m+1}}}(\alpha_{m+1}-1)^{\frac{1}{p_{m+1}}}}{(n-\gamma)^{1-\frac{1}{p_{m+1}}}(n-\alpha_{m+1})^{\frac{1}{p_{m+1}}} + (\gamma-1)^{1-\frac{1}{p_{m+1}}}(\alpha_{m+1}-1)^{\frac{1}{p_{m+1}}}}.$$

It is easy to verify that

$$\beta = 1 + \frac{(n-1)\prod_{j=1}^{m+1}(\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^{m+1}(n-\alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^{m+1}(\alpha_j - 1)^{\frac{1}{p_j}}}.$$

Thus, by the principle of mathematical induction, we conclude that

$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{M}_n^*(\beta),$$

where β is given by (3.2).

Finally, by taking the functions $f_j(z)$ $(j = 1, \ldots, m)$ given by (3.3), we have

$$(f_1 \bullet \cdots \bullet f_m)(z) = z + \left(\prod_{j=1}^{m}\left(\frac{\alpha_j - 1}{n - \alpha_j}\right)\right) z^n,$$

which shows that

$$\left(\frac{n-\beta}{\beta-1}\right)\left(\prod_{j=1}^{m}\left(\frac{\alpha_j - 1}{n - \alpha_j}\right)^{\frac{1}{p_j}}\right) = 1.$$

Therefore, Theorem 3.1 is sharp for the functions $f_j(z)$ $(j = 1, \ldots, m)$ given by (3.3). This completes the proof of Theorem 3.1. □

By putting $\alpha_j = \alpha$ $(j = 1, \ldots, m)$ in Theorem 3.1, we obtain

Corollary 3.2. *If $f_j(z) \in \mathcal{M}_n^*(\alpha)$ $(j = 1, \ldots, m)$, then*

$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{M}_n^*(\alpha).$$

The result is sharp for the functions $f_j(z)$ $(j = 1, \ldots, m)$ given by

$$f_j(z) = z + \left(\frac{\alpha - 1}{n - \alpha}\right) z^n \qquad (j = 1, \ldots, m).$$

Similarly, for the generalized convolution defined by (3.1) for functions in the class $\mathcal{N}_n^*(\alpha)$, we derive

Theorem 3.3. *If* $f_j(z) \in \mathcal{N}_n^*(\alpha_j)$ $(j = 1, \ldots, m)$, *then*

$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{N}_n^*(\beta),$$

where

$$\beta = 1 + \frac{(n-1)\prod_{j=1}^{m}(\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^{m}(n-\alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^{m}(\alpha_j - 1)^{\frac{1}{p_j}}}. \tag{3.6}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, \ldots, m)$ *given by*

$$f_j(z) = z + \left(\frac{\alpha_j - 1}{n(n-\alpha_j)}\right) z^n \qquad (j = 1, \ldots, m). \tag{3.7}$$

Proof. By applying the same technique as in the proof of Theorem 3.1, we find that $(f_1 \bullet f_2)(z) \in \mathcal{N}_n^*(\delta)$, where

$$\delta \geqq 1 + \frac{(k-1)\prod_{j=1}^{2}(\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^{2}(k-\alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^{2}(\alpha_j - 1)^{\frac{1}{p_j}}} \qquad (k = n, n+1, n+2, \ldots),$$

for $f_1(z) \in \mathcal{N}_n^*(\alpha_1)$ and $f_2(z) \in \mathcal{N}_n^*(\alpha_2)$. Therefore, we have

$$\delta = 1 + \frac{(n-1)\prod_{j=1}^{2}(\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^{2}(k-\alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^{2}(\alpha_j - 1)^{\frac{1}{p_j}}}. \tag{3.8}$$

Furthermore, by assuming that

$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{N}_n^*(\gamma),$$

where

$$\gamma = 1 + \frac{(n-1)\prod_{j=1}^{m}(\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^{m}(k-\alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^{m}(\alpha_j - 1)^{\frac{1}{p_j}}},$$

we can show that

$$(f_1 \bullet \cdots \bullet f_{m+1})(z) \in \mathcal{N}_n^*(\beta),$$

where

$$\beta = 1 + \frac{(n-1)\prod_{j=1}^{m+1}(\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^{m+1}(k-\alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^{m+1}(\alpha_j - 1)^{\frac{1}{p_j}}}.$$

Therefore, using the principle of mathematical induction once again, we conclude that

$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{N}_n^*(\beta)$$

with β given by (3.6).

It is clear that the result of Theorem 3.3 is sharp for the functions $f_j(z)$ $(j = 1, \ldots, m)$ given by (3.7). □

Finally, by letting $\alpha_j = \alpha$ $(j = 1, \ldots, m)$ in Theorem 3.3, we deduce

Corollary 3.4. *If* $f_j(z) \in \mathcal{N}_n^*(\alpha)$ $(j = 1, \ldots, m)$, *then*

$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{N}_n^*(\alpha).$$

The result is sharp for the functions $f_j(z)$ $(j = 1, \ldots, m)$ *given by*

$$f_j(z) = z + \left(\frac{\alpha - 1}{n(n-\alpha)}\right) z^n \qquad (j = 1, \ldots, m). \tag{3.9}$$

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