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NORMS OF CERTAIN OPERATORS ON WEIGHTED ℓ_p SPACES AND LORENTZ SEQUENCE SPACES

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ABSTRACT. The problem addressed is the exact determination of the norms of the classical Hilbert, Copson and averaging operators on weighted ℓ_p spaces and the corresponding Lorentz sequence spaces $d(w,p)$, with the power weighting sequence $w_n = n^{-\alpha}$ or the variant defined by $w_1 + \cdots + w_n = n^{1-\alpha}$. Exact values are found in each case except for the averaging operator with $w_n = n^{-\alpha}$, for which estimates deriving from various different methods are obtained and compared.

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1. INTRODUCTION

In [13], the first author determined the norms and so-called "lower bounds" of the Hilbert, Copson and averaging operators on $\ell_1(w)$ and on the Lorentz sequence space $d(w,1)$, with the power weighting sequence $w_n = 1/n^\alpha$ or the closely related sequence (equally natural in the context of Lorentz spaces) given by $W_n = n^{1-\alpha}$, where $W_n = w_1 + \cdots + w_n$. In the present paper, we address the problem of finding the norms of these operators in the case $p > 1$. The problem of lower bounds was considered in a companion paper [14].

The classical inequalities of Hilbert, Copson and Hardy describe the norms of these operators on ℓ_p (where $p > 1$). Solutions to our problem need to reproduce these inequalities when we take $w_n = 1$, and the results of [13] when we take $p = 1$. The methods used for the case $p = 1$

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no longer apply. The norms of these operators on $d(w,p)$ are determined by their action on *decreasing*, non-negative sequences in $\ell_p(w)$: we denote this quantity by $\Delta_{p,w}$. In most cases, it turns out to coincide with the norm on $\ell_p(w)$ itself. In the context of $\ell_p(w)$, we also consider the increasing weight $w_n = n^\alpha$, although such weights do not generate a Lorentz sequence space. This case cannot always be treated together with $1/n^\alpha$, because of the reversal of some inequalities at $\alpha = 0$.

Our two special choices of w are alternative analogues of the weighting function $1/x^\alpha$ in the continuous case. The solutions of the continuous analogues of our problems are well known and quite simple to establish. Best-constant estimations are notoriously harder for the discrete case, essentially because discrete sums may be greater or less than their approximating integrals.

There is an extensive literature on boundedness of various classes of operators on ℓ_p spaces, with or without weights. Less attention has been given to the exact evaluation of norms. Our study aims to do this for the most "natural" operators and weights: as we shall see, the problem is already quite hard enough for these specific cases without attempting anything more general. Indeed, we fail to reach an exact solution in one important case. Problems involving two indices p, q, or two weights, lead rapidly to intractable supremum evaluations. Though we do formulate some estimates applying to general weights, our main objective is not to present new results of a general nature. Rather, given the wealth of known results and methods, the task is to identify the ones that lead to a solution, or at least a sharp estimate, for the problems under consideration. Any particular theorem can be effective in one context and ineffective in another.

For the Hilbert operator H, the "Schur" method can be adapted to show that the value from the continuous case is reproduced: for either choice of w, we have

$$\|H\|_{p,w} = \Delta_{p,w}(H) = \frac{\pi}{\sin[(1-\alpha)\pi/p]}.$$

The Copson operator C and the averaging operator A are triangular instead of symmetric, and other methods are needed to deliver the right constant even when $w_n = 1$. A better starting point is Bennett's systematic set of theorems on "factorable" triangular matrices [4, 5, 6]. For C, one such theorem can be applied to show that (for general w), $\|C\|_{p,w} \le p \sup_{n\ge 1}(W_n/nw_n)$, and hence that $\|C\|_{p,w} = \Delta_{p,w}(C) = p/(1-\alpha)$ for both our decreasing weights (reproducing the value in the continuous case).

For the averaging operator A, a similar method gives the value $p/(p-1-\alpha)$ (reproducing the continuous case) for the *increasing* weight n^α (where $\alpha < p-1$). For $W_n = n^{1-\alpha}$, classical methods can be adapted to show that $\Delta_{p,w}(A) = p/(p-1+\alpha)$, suggesting that this weight is the "right" analogue of $1/x^\alpha$ in this context (though we do not know whether $\|A\|_{p,w}$ has the same value). However, for $w_n = 1/n^\alpha$, the problem is much more difficult. A simple example shows that the above value is not correct. We can only identify and compare the estimates deriving from the various theorems and methods available; different estimates are sharper in different cases. The best estimate provided by the factorable-matrix theorems is $p\zeta(p+\alpha)$, and we show that this can be replaced by the scale of estimates $[r\zeta(r+\alpha)]^{r/p}$ for $1 \le r \le p$. The case $r = 1$ occurs as a point on another scale of estimates derived by the Schur method. A precise solution would have to reproduce the known values p^* when $\alpha = 0$ and $\zeta(1+\alpha)$ when $p = 1$: it seems unlikely that it can be expressed by a single reasonably simple formula in terms of p and α.

2. PRELIMINARIES

Let $w = (w_n)$ be a sequence of positive numbers. We write $W_n = w_1 + \cdots + w_n$ (and similarly for sequences denoted by (x_n), (y_n), etc.). Let $p \ge 1$. By $\ell_p(w)$ we mean the space of sequences $x = (x_n)$ with

$$S_p = \sum_{n=1}^{\infty} w_n |x_n|^p$$

convergent, with norm $\|x\|_{p,w} = S_p^{1/p}$. When $w_n = 1$ for all n, we denote the norm by $\|.\|_p$.

Now suppose that (w_n) is decreasing, $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n$ is divergent. The Lorentz sequence space $d(w,p)$ is then defined as follows. Given a null sequence $x = (x_n)$, let (x_n^*) be the decreasing rearrangement of $|x_n|$. Then $d(w,p)$ is the space of null sequences x for which x^* is in $\ell_p(w)$, with norm $\|x\|_{d(w,p)} = \|x^*\|_{p,w}$.

We denote by e_n the sequence having 1 in place n and 0 elsewhere.

Let A be the operator defined by $Ax = y$, where $y_i = \sum_{j=1}^{\infty} a_{i,j} x_j$. We write $\|A\|_p$ for the norm of A as an operator on ℓ_p, and $\|A\|_{p,w,v}$ for its norm as an operator from $\ell_p(w)$ to $\ell_p(v)$ (or just $\|A\|_{p,w}$ when $v = w$). This norm equates to the norm of another operator on ℓ_p itself: by substitution, one has $\|A\|_{p,w,v} = \|B\|_p$, where B is the operator with matrix $b_{i,j} = v_i^{1/p} a_{i,j} w_j^{-1/p}$.

We assume throughout that $a_{i,j} \geq 0$ for all i, j, which implies in each case that the norm is determined by the action of A on non-negative sequences. Next, we establish conditions, adequate for the operators considered below, ensuring that $\|A\|_{d(w,p)}$ is determined by *decreasing*, non-negative sequences (more general conditions are given in [13, Theorem 2]). Denote by $\delta_p(w)$ the set of decreasing, non-negative sequences in $\ell_p(w)$, and define

$$\Delta_{p,w}(A) = \sup\{\|Ax\|_{p,w} : x \in \delta_p(w) : \|x\|_{p,w} = 1\}.$$

Lemma 2.1. *Suppose that (w_n) is decreasing, that $a_{i,j} \geq 0$ for all i,j, and A maps $\delta_p(w)$ into $\ell_p(w)$. Write $c_{m,j} = \sum_{i=1}^{m} a_{i,j}$. Suppose further that:*

(i) $\lim_{j\to\infty} a_{i,j} = 0$ *for each i;*

and either (ii) $a_{i,j}$ *decreases with j for each i,*

or (iii) $a_{i,j}$ *decreases with i for each j and* $c_{m,j}$ *decreases with j for each m.*

Then $\|A(x^*)\|_{d(w,p)} \geq \|A(x)\|_{d(w,p)}$ for non-negative elements x of $d(w,p)$. Hence $\|A\|_{d(w,p)} = \Delta_{p,w}(A)$.

Proof. Let $y = Ax$ and $z = Ax^*$. As before, write $X_j = x_1 + \cdots + x_j$, etc. First, assume condition (ii). By Abel summation and (ii), we have

$$y_i = \sum_{j=1}^{\infty} a_{i,j} x_j = \sum_{j=1}^{\infty} (a_{i,j} - a_{i,j+1}) X_j,$$

and similarly for z_i with X_j^* replacing X_j. Since $X_j \leq X_j^*$ for all j, we have $y_i \leq z_i$ for all i, which implies that $\|y\|_{d(w,p)} \leq \|z\|_{d(w,p)}$.

Now assume (iii). Then y_i and z_i decrease with i, and

$$Y_m = \sum_{i=1}^{m} \sum_{j=1}^{\infty} a_{i,j} x_j = \sum_{j=1}^{\infty} c_{m,j} x_j = \sum_{j=1}^{\infty} (c_{m,j} - c_{m,j+1}) X_j$$

and similarly for Z_m. Hence $Y_m \leq Z_m$ for all m. By the majorization principle (e.g. [3, 1.30]), this implies that $\sum_{i=1}^{m} y_i^p \leq \sum_{i=1}^{m} z_i^p$ for all m, and hence by Abel summation that $\|y\|_{d(w,p)} \leq \|z\|_{d(w,p)}$. □

The evaluations in [13] are based on the property, special for $p = 1$, that $\|A\|_{1,w}$ is determined by the elements e_n, and $\Delta_{1,w}(A)$ by the elements $e_1 + \cdots + e_n$. These statements fail when $p > 1$ (with or without weights). For $\|A\|_p$ this is very well known. For Δ_p, let A be the averaging operator on ℓ_p. The lower-bound estimation in Hardy's inequality shows that $\Delta_p(A) = p^*$, while integral estimation shows that if $x_n = e_1 + \cdots + e_n$, then

$$\sup_{n\geq 1} \frac{\|Ax_n\|_p}{\|x_n\|_p} = (p^*)^{1/p}.$$

The "Schur" method. By this (taking a slight historical liberty) we mean the following technique. It can be used to give a straightforward solution of the continuous analogues of all the problems considered here (cf. [11, Sections 9.2 and 9.3]). We state a slightly generalized form of the method for the discrete case.

Lemma 2.2. *Let $p > 1$ and $p^* = p/(p-1)$. Let B be the operator with matrix $(b_{i,j})$, where $b_{i,j} \geq 0$ for all i, j. Suppose that $(s_i),\ (t_j)$ are two sequences of strictly positive numbers such that for some $C_1,\ C_2$:*

$$s_i^{1/p} \sum_{j=1}^{\infty} b_{i,j} t_j^{-1/p} \leq C_1 \quad \textit{for all } i, \qquad t_j^{1/p^*} \sum_{i=1}^{\infty} b_{i,j} s_i^{-1/p^*} \leq C_2 \quad \textit{for all } j.$$

Then $\|B\|_p \leq C_1^{1/p^*} C_2^{1/p}$.

Proof. Let $y_i = \sum_{j=1}^{\infty} b_{i,j} x_j$. By Hölder's inequality,

$$\begin{aligned} y_i &= \sum_{j=1}^{\infty} (b_{i,j}^{1/p^*} t_j^{-1/pp^*})(b_{i,j}^{1/p} t_j^{1/pp^*} x_j) \\ &\leq \left(\sum_{j=1}^{\infty} b_{i,j} t_j^{-1/p} \right)^{1/p^*} \left(\sum_{j=1}^{\infty} b_{i,j} t_j^{1/p^*} x_j^p \right)^{1/p} \\ &\leq \left(C_1 s_i^{-1/p} \right)^{1/p^*} \left(\sum_{j=1}^{\infty} b_{i,j} t_j^{1/p^*} x_j^p \right)^{1/p}, \end{aligned}$$

so

$$\begin{aligned} \sum_{i=1}^{\infty} y_i^p &\leq C_1^{p/p^*} \sum_{j=1}^{\infty} x_j^p t_j^{1/p^*} \sum_{i=1}^{\infty} b_{i,j} s_i^{-1/p^*} \\ &\leq C_1^{p/p^*} C_2 \sum_{j=1}^{\infty} x_j^p. \end{aligned}$$

□

This result has usually been applied with $s_j = t_j = j$. As we will see, useful consequences can be derived from other choices of $s_j,\ t_j$.

Theorems on "factorable" triangular matrices. These theorems appeared in [4, 5, 6], formulated for the more general case of operators from ℓ_p to ℓ_q. They can be summarized as follows. In each case, operators S, T are defined by

$$(Sx)_i = a_i \sum_{j=i}^{\infty} b_j x_j, \qquad (Tx)_i = a_i \sum_{j=1}^{i} b_j x_j,$$

where $a_i, b_j \geq 0$ for all i, j. Note that S is upper-triangular, T lower-triangular. For $p > 1$, we define

$$\alpha_m = \sum_{i=1}^{m} a_i^p, \quad \tilde{\alpha}_m = \sum_{i=m}^{\infty} a_i^p, \quad \beta_m = \sum_{j=1}^{m} b_j^{p^*}, \quad \tilde{\beta}_m = \sum_{j=m}^{\infty} b_j^{p^*}.$$

Proposition 2.3. *("head" version).*

(i) *If* $\sum_{j=1}^{m}(b_j\alpha_j)^{p^*} \le K_1^{p^*}\alpha_m$ *for all* m *(in particular, if* $b_j\alpha_j \le K_1 a_j^{p-1}$ *for all* j*),then* $\|S\|_p \le pK_1$.

(ii) *If* $\sum_{i=1}^{m}(a_i\beta_i)^{p} \le K_1^{p}\beta_m$ *for all* m *(in particular, if* $a_j\beta_j \le K_1 b_j^{p^*-1}$ *for all* j*), then* $\|T\|_p \le p^*K_1$.

Proposition 2.4. *("tail" version).*

(i) *If* $\sum_{i=m}^{\infty}(a_i\tilde{\beta}_i)^{p} \le K_2^{p}\tilde{\beta}_m$ *for all* m*, (in particular, if* $a_j\tilde{\beta}_j \le K_2 b_j^{p^*-1}$ *for all* j*), then* $\|S\|_p \le p^*K_2$.

(ii) *If* $\sum_{j=m}^{\infty}(b_j\tilde{\alpha}_j)^{p^*} \le K_2^{p^*}\tilde{\alpha}_m$ *for all* m*, (in particular, if* $b_j\tilde{\alpha}_j \le K_2 a_j^{p-1}$ *for all* j*) then* $\|T\|_p \le pK_2$.

Proposition 2.5. *("mixed" version).*

(i) *If* $\alpha_m^{1/p}\tilde{\beta}_m^{1/p^*} \le K_3$ *for all* m*, then* $\|S\|_p \le p^{1/p}(p^*)^{1/p^*}K_3$.

(ii) *If* $\tilde{\alpha}_m^{1/p}\beta_m^{1/p^*} \le K_3$ *for all* m*, then* $\|T\|_p \le p^{1/p}(p^*)^{1/p^*}K_3$.

In each case, (ii) is equivalent to (i), by duality. Propositions 2.3 and 2.4 are [4, Theorem 2] and (with suitable substitutions) [5, Theorem $2'$] (note: the right-hand exponent in Bennett's formula (18) should be $(r-1)/(r-s)$). For the reader's convenience, we include here a direct proof of Proposition 2.3(i), simplified from the more general theorem in [5]; the proof of 2.4(ii) is almost the same.

Proof of Proposition 2.3. (i) Note first the following fact, easily proved by Abel summation: if $s_j, : t_j$ $(1 \le j \le N)$ are real numbers such that $\sum_{j=1}^{m} s_j \le \sum_{j=1}^{m} t_j$ for $1 \le m \le N$, and $u_1 \ge \ldots \ge u_N \ge 0$, then $\sum_{j=1}^{N} s_j u_j \le \sum_{j=1}^{N} t_j u_j$.

It is enough to consider finite sums with $i, j \le N$, and to take $x_i \ge 0$. Let $y = Sx$. Write $R_i = \sum_{j=i}^{N} b_j x_j$, so that $y_i = a_i R_i$. By Abel summation,

$$\sum_{i=1}^{N} y_i^p = \sum_{i=1}^{N} a_i^p R_i^p = \sum_{i=1}^{N-1} \alpha_i (R_i^p - R_{i+1}^p) + \alpha_N R_N^p.$$

By the mean-value theorem, $y^p - x^p \le p(y-x)y^{p-1}$ for any $x, y \ge 0$. Since $R_i - R_{i+1} = b_i x_i$ for $1 \le i \le N-1$, we have

$$R_i^p - R_{i+1}^p \le p b_i x_i R_i^{p-1}$$

for such i. Also, $R_N^p \le p b_N x_N R_N^{p-1}$, since $R_N = b_N x_N$. Hence

$$\sum_{i=1}^{N} y_i^p \le p \sum_{i=1}^{N} \alpha_i b_i x_i R_i^{p-1} \le p \left(\sum_{i=1}^{N} x_i^p\right)^{1/p} \left(\sum_{i=1}^{N} (b_i\alpha_i)^{p^*} R_i^p\right)^{1/p^*}$$

(note that $p^*(p-1) = p$). By the "fact" noted above and our hypothesis (or directly from the alternative hypothesis),

$$\sum_{i=1}^{N} (b_i\alpha_i)^{p^*} R_i^p \le K_1^{p^*} \sum_{i=1}^{N} a_i^p R_i^p = K_1^{p^*} \sum_{i=1}^{N} y_i^p.$$

Together, these inequalities give $\|y\|_p \le pK_1\|x\|_p$. □

Proposition 2.5 is [6, Theorem 9]; with standard substitutions, it is also a special case of [1, Theorems 4.1 and 4.2]. The corresponding result for the continuous case was given in [16].

We shall be particularly concerned with two choices of w, defined respectively by $w_n = 1/n^\alpha$ (for $\alpha \geq 0$) and by $W_n = n^{1-\alpha}$ (for $0 < \alpha < 1$). Note that in the second case,

$$w_n = n^{1-\alpha} - (n-1)^{1-\alpha} = \int_{n-1}^{n} \frac{1-\alpha}{t^\alpha}\, dt,$$

and hence

$$\frac{1-\alpha}{n^\alpha} \leq w_n \leq \frac{1-\alpha}{(n-1)^\alpha}.$$

Several of our estimations will be expressed in terms of the zeta function. It will be helpful to recall that $\zeta(1+\alpha) = 1/\alpha + r(\alpha)$ for $\alpha > 0$, where $\frac{1}{2} \leq r(\alpha) \leq 1$ and $r(\alpha) \to \gamma$ (Euler's constant) as $\alpha \to 0$. Also, for the evaluation of various suprema that arise, we will need the following lemmas.

Lemma 2.6. [9, Proposition 3]. *Let $A_n = \sum_{j=1}^n j^\alpha$. Then $A_n/n^{1+\alpha}$ decreases with n if $\alpha \geq 0$, and increases if $\alpha < 0$. If $\alpha > -1$, it tends to $1/(1+\alpha)$ as $n \to \infty$.*

Lemma 2.7. [8, Remark 4.10] *(without proof)*, [13, Proposition 6]. *Let $\alpha > 0$ and let $C_n = \sum_{k=n}^\infty 1/k^{1+\alpha}$. Then $n^\alpha C_n$ decreases with n, and $(n-1)^\alpha C_n$ increases.*

Lemma 2.8. [7, Lemma 8], *more simply in* [9, Proposition 1]. *The expression*

$$\frac{1}{n(n+1)^\alpha} \sum_{k=1}^{n} k^\alpha$$

increases with n when $\alpha \geq 1$ or $\alpha \leq 0$, and decreases with n if $0 \leq \alpha \leq 1$.

3. THE HILBERT OPERATOR

We consider the Hilbert operator H, with matrix $a_{i,j} = 1/(i+j)$. This satisfies conditions (i) and (ii) of Lemma 2.1. Hilbert's classical inequality states that $\|H\|_p = \pi/\sin(\pi/p)$ for $p > 1$. For the case $p = 1$, with either of our choices of w, it was shown in [13] that $\|H\|_{1,w} = \Delta_{1,w}(H) = \pi/\sin \alpha\pi$.

For the analogous operator in the continuous case, with $w(x) = 1/x^\alpha$, it is quite easily shown by the Schur method that $\|H\|_{p,w} = \pi/\sin[(1-\alpha)\pi/p]$. We show that the method adapts to the discrete case, giving this value again for either of our choices of w. This is straightforward for $w_n = 1/n^\alpha$, but rather more delicate for $W_n = n^{1-\alpha}$.

Let $0 < a < 1$. As with most studies of the Hilbert operator, we use the well-known integral

$$\int_0^\infty \frac{1}{t^a(t+c)}\, dt = \frac{\pi}{c^a \sin a\pi}.$$

Write

$$g_n(a) = \int_{n-1}^{n} \frac{1}{t^a}\, dt.$$

Note that $g_n(a) \geq 1/n^a$.

Lemma 3.1. *With this notation, we have for each $j \geq 1$,*

$$\sum_{i=1}^\infty \frac{1}{i^a(i+j)} \leq \sum_{i=1}^\infty \frac{g_i(a)}{i+j} \leq \frac{\pi}{j^a \sin a\pi}.$$

Proof. Clearly,

$$\frac{g_i(a)}{i+j} = \frac{1}{i+j}\int_{i-1}^{i} \frac{1}{t^a}\, dt \leq \int_{i-1}^{i} \frac{1}{t^a(t+j)}\, dt.$$

The statement follows, by the integral quoted above. □

Now let $0 < \alpha < 1$. Write $v_n = g_n(\alpha)$ and $v'_n = (1-\alpha)v_n$. In our previous terminology, v' is our "second choice of w". Clearly, an operator will have the same norm on $d(v', p)$ and on $d(v, p)$. Note that by Hölder's inequality for integrals, $v_n^r \le g_n(\alpha r)$ for $r > 1$.

Theorem 3.2. *Let H be the operator with matrix $a_{i,j} = 1/(i+j)$, and let $p > 1$. Let $w_n = n^{-\alpha}$, where $1 - p < \alpha < 1$. Then*

$$\|H\|_{p,w} = \Delta_{p,w}(H) = \frac{\pi}{\sin[(1-\alpha)\pi/p]}.$$

If $0 < \alpha < 1$ and $v_n = \int_{n-1}^n t^{-\alpha} dt$, then $\|H\|_{p,v}$ and $\Delta_{p,v}(H)$ also have the value stated.

Proof. Write $M = \pi/\sin[(1-\alpha)\pi/p]$. Let $w_n = n^{-\alpha}$, where $1-p < \alpha < 1$. Now $\|H\|_{p,w} = \|B\|_p$, where B has matrix $b_{i,j} = (j/i)^{\alpha/p}/(i+j)$. In Lemma 2.2, take $s_i = t_i = i$, and let C_1, C_2 be defined as before. Then

$$b_{i,j} s_i^{1/p} t_j^{-1/p} = \frac{1}{i+j} \left(\frac{i}{j}\right)^{(1-\alpha)/p}.$$

By Lemma 3.1, it follows that $C_1 \le M$. Similarly, $C_2 \le M$.

Now let $0 < \alpha < 1$, and let v, w be as stated. We show in fact that $\|H\|_{p,w,v} \le M$. It then follows that $\|H\|_{p,v} \le M$, since $\|x\|_{p,w} \le \|x\|_{p,v}$ for all x. Note that $\|H\|_{p,w,v} = \|B\|_p$, where now

$$b_{i,j} = \frac{1}{i+j} (j^\alpha v_i)^{1/p}.$$

Take $s_i = v_i^{-1/\alpha}$ and $t_j = j$. Then

$$b_{i,j}(s_i/t_j)^{1/p} = \frac{1}{i+j} (j^\alpha v_i)^{(\alpha-1)/\alpha p}.$$

By Lemma 3.1,

$$\sum_{j=1}^\infty \frac{j^{(\alpha-1)/p}}{i+j} \le i^{(\alpha-1)/p} M.$$

Since $i^{-1} \le v_i^{1/\alpha}$, we have $i^{(\alpha-1)/p} \le v_i^{(1-\alpha)/\alpha p}$, and hence $C_1 \le M$.

Also,

$$b_{i,j}(t_j/s_i)^{1/p^*} = \frac{1}{i+j} (j^\alpha v_i)^t,$$

where

$$t = \frac{1}{p} + \frac{1}{\alpha p^*}.$$

Note that $t > 1$, so as remarked above, we have $v_i^t \le g_i(\alpha t)$. By Lemma 3.1 again,

$$\sum_{i=1}^\infty \frac{g_i(\alpha t)}{i+j} \le \frac{M'}{j^{\alpha t}},$$

where $M' = \pi/\sin \alpha t \pi$. Now

$$\alpha t = \frac{\alpha}{p} + \frac{1}{p^*} = 1 - \frac{1-\alpha}{p},$$

so that $M' = M$, and hence $C_2 \le M$. The statement follows.

To show that $\Delta_{p,w}(H) \ge M$, take $r = (1-\alpha)/p$, so that $\alpha + rp = 1$. Fix N, and let

$$x_j = \begin{cases} 1/j^r & \text{for } j \le N, \\ 0 & \text{for } j > n. \end{cases}$$

Then (x_j) is decreasing and $\sum_{j=1}^{\infty} w_j x_j^p = \sum_{j=1}^{N} \frac{1}{j}$. Let $y = Hx$. By routine methods (we omit the details), one finds that

$$\sum_{i=1}^{N} w_i y_i^p \geq M^p \sum_{i=1}^{N} \frac{1}{i} - g(r),$$

where $g(r)$ is independent of N. Clearly, the required statement follows. Minor modifications give the same conclusion for v. □

Note. A variant H_0 of the discrete Hilbert operator has matrix $1/(i+j-1)$. This is decidedly more difficult! Already in the case $p = 1$, the norms do not coincide for our two weights, and it seems unlikely that there is a simple formula for $\|H_0\|_{1,w}$: see [13].

4. THE COPSON OPERATOR

The "Copson" operator C is defined by $y = Cx$, where $y_i = \sum_{j=i}^{\infty}(x_j/j)$. It is given by the transpose of the Cesaro matrix:

$$a_{i,j} = \begin{cases} 1/j & \text{for } i \leq j \\ 0 & \text{for } i > j \end{cases}.$$

This matrix satisfies conditions (i) and (iii) of Lemma 2.1. Copson's inequality [11, Theorem 331] states that $\|C\|_p \leq p$ (Copson's original result [10] was in fact the reverse inequality for the case $0 < p < 1$).

The corresponding operator in the continuous case is defined by $(Cf)(x) = \int_x^{\infty}[f(y)/y]\,dy$. When $w(x) = 1/x^{\alpha}$, it is quite easily shown, for example by the Schur method, that $\Delta_{p,w}(C) = \|C\|_{p,w} = p/(1-\alpha)$.

For the discrete case, we will show that this value is correct for either decreasing choice of w. However, the Schur method does not lead to the right constant in the discrete version of Copson's inequality, and instead we use Proposition 2.3. First we formulate a result for general weights. The *1-regularity constant* of a sequence (w_n) is defined to be $r_1(w) = \sup_{n\geq 1} W_n/(nw_n)$.

Theorem 4.1. *Suppose that (w_n) is 1-regular and $p \geq 1$. Then the Copson operator C maps $d(w,p)$ into itself, and $\|C\|_{p,w} \leq pr_1(w)$.*

Proof. We have $\|C\|_{p,w} = \|S\|_p$, where S is as in Proposition 2.3, with $a_i = w_i^{1/p}$ and $b_j = 1/(jw_j^{1/p})$. In the notation of Proposition 2.3, $\alpha_m = W_m$, so

$$b_j\alpha_j = \frac{W_j}{jw_j^{1/p}} = \frac{W_j}{jw_j} w_j^{1/p^*} \leq r_1(w) w_j^{1/p^*} = r_1(w) a_j^{p/p^*}.$$

Since $p/p^* = p - 1$, the simpler hypothesis of Proposition 2.3(i) holds with $K_1 = r_1(w)$. □

Note. The same reasoning shows that $\|C\|_{p,w,v} \leq pK_1$, where K is such that $V_n \leq K_1 n w_n^{1/p} v_n^{1/p^*}$ for all n.

An example in [15, Section 2.3] shows that $\|C\|_{p,w}$ is not necessarily equal to $pr_1(w)$. However, equality does hold for our special choices of decreasing w, as we now show.

Theorem 4.2. *Let C be the Copson operator. Suppose that $p \geq 1$ and that w is defined either by $w_n = 1/n^{\alpha}$ or by $W_n = n^{1-\alpha}$, where $0 \leq \alpha < 1$. Then*

$$\Delta_{p,w}(C) = \|C\|_{p,w} = \frac{p}{1-\alpha}.$$

Proof. We show first that in either case, $r_1(w) = 1/(1-\alpha)$, so that $\|C\|_{p,w} \le p/(1-\alpha)$. First, consider $w_n = 1/n^\alpha$. By comparison with the integrals of $1/t^\alpha$ on $[1,n]$ and $[0,n]$, we have

$$\frac{1}{1-\alpha}(n^{1-\alpha} - 1) \le W_n \le \frac{n^{1-\alpha}}{1-\alpha},$$

from which the required statement follows easily. Now consider the case $W_n = n^{1-\alpha}$. Then $nw_n/W_n = n^\alpha w_n$. By the inequalities for w_n mentioned in Section 2,

$$1 - \alpha \le n^\alpha w_n \le (1-\alpha)\frac{n^\alpha}{(n-1)^\alpha},$$

from which it is clear that again $r_1(w) = 1/(1-\alpha)$.

We now show that $\Delta_{p,w}(C) \ge 1 - \alpha$. Choose $\varepsilon > 0$ and define r by $\alpha + rp = 1 + \varepsilon$. Let $x_n = 1/n^r$ for all n; note that (x_n) is decreasing. Then $n^{-\alpha} x_n^p = 1/n^{\alpha+rp} = 1/n^{1+\varepsilon}$, so x is in $\ell_p(w)$ for either choice of (w_n) (note that for the second choice, $w_n \le c/n^\alpha$ for a constant c). Also, again by integral estimation,

$$y_n = \sum_{k=n}^{\infty} \frac{1}{k^{1+r}} \ge \frac{1}{rn^r} = \frac{x_n}{r}$$

for all n, so

$$\|y\|_{p,w} \ge \frac{1}{r}\|x\|_{p,w} = \frac{p}{1-\alpha+\varepsilon}\|x\|_{p,w}.$$

The statement follows. □

Theorem 4.1 has a very simple consequence for *increasing* weights:

Proposition 4.3. *Let (w_n) be any increasing weight sequence. Then $\|C\|_{p,w} \le p$.*

Proof. If (w_n) is increasing, then $W_n \le nw_n$, with equality when $n = 1$. Hence $r_1(w) = 1$. □

Clearly, $\Delta_{p,w}(C) \ge 1$, since $C(e_1) = e_1$. For the case $w_n = n^\alpha$, the method of Theorem 4.2 shows that also $\Delta_{p,w}(C) \ge p/(1+\alpha)$. A simple example shows that $\Delta_{p,w}(C)$ can be greater than both 1 and $p/(1+\alpha)$.

Example 4.1. Let $p = 2$ and $\alpha = 1$, so that $p/(1+\alpha) = 1$. Take $x = (4, 2, 0, 0, \ldots)$. Then $y = (5, 1, 0, 0, \ldots)$, and $\|x\|_{2,w}^2 = 24$, while $\|y\|_{2,w}^2 = 27$.

We leave further investigation of this case to another study; it is analogous to the problem of the averaging operator with $w_n = 1/n^\alpha$, considered in more detail below.

5. THE AVERAGING OPERATOR: RESULTS FOR GENERAL WEIGHTS

Henceforth, A will mean the averaging operator, defined by $y = Ax$, where $y_n = \frac{1}{n}(x_1 + \cdots + x_n)$. It is given by the Cesaro matrix

$$a_{i,j} = \begin{cases} 1/i & \text{for } j \le i \\ 0 & \text{for } j > i \end{cases},$$

which satisfies conditions (i) and (ii) of Lemma 2.1. In this section, we give some results for general weighting sequences. We consider upper estimates first. Hardy's inequality states that $\|A\|_p = p^*$ for $p > 1$. It is easy to deduce that the same upper estimate applies for any decreasing w:

Proposition 5.1. *If (w_n) is any decreasing, non-negative sequence, then $\|A\|_{p,w} \le p^*$.*

Proof. Let x be a non-negative element of $\ell_p(w)$, and let $u_j = w_j^{1/p} x_j$. Then $\|u\|_p = \|x\|_{p,w}$, and since (w_j) is decreasing,

$$w_n^{1/p} X_n \le \sum_{j=1}^{n} w_j^{1/p} x_j = U_n$$

(where, as usual, X_n means $x_1 + \cdots + x_n$). Hence

$$\begin{aligned} \|Ax\|_{p,w}^p &= \sum_{n=1}^{\infty} \frac{w_n}{n^p} X_n^p \\ &\le \sum_{n=1}^{\infty} \frac{1}{n^p} U_n^p \\ &\le (p^*)^p \sum_{n=1}^{\infty} u_n^p \quad \text{by Hardy's inequality} \\ &= (p^*)^p \|x\|_{p,w}^p. \end{aligned}$$

□

The next result records the estimates for the averaging operator derived from Propositions 2.3, 2.4 and 2.5. Instead of the fully general statements, we give simpler forms pertinent to our objectives. For $1 \le r \le p$, define

$$U_m(r) = \sum_{j=m}^{\infty} \frac{w_j}{j^r}, \qquad V(r) = \sup_{m \ge 1} \frac{m^{r-1} U_m(r)}{w_m}.$$

Proposition 5.2. *Let A be the averaging operator and let $p > 1$. Then:*

(i) $\|A\|_{p,w} \le p^* K_1$, where $\sum_{j=1}^{m} w_j^{1-p^*} \le K_1 m w_m^{1-p^*}$ for all m;
(ii) $\|A\|_{p,w} \le pV(p)$;
(iii) if (w_n) is decreasing, then $\|A\|_{p,w} \le p^{1/p} (p^*)^{1/p^*} V(p)^{1/p}$.

Proof. Recall that $\|A\|_{p,w} = \|T\|_p$, where T is as in Proposition 2.3, with $a_i = w_i^{1/p}/i$ and $b_j = w_j^{-1/p}$. With notation as before, we have $\tilde{\alpha}_m = \sum_{j \ge m} w_j / j^p = U_m(p)$ and $\beta_m = \sum_{j=1}^{m} w_j^{-p^*/p}$. Noting that $p^*/p = p^* - 1$, one checks easily that Propositions 2.3(ii) and 2.4(ii) (with the simpler, alternative hypotheses) translate into (i) and (ii).

For (iii), note that if (w_n) is decreasing, then $\beta_m \le m w_m^{-p^*/p}$. Hence $\tilde{\alpha}_m \beta_m^{p-1} \le m^{p-1} U_m(p)/w_m$, so the condition of Proposition 2.5(iii) is satisfied with $K_3 = V(p)^{1/p}$. □

For decreasing (w_n), (i) will give an estimate no less than p^*, hence no advance on Proposition 5.1 (one need only take $m = 1$ to see that K_1 is at least 1). Also, (iii) adds nothing to (ii), since

$$[pV(p)]^{1/p} (p^*)^{1/p^*} \ge \min[pV(p), p^*].$$

In the specific case we consider below, the supremum defining $V(p)$ is attained at $m = 1$. In such a case, nothing is lost by the inequality used in (iii) for β_m. Furthermore, nothing would be gained by using the more general [1, Theorem 4.1] instead of Proposition 2.5.

Another result relating to our $V(p)$ is [12, Corollary of Theorem 3.4] (repeated, with an extra condition removed, in [2, Corollary 4.3]). The quantity considered is $C = \sup_{n \ge 1} n^p U_n(p)/W_n$. Again, in our case, C coincides with $V(p)$. It is shown in [12] that if C is finite, then so is $\Delta_{p,w}(A)$, without giving an explicit relationship.

The next theorem improves on the estimate in (ii) by exhibiting it as one point in a scale of estimates. We are not aware that it is a case of any known result.

Theorem 5.3. *Suppose that* $1 \le r \le p$ *and* $\sum_{n=1}^\infty w_n/n^r$ *is convergent. Define* $V(r)$ *as above. Then* $\|A\|_{p,w} \le [rV(r)]^{r/p}$.

Proof. Write $U_n(r) = U_n$ and $V(r) = V$. Let $z_n = x_n^{p/r}$ and (as usual) $Z_n = z_1 + \cdots + z_n$. By Hölder's inequality,

$$X_n \le n^{1-r/p} Z_n^{r/p},$$

hence $X_n^p \le n^{p-r} Z_n^r$. If $y = Ax$, then $y_n = X_n/n$, so $y_n^p \le n^{-r} Z_n^r$. For any N, we have

$$\begin{aligned}
\sum_{n=1}^N \frac{w_n}{n^r} Z_n^r &= \sum_{n=1}^N (U_n - U_{n+1}) Z_n^r \\
&= \sum_{n=1}^N U_n (Z_n^r - Z_{n-1}^r) - U_{N+1} Z_N^r \\
&\le r \sum_{n=1}^N U_n z_n Z_n^{r-1} \quad \text{by the mean-value theorem} \\
&\le rV \sum_{n=1}^N \frac{w_n}{n^{r-1}} z_n Z_n^{r-1} \quad \text{by the definition of } V \\
&\le rV \left(\sum_{n=1}^N w_n z_n^r \right)^{1/r} \left(\sum_{n=1}^N \frac{w_n}{n^r} Z_n^r \right)^{1/r^*} \quad \text{by Hölder's inequality.}
\end{aligned}$$

Since $z_n^r = x_n^p$, it follows that (for all N)

$$\sum_{n=1}^N w_n y_n^p \le \sum_{n=1}^N \frac{w_n}{n^r} Z_n^r \le (rV)^r \sum_{n=1}^N w_n x_n^p.$$

□

The case $r = 1$ (for which the proof, of course, becomes simpler), gives $\|A\|_{p,w} \le V(1)^{1/p}$, in which

$$V(1) = \sup_{m \ge 1} \frac{1}{w_m} \sum_{j=m}^\infty \frac{w_j}{j}.$$

Among the above results (including the Schur method) only Proposition 5.2(i) delivers the right constant p^* in Hardy's original inequality. Another method that does so is the classical one of [11, Theorem 326]. The next result shows what is obtained by adapting this method to the weighted case. Note that it applies to $\Delta_{p,w}(A)$, not $\|A\|_{p,w}$.

Theorem 5.4. *Let* $p \ge 1$*, and let*

$$c = \sup_{n \ge 1} \frac{n w_{n+1}}{W_n}.$$

Assume that $c < p$*. Then*

$$\Delta_{p,w}(A) \le \frac{p}{p-c}.$$

Proof. Let $x = (x_n)$ be a decreasing, non-negative element of $\ell_p(w)$, and let $y = Ax$. Fix a positive integer N. By Abel summation,

$$\sum_{n=1}^N w_n y_n^p = \sum_{n=2}^N W_{n-1}(y_{n-1}^p - y_n^p) + W_N y_N^p.$$

By the mean-value theorem, $y_{n-1}^p - y_n^p \geq p y_n^{p-1}(y_{n-1} - y_n)$. Also, for $n \geq 2$,

$$x_n = n y_n - (n-1) y_{n-1} = y_n - (n-1)(y_{n-1} - y_n),$$

so

$$y_{n-1} - y_n = \frac{1}{n-1}(y_n - x_n).$$

Hence

$$\begin{aligned} \sum_{n=1}^{N} w_n y_n^p &\geq p \sum_{n=2}^{N} W_{n-1} y_n^{p-1}(y_{n-1} - y_n) \\ &= p \sum_{n=2}^{N} \frac{W_{n-1}}{n-1} y_n^{p-1}(y_n - x_n). \end{aligned}$$

Now since (x_n) is decreasing, $y_n \geq x_n$ for all n. Also, $y_1 = x_1$. Since $w_n \leq c W_{n-1}/(n-1)$, we deduce that

$$\sum_{n=1}^{N} w_n y_n^p \geq \frac{p}{c} \sum_{n=2}^{N} w_n y_n^{p-1}(y_n - x_n) = \frac{p}{c} \sum_{n=1}^{N} w_n y_n^{p-1}(y_n - x_n),$$

so that

$$(p - c) \sum_{n=1}^{N} w_n y_n^p \leq p \sum_{n=1}^{N} w_n y_n^{p-1} x_n.$$

A standard application of Hölder's inequality to the right-hand side finishes the proof. □

We now consider lower estimates. First, an obvious one:

Proposition 5.5. *We have*

$$\Delta_{p,w}(A) \geq \left(\frac{1}{w_1} \sum_{n=1}^{\infty} \frac{w_n}{n^p} \right)^{1/p}.$$

Proof. Take $x = e_1$. Then $y_n = 1/n$ for all n, so the statement follows. (In particular, the stated series must converge in order for $A(e_1)$ to be in $\ell_p(w)$.) □

Secondly, we formulate the lower estimate derived by the classical method of considering $x_n = n^{-s}$ for a suitable s.

Lemma 5.6. *Suppose that a_n, b_n are non-negative numbers such that $\sum_{n=1}^{\infty} a_n$ is divergent and $\lim_{n\to\infty} b_n = 0$. Then*

$$\frac{\sum_{n=1}^{N} a_n b_n}{\sum_{n=1}^{N} a_n} \to 0 \quad \textit{as } N \to \infty.$$

Proof. Elementary. □

Theorem 5.7. *Let A be the averaging operator and let $p > 1$. Let (w_n) be any positive sequence, and let $q < p$ be such that $\sum_{n=1}^{\infty} w_n/n^q$ is divergent. Then*

$$\Delta_{w,p}(A) \geq \frac{p}{p-q}.$$

Proof. Write $p/(p-q) = r$, so that $1/r = 1 - q/p$. Fix N, and let

$$x_n = \begin{cases} n^{-q/p} & \text{for } 1 \leq n \leq N \\ 0 & \text{for } n > N \end{cases}.$$

Then

$$\sum_{n=1}^{\infty} w_n x_n^p = \sum_{n=1}^{N} \frac{w_n}{n^q}. \tag{5.1}$$

Also, for $n \le N$,

$$X_n \ge \int_1^n t^{-q/p}\, dt = r(n^{1/r} - 1),$$

so that

$$y_n = \frac{X_n}{n} \ge \frac{r}{n^{q/p}}(1 - n^{-1/r}).$$

Since $(1-t)^p \ge 1 - pt$ for $0 < t < 1$, we have

$$y_n^p \ge \frac{r^p}{n^q}(1 - pn^{-1/r}),$$

and hence

$$w_n y_n^p \ge r^p \frac{w_n}{n^q} - pr^p \frac{w_n}{n^{q+1/r}}(2). \tag{5.2}$$

Take $\varepsilon > 0$. By (5.1) and (5.2), together with Lemma 5.6 (with $a_n = w_n/n^q$ and $b_n = n^{-1/r}$), it is clear that for all large enough N,

$$\sum_{n=1}^{N} w_n y_n^p \ge (1-\varepsilon) r^p \sum_{n=1}^{\infty} w_n x_n^p.$$

□

Corollary 5.8. *If $\sum_{n=1}^{\infty} w_n/n$ is divergent (in particular, if (w_n) is increasing), then $\Delta_{p,w}(A) \ge p^*$.*

Remarks on the relation between $\|A\|_{p,w}$ and $\Delta_{p,w}(A)$. If (w_n) is *increasing*, then $\|A\|_{p,w} = \Delta_{p,w}(A)$, for if $x^* = (x_n^*)$ is the decreasing rearrangement of x, then it is easily seen that $\|x^*\|_{p,w} \le \|x\|_{p,w}$, while $\|Ax^*\|_{p,w} \ge \|Ax\|_{p,w}$. The following example shows that this is not true for *decreasing* weights in general (though it may possibly be true for $1/n^\alpha$), and indeed that the estimate in Theorem 5.4 does not apply to $\|A\|_{p,w}$.

Example 5.1. Let $E_k = \{i : k! \le i < (k+1)!\}$, and let $w_i = 1/(k!k)$ for $i \in E_k$. Then $\sum_{i\in E_k} w_i = 1$ and, by integral estimation, $\sum_{i\in E_k}(1/i) > \log(k+1)$. Now fix k, and choose p close enough to 1 to have $\sum_{i\in E_k} i^{-p} > \log k$. Write $\|\ \|$ for $\|\ \|_{p,w}$. Take $n = k!$. Then $\|e_n\|^p = w_n = 1/(k!k)$, while

$$\|Ae_n\|^p > \sum_{i\in E_k} \frac{w_i}{i^p} > \frac{\log k}{k!k}.$$

Hence $\|A\|_{p,w}^p > \log k$. We show that $nw_{n+1}/W_n \le \frac{2}{3}$ for all n, so that $\Delta_{p,w}(A) \le 3$, by Theorem 5.4. If $k \ge 3$ and $n \in E_k$, then $W_n \ge k-1$, so

$$\frac{nw_{n+1}}{W_n} \le \frac{(k+1)!}{k!k}\frac{1}{k-1} = \frac{k+1}{k(k-1)} \le \frac{2}{3}.$$

The required inequality is easily checked for n in E_1 and E_2.

6. THE AVERAGING OPERATOR: RESULTS FOR SPECIFIC WEIGHTS

We now explore the extent to which the results of Section 5 solve the problem for our chosen weighting sequences. The analogous operator in the continuous case is given by $(Af)(x) = \frac{1}{x}\int_0^x f$. When $w(x) = x^{-\alpha}$, one can show by the Schur method (or as in [17, Theorem 1.9.16]) that

$$\|A\|_{p,w} = \Delta_{p,w}(A) = p/(p-1+\alpha).$$

In the discrete case, when $p = 1$, it was shown in [13] that:

(i) if $w_n = 1/n^\alpha$, then $\|A\|_{1,w} = \Delta_{1,w}(A) = \zeta(1+\alpha)$,
(ii) if $W_n = n^{1-\alpha}$, then $\Delta_{1,w}(A) = 1/\alpha$.

As suggested by (ii) and Hardy's inequality, the continuous case is reproduced when $W_n = n^{1-\alpha}$:

Theorem 6.1. *Let A be the averaging operator and let $p \geq 1$. Let (w_n) be defined by $W_n = n^{1-\alpha}$, where $0 \leq \alpha < 1$. Then*

$$\Delta_{p,w}(A) = \frac{p}{p-1+\alpha}.$$

Proof. Recall that

$$w_{n+1} \leq \frac{1-\alpha}{n^\alpha} \leq w_n.$$

Hence

$$\frac{nw_{n+1}}{W_n} = n^\alpha w_{n+1} \leq 1-\alpha$$

for $n \geq 1$, so Theorem 5.4 applies with $c = 1-\alpha$. Also, Theorem 5.7 applies with $q = 1-\alpha$. The stated equality follows. □

We do not know whether $\|A\|_{p,w}$ has the same value; however, $\Delta_{p,w}(A)$ is of more interest for this choice of w, since it is motivated by Lorentz spaces.

For the increasing weight $w_n = n^\alpha$, our problem is solved by the method of Proposition 2.3 (which, it will be recalled, was effective for the Copson operator with *decreasing* weights).

Theorem 6.2. *Let $w_n = n^\alpha$, where $0 \leq \alpha < p-1$. Then*

$$\|A\|_{p,w} = \Delta_{p,w}(A) = \frac{p}{p-1-\alpha}.$$

Proof. By Theorem 5.7, we have $\Delta_{p,w}(A) \geq p/(p-1-\alpha)$. We use Proposition 5.2(i) to prove the reverse inequality. The condition is

$$\sum_{j=1}^m \frac{1}{j^{\alpha(p^*-1)}} \leq K_1 \frac{m}{m^{\alpha(p^*-1)}}.$$

Note that $\alpha(p^*-1) = \alpha/(p-1) < 1$. By Lemma 2.6, the condition is satisfied with

$$K_1 = \frac{1}{1-\alpha(p^*-1)} = \frac{p-1}{p-1-\alpha}.$$

So $\|A\|_{p,w} \leq p^* K_1 = p/(p-1-\alpha)$. □

Note. For $\alpha < 1$, this result also follows from Theorem 5.4 and Lemma 2.8.

We now come to the hard case, $w_n = 1/n^\alpha$. The trivial lower estimate 5.5 is enough to show that $\Delta_{p,w}(A)$ can be greater than $p/(p-1+\alpha)$. Indeed, we have at once from Proposition 5.5 and Theorem 5.7:

Proposition 6.3. *Let $p > 1$ and $w_n = 1/n^\alpha$, where $\alpha \geq 0$. Then $\Delta_{p,w}(A) \geq \max(m_1, m_2)$, where*

$$m_1 = \frac{p}{p-1+\alpha}, \quad m_2 = \zeta(p+\alpha)^{1/p}.$$

Either lower estimate can be larger (even when $\alpha < 1$), as the following table shows:

p	α	m_1	m_2
1.1	0.9	1.1	1.572
2	0.1	1.818	1.249

No improvement on m_2 is obtained by considering elements of the form e_n or $e_1 + \cdots + e_n$. However, the next example shows that $\Delta_{p,w}(A)$ can be greater than both m_1 and m_2.

Example 6.1. Let $p = 2$, $\alpha = 1$. Then $m_1 = 1$ and $m_2 = \zeta(3)^{1/2} \approx 1.096$. Let $x = (2, 1, 0, 0, \ldots)$. Then $y_1 = 2$ and $y_n = 3/n$ for $n \geq 2$. Hence $\|x\|_{2,w}^2 = 4\frac{1}{2}$, while $\|y\|_{2,w}^2 = 4 + 9[\zeta(3) - 1] \approx 5.818$, so that $\|y\|_{2,w}/\|x\|_{2,w} \approx 1.137$.

(One could formulate another general lower estimate using this choice of x, but it is too unpleasant to be worth stating explicitly.)

We now record the upper estimates derived from the various results above. Of course, by Proposition 5.1, we have $\|A\|_{p,w} \leq p^*$.

Proposition 6.4. *Let $p > 1$, and let $w_n = 1/n^\alpha$, where $\alpha \geq 0$. Then*

$$\Delta_{p,w}(A) \leq M_1 =: \frac{p}{p - 2^{-\alpha}}.$$

Proof. We have

$$\frac{W_n}{nw_{n+1}} = \frac{(n+1)^\alpha}{n} \sum_{k=1}^{n} \frac{1}{k^\alpha}.$$

By Lemma 2.8, this expression increases with n, so has its least value 2^α when $n = 1$. Hence Theorem 5.4 applies with $c = 2^{-\alpha}$. ☐

Proposition 6.5. *Let $p > 1$ and $1 \leq r \leq p$, and let $w_n = 1/n^\alpha$, where $\alpha \geq 0$. Then*

$$\|A\|_{p,w} \leq M_2(r) =: [r\zeta(r+\alpha)]^{r/p}.$$

Proof. Apply Theorem 5.3. In the notation used there, we have

$$\frac{m^{r-1}U_m(r)}{w_m} = m^{r+\alpha-1} \sum_{j=m}^{\infty} \frac{1}{j^{r+\alpha}}.$$

By Lemma 2.7, this expression decreases with m, so is greatest when $m = 1$, with the value $\zeta(r+\alpha)$. ☐

Note that in particular, $M_2(1) = \zeta(1+\alpha)^{1/p}$ and $M_2(p) = p\zeta(p+\alpha)$. By the remark after Proposition 5.2, $\min[M_2(p), p^*] \leq p^{1/p}(p^*)^{1/p^*} m_2 \leq 2m_2$.

To optimize the estimate in Proposition 6.5, we have to choose r (in the interval $[1, p]$) to minimize $[r\zeta(r+\alpha)]^r$. Computations show that when $\alpha \geq 0.4$, the least value occurs when $r = 1$, while for $\alpha = 0.1$, it occurs when $r \approx 1.35$.

The Schur method provides another scale of estimates, as follows.

Proposition 6.6. *Let $w_n = 1/n^\alpha$, where $\alpha > 0$. Let $\alpha \leq r < p + \alpha$. Then $\|A\|_{p,w} \leq M_3(r)$, where*

$$M_3(r) = \left(\frac{p}{p-r+\alpha}\right)^{1/p^*} \zeta\left(1 + \frac{r}{p^*} + \frac{\alpha}{p}\right)^{1/p}.$$

Proof. In Lemma 2.2, take $s_j = t_j = j^r$. We have $b_{i,j} = a_i b_j$ for $j \leq i$, where $a_i = i^{-\alpha/p-1}$ and $b_j = j^{\alpha/p}$. So C_1 is the supremum (as i varies) of

$$i^{(r-\alpha)/p-1} \sum_{j=1}^{i} \frac{1}{j^{(r-\alpha)/p}}.$$

By Lemma 2.6 (or trivially when $r = \alpha$),

$$C_1 = \frac{1}{1-(r-\alpha)/p} = \frac{p}{p-r+\alpha}.$$

Also, C_2 is the supremum (as j varies) of

$$j^{\alpha/p+r/p^*} \sum_{i=j}^{\infty} \frac{1}{i^{1+\alpha/p+r/p^*}}.$$

By Lemma 2.7, this is greatest when $j = 1$, with value $\zeta(1 + r/p^* + \alpha/p)$. □

Note that $M_3(\alpha) = M_2(1) = \zeta(1+\alpha)^{1/p}$. Hence M_3 will always be at least as good as M_2 when $\alpha > 0.4$.

The following table compares the estimates in some particular cases. We denote by m the greater of the lower estimates m_1, m_2. Recall that M_1 estimates $\Delta_{p,w}(A)$. The values of r used for $M_2(r)$ and $M_3(r)$ are indicated.

p	α	m	p^*	M_1	$M_2(r)$	$M_3(r)$
1.1	0.1	5.5	11	6.587	6.151 (1.1)	6.031 (0.98)
1.1	0.9	1.572	11	1.950	1.663 (1)	1.663 (0.9)
1.2	0.3	2.4	6	3.095	3.084 (1.1)	2.886 (0.9)
2	0.5	1.333	2	1.547	1.616 (1)	1.593 (0.75)

Table 6.1: Numerical values of upper and lower estimates

In most, if not all cases, M_3 is a better estimate than M_2, at the cost of being more complicated. Both M_2 and M_3 reproduce the correct value $\zeta(1+\alpha)$ when $p = 1$. On the other hand, both can be larger than p^* in other cases. Some product of the type $M_3(1)$ would reproduce the values $\zeta(1+\alpha)$ when $p = 1$ and p^* when $\alpha = 0$; the exponent would need to be of the form $f(\alpha, p)$, where $f(0, p) = 1$ for $p > 1$ and $f(\alpha, 1) = 0$ for $\alpha > 0$.

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