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ON A REVERSE OF JESSEN'S INEQUALITY FOR ISOTONIC LINEAR FUNCTIONALS

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ABSTRACT. A reverse of Jessen's inequality and its version for $m - \Psi-$convex and $M - \Psi-$convex functions are obtained. Some applications for particular cases are also pointed out.

Key words and phrases: Jessen's Inequality, Isotonic Linear Functionals.

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1. INTRODUCTION

Let L be a linear class of real-valued functions $g : E \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L,$ i.e., if $f_0(t) = 1, t \in E$ then $f_0 \in L.$

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}.$
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0.$
The mapping A is said to be *normalised if*
(A3) $A(\mathbf{1}) = 1.$

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2] and [10]).

We recall Jessen's inequality (see also [8]).

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Theorem 1.1. *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ (I is an interval), be a convex function and $f : E \to I$ such that $\phi \circ f$, $f \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$\phi(A(f)) \leq A(\phi \circ f). \tag{1.1}$$

A counterpart of this result was proved by Beesack and Pečarić in [2] for compact intervals $I = [\alpha, \beta]$.

Theorem 1.2. *Let $\phi : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $f : E \to [\alpha, \beta]$ such that $\phi \circ f$, $f \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$A(\phi \circ f) \leq \frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\phi(\beta). \tag{1.2}$$

Remark 1.3. Note that (1.2) is a generalisation of the inequality

$$A(\phi) \leq \frac{b - A(e_1)}{b - a}\phi(a) + \frac{A(e_1) - a}{b - a}\phi(b) \tag{1.3}$$

due to Lupaş [9] (see for example [2, Theorem A]), which assumed $E = [a, b]$, L satisfies (L1), (L2), $A : L \to \mathbb{R}$ satisfies (A1), (A2), $A(\mathbf{1}) = 1$, ϕ is convex on E and $\phi \in L$, $e_1 \in L$, where $e_1(x) = x$, $x \in [a, b]$.

The following inequality is well known in the literature as the Hermite-Hadamard inequality

$$\varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \varphi(t)\,dt \leq \frac{\varphi(a) + \varphi(b)}{2}, \tag{1.4}$$

provided that $\varphi : [a, b] \to \mathbb{R}$ is a convex function.

Using Theorem 1.1 and Theorem 1.2, we may state the following generalisation of the Hermite-Hadamard inequality for isotonic linear functionals ([11] and [12]).

Theorem 1.4. *Let $\phi : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $e : E \to [a, b]$ with e, $\phi \circ e \in L$. If $A \to \mathbb{R}$ is an isotonic linear and normalised functional, with $A(e) = \frac{a+b}{2}$, then*

$$\varphi\left(\frac{a+b}{2}\right) \leq A(\phi \circ e) \leq \frac{\varphi(a) + \varphi(b)}{2}. \tag{1.5}$$

For other results concerning convex functions and isotonic linear functionals, see [3] – [6], [12] – [14] and the recent monograph [7].

2. The Concepts of $m - \Psi-$Convex and $M - \Psi-$Convex Functions

Assume that the mapping $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ (I is an interval) is convex on I and $m \in \mathbb{R}$. We shall say that the mapping $\phi : I \to \mathbb{R}$ is $m - \Psi-$ *lower convex* if $\phi - m\Psi$ is a convex mapping on I (see [4]). We may introduce the class of functions

$$\mathcal{L}(I, m, \Psi) := \{\phi : I \to \mathbb{R} | \phi - m\Psi \text{ is convex on } I\}. \tag{2.1}$$

Similarly, for $M \in \mathbb{R}$ and Ψ as above, we can introduce the class of $M - \Psi-$*upper convex* functions by (see [4])

$$\mathcal{U}(I, M, \Psi) := \{\phi : I \to \mathbb{R} | M\Psi - \phi \text{ is convex on } I\}. \tag{2.2}$$

The intersection of these two classes will be called the class of $(m, M) - \Psi-$*convex functions* and will be denoted by

$$\mathcal{B}(I, m, M, \Psi) := \mathcal{L}(I, m, \Psi) \cap \mathcal{U}(I, M, \Psi). \tag{2.3}$$

Remark 2.1. If $\Psi \in \mathcal{B}(I, m, M, \Psi)$, then $\phi - m\Psi$ and $M\Psi - \phi$ are convex and then $(\phi - m\Psi) + (M\Psi - \phi)$ is also convex which shows that $(M - m)\Psi$ is convex, implying that $M \geq m$ (as Ψ is assumed not to be the trivial convex function $\Psi(t) = 0$, $t \in I$).

The above concepts may be introduced in the general case of a convex subset in a real linear space, but we do not consider this extension here.

In [6], S.S. Dragomir and N.M. Ionescu introduced the concept of $g-$convex dominated mappings, for a mapping $f: I \to \mathbb{R}$. We recall this, by saying, for a given convex function $g: I \to \mathbb{R}$, the function $f: I \to \mathbb{R}$ is $g-$convex dominated iff $g+f$ and $g-f$ are convex mappings on I. In [6], the authors pointed out a number of inequalities for convex dominated functions related to Jensen's, Fuchs', Pečarić's, Barlow-Marshall-Proschan and Vasić-Mijalković results, etc.

We observe that the concept of $g-$convex dominated functions can be obtained as a particular case from $(m,M)-\Psi-$convex functions by choosing $m=-1$, $M=1$ and $\Psi=g$.

The following lemma holds (see also [4]).

Lemma 2.2. *Let $\Psi, \phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable functions on $\mathring{I}$ and Ψ is a convex function on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ iff*

$$m\left[\Psi(x) - \Psi(y) - \Psi'(y)(x-y)\right] \leq \phi(x) - \phi(y) - \phi'(y)(x-y) \tag{2.4}$$

for all $x, y \in \mathring{I}$.

(ii) *For $M \in \mathbb{R}$, the function $\phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ iff*

$$\phi(x) - \phi(y) - \phi'(y)(x-y) \leq M\left[\Psi(x) - \Psi(y) - \Psi'(y)(x-y)\right] \tag{2.5}$$

for all $x, y \in \mathring{I}$.

(iii) *For $M, m \in \mathbb{R}$ with $M \geq m$, the function $\phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ iff both (2.4) and (2.5) hold.*

Proof. Follows by the fact that a differentiable mapping $h: I \to \mathbb{R}$ is convex on $\mathring{I}$ iff $h(x) - h(y) \geq h'(y)(x-y)$ for all $x, y \in \mathring{I}$. □

Another elementary fact for twice differentiable functions also holds (see also [4]).

Lemma 2.3. *Let $\Psi, \phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathring{I}$ and Ψ is convex on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ iff*

$$m\Psi''(t) \leq \phi''(t) \quad \text{for all } t \in \mathring{I}. \tag{2.6}$$

(ii) *For $M \in \mathbb{R}$, the function $\phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ iff*

$$\phi''(t) \leq M\Psi''(t) \quad \text{for all } t \in \mathring{I}. \tag{2.7}$$

(iii) *For $M, m \in \mathbb{R}$ with $M \geq m$, the function $\phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ iff both (2.6) and (2.7) hold.*

Proof. Follows by the fact that a twice differentiable function $h: I \to \mathbb{R}$ is convex on $\mathring{I}$ iff $h''(t) \geq 0$ for all $t \in \mathring{I}$. □

We consider the $p-$logarithmic mean of two positive numbers given by

$$L_p(a,b) := \begin{cases} a & \text{if} \quad b = a, \\ \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a \neq b \end{cases}$$

and $p \in \mathbb{R} \diagdown \{-1, 0\}$.

The following proposition holds (see also [4]).

Proposition 2.4. *Let $\phi : (0, \infty) \to \mathbb{R}$ be a differentiable mapping.*

(i) *For* $m \in \mathbb{R}$*, the function* $\phi \in \mathcal{L}\left((0,\infty), m, (\cdot)^p\right)$ *with* $p \in (-\infty, 0) \cup (1,\infty)$ *iff*

$$mp(x-y)\left[L_{p-1}^{p-1}(x,y) - y^{p-1}\right] \leq \phi(x) - \phi(y) - \phi'(y)(x-y) \tag{2.8}$$

for all $x, y \in (0,\infty)$.

(ii) *For* $M \in \mathbb{R}$*, the function* $\phi \in \mathcal{U}\left((0,\infty), M, (\cdot)^p\right)$ *with* $p \in (-\infty, 0) \cup (1,\infty)$ *iff*

$$\phi(x) - \phi(y) - \phi'(y)(x-y) \leq Mp(x-y)\left[L_{p-1}^{p-1}(x,y) - y^{p-1}\right] \tag{2.9}$$

for all $x, y \in (0,\infty)$.

(iii) *For* $M, m \in \mathbb{R}$ *with* $M \geq m$*, the function* $\phi \in \mathcal{B}\left((0,\infty), M, (\cdot)^p\right)$ *with* $p \in (-\infty, 0) \cup (1,\infty)$ *iff both (2.8) and (2.9) hold.*

The proof follows by Lemma 2.2 applied for the convex mapping $\Psi(t) = t^p$, $p \in (-\infty, 0) \cup (4,\infty)$ and via some elementary computation. We omit the details.

The following corollary is useful in practice.

Corollary 2.5. *Let* $\phi : (0,\infty) \to \mathbb{R}$ *be a differentiable function.*

(i) *For* $m \in \mathbb{R}$*, the function* ϕ *is* $m-$*quadratic-lower convex (i.e., for* $p = 2$*) iff*

$$m(x-y)^2 \leq \phi(x) - \phi(y) - \phi'(y)(x-y) \tag{2.10}$$

for all $x, y \in (0,\infty)$.

(ii) *For* $M \in \mathbb{R}$*, the function* ϕ *is* $M-$*quadratic-upper convex iff*

$$\phi(x) - \phi(y) - \phi'(y)(x-y) \leq M(x-y)^2 \tag{2.11}$$

for all $x, y \in (0,\infty)$.

(iii) *For* $m, M \in \mathbb{R}$ *with* $M \geq m$*, the function* ϕ *is* $(m, M)-$*quadratic convex if both (2.10) and (2.11) hold.*

The following proposition holds (see also [4]).

Proposition 2.6. *Let* $\phi : (0,\infty) \to \mathbb{R}$ *be a twice differentiable function.*

(i) *For* $m \in \mathbb{R}$*, the function* $\phi \in \mathcal{L}\left((0,\infty), m, (\cdot)^p\right)$ *with* $p \in (-\infty, 0) \cup (1,\infty)$ *iff*

$$p(p-1)mt^{p-2} \leq \phi''(t) \quad \text{for all } t \in (0,\infty). \tag{2.12}$$

(ii) *For* $M \in \mathbb{R}$*, the function* $\phi \in \mathcal{U}\left((0,\infty), M, (\cdot)^p\right)$ *with* $p \in (-\infty, 0) \cup (1,\infty)$ *iff*

$$\phi''(t) \leq p(p-1)Mt^{p-2} \quad \text{for all } t \in (0,\infty). \tag{2.13}$$

(iii) *For* $m, M \in \mathbb{R}$ *with* $M \geq m$*, the function* $\phi \in \mathcal{B}\left((0,\infty), m, M, (\cdot)^p\right)$ *with* $p \in (-\infty, 0) \cup (1,\infty)$ *iff both (2.12) and (2.13) hold.*

As can be easily seen, the above proposition offers the practical criterion of deciding when a twice differentiable mapping is $(\cdot)^p$ −lower or $(\cdot)^p$ −upper convex and which weights the constant m and M are.

The following corollary is useful in practice.

Corollary 2.7. *Assume that the mapping* $\phi : (a,b) \subseteq \mathbb{R} \to \mathbb{R}$ *is twice differentiable.*

(i) *If* $\inf_{t\in(a,b)} \phi''(t) = k > -\infty$*, then* ϕ *is* $\frac{k}{2}-$*quadratic lower convex on* (a,b);

(ii) *If* $\sup_{t\in(a,b)} \phi''(t) = K < \infty$*, then* ϕ *is* $\frac{K}{2}-$*quadratic upper convex on* (a,b).

3. A REVERSE INEQUALITY

We start with the following result which gives another counterpart for $A(\phi \circ f)$, as did the Lupaş-Beesack-Pečarić result (1.2).

Theorem 3.1. *Let $\phi : (\alpha, \beta) \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (α, β), $f : E \to (\alpha, \beta)$ such that $\phi \circ f$, f, $\phi' \circ f$, $\phi' \circ f \cdot f \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$\begin{aligned} 0 &\leq A(\phi \circ f) - \phi(A(f)) \\ &\leq A(\phi' \circ f \cdot f) - A(f) \cdot A(\phi' \circ f) \\ &\leq \frac{1}{4}[\phi'(\beta) - \phi'(\alpha)](\beta - \alpha) \quad \textit{(if } \alpha, \beta \textit{ are finite).} \end{aligned} \tag{3.1}$$

Proof. As ϕ is differentiable convex on (α, β), we may write that

$$\phi(x) - \phi(y) \geq \phi'(y)(x - y), \text{ for all } x, y \in (\alpha, \beta), \tag{3.2}$$

from where we obtain

$$\phi(A(f)) - (\phi \circ f)(t) \geq (\phi' \circ f)(t)(A(f) - f(t)) \tag{3.3}$$

for all $t \in E$, as, obviously, $A(f) \in (\alpha, \beta)$.

If we apply to (3.3) the functional A, we may write

$$\phi(A(f)) - A(\phi \circ f) \geq A(f) \cdot A(\phi' \circ f) - A(\phi' \circ f \cdot f),$$

which is clearly equivalent to the first inequality in (3.1).

It is well known that the following Grüss inequality for isotonic linear and normalised functionals holds (see [1])

$$|A(hk) - A(h)A(k)| \leq \frac{1}{4}(M - m)(N - n), \tag{3.4}$$

provided that $h, k \in L$, $hk \in L$ and $-\infty < m \leq h(t) \leq M < \infty$, $-\infty < n \leq k(t) \leq N < \infty$, for all $t \in E$.

Taking into account that for finite α,β we have $\alpha < f(t) < \beta$ with ϕ' being monotonic on (α, β), we have $\phi'(\alpha) \leq \phi' \circ f \leq \phi'(\beta)$, and then by the Grüss inequality, we may state that

$$A(\phi' \circ f \cdot f) - A(f) \cdot A(\phi' \circ f) \leq \frac{1}{4}[\phi'(\beta) - \phi'(\alpha)](\beta - \alpha)$$

and the theorem is completely proved. □

The following corollary holds.

Corollary 3.2. *Let $\phi : [a, b] \subset \mathring{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on $\mathring{I}$. If ϕ, e_1, ϕ', $\phi' \cdot e_1 \in L$ ($e_1(x) = x$, $x \in [a, b]$) and $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then:*

$$\begin{aligned} 0 &\leq A(\phi) - \phi(A(e_1)) \\ &\leq A(\phi' \cdot e_1) - A(e_1) \cdot A(\phi') \\ &\leq \frac{1}{4}[\phi'(b) - \phi'(a)](b - a). \end{aligned} \tag{3.5}$$

There are some particular cases which can naturally be considered.

(1) Let $\phi(x) = \ln x$, $x > 0$. If $\ln f$, f, $\frac{1}{f} \in L$ and $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then:

$$0 \leq \ln[A(f)] - A[\ln(f)] \leq A(f)A\left(\frac{1}{f}\right) - 1, \tag{3.6}$$

provided that $f(t) > 0$ for all $t \in E$ and $A(f) > 0$.

If $0 < m \leq f(t) \leq M < \infty$, $t \in E$, then, by the second part of (3.1) we have:

$$A(f)\,A\left(\frac{1}{f}\right) - 1 \leq \frac{(M-m)^2}{4mM} \qquad \text{(which is a known result).} \tag{3.7}$$

Note that the inequality (3.6) is equivalent to

$$1 \leq \frac{A(f)}{\exp[A[\ln(f)]]} \leq \exp\left[A(f)\,A\left(\frac{1}{f}\right) - 1\right]. \tag{3.8}$$

(2) Let $\phi(x) = \exp(x)$, $x \in \mathbb{R}$. If $\exp(f)$, f, $f \cdot \exp(f) \in L$ and $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then

$$\begin{aligned} 0 &\leq A[\exp(f)] - \exp[A(f)] \\ &\leq A[f\exp(f)] - A(f)\exp[A(f)] \\ &\leq \frac{1}{4}[\exp(M) - \exp(m)](M-m) \quad (\text{if } m \leq f \leq M \text{ on } E). \end{aligned} \tag{3.9}$$

4. A Further Result for $m-\Psi-$Convex and $M-\Psi-$Convex Functions

In [4], S.S. Dragomir proved the following inequality of Jessen's type for $m-\Psi-$convex and $M-\Psi-$convex functions.

Theorem 4.1. *Let $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $f : E \to I$ such that $\Psi \circ f$, $f \in L$ and $A : L \to \mathbb{R}$ be an isotonic linear and normalised functional.*

(i) *If $\phi \in \mathcal{L}(I, m, \Psi)$ and $\phi \circ f \in L$, then we have the inequality*

$$m[A(\Psi \circ f) - \Psi(A(f))] \leq A(\phi \circ f) - \phi(A(f)). \tag{4.1}$$

(ii) *If $\phi \in \mathcal{U}(I, M, \Psi)$ and $\phi \circ f \in L$, then we have the inequality*

$$A(\phi \circ f) - \phi(A(f)) \leq M[A(\Psi \circ f) - \Psi(A(f))]. \tag{4.2}$$

(iii) *If $\phi \in \mathcal{B}(I, m, M, \Psi)$ and $\phi \circ f \in L$, then both (4.1) and (4.2) hold.*

The following corollary is useful in practice.

Corollary 4.2. *Let $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$, $f : E \to I$ such that $\Psi \circ f$, $f \in L$ and $A : L \to \mathbb{R}$ be an isotonic linear and normalised functional.*

(i) *If $\phi : I \to \mathbb{R}$ is twice differentiable and $\phi''(t) \geq m\Psi''(t)$, $t \in \mathring{I}$ (where m is a given real number), then (4.1) holds, provided that $\phi \circ f \in L$.*

(ii) *If $\phi : I \to \mathbb{R}$ is twice differentiable and $\phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$ (where M is a given real number), then (4.2) holds, provided that $\phi \circ f \in L$.*

(iii) *If $\phi : I \to \mathbb{R}$ is twice differentiable and $m\Psi''(t) \leq \phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$, then both (4.1) and (4.2) hold, provided $\phi \circ f \in L$.*

In [5], S.S. Dragomir obtained the following result of Lupaş-Beesack-Pečarić type for $m-\Psi-$convex and $M-\Psi-$convex functions.

Theorem 4.3. *Let $\Psi : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $f : I \to [\alpha, \beta]$ such that $\Psi \circ f$, $f \in L$ and $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional.*

(i) *If $\phi \in \mathcal{L}(I, m, \Psi)$ and $\phi \circ f \in L$, then we have the inequality*

$$\begin{aligned} m&\left[\frac{\beta - A(f)}{\beta - \alpha}\Psi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\Psi(\beta) - A(\Psi \circ f)\right] \\ &\leq \frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\phi(\beta) - A(\phi \circ f). \end{aligned} \tag{4.3}$$

(ii) *If $\phi \in \mathcal{U}(I, M, \Psi)$ and $\phi \circ f \in L$, then*

$$\frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\phi(\beta) - A(\phi \circ f) \leq M\left[\frac{\beta - A(f)}{\beta - \alpha}\Psi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\Psi(\beta) - A(\Psi \circ f)\right]. \tag{4.4}$$

(iii) *If $\phi \in \mathcal{B}(I, m, M, \Psi)$ and $\phi \circ f \in L$, then both (4.3) and (4.4) hold.*

The following corollary is useful in practice.

Corollary 4.4. *Let $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$, $f : E \to I$ such that $\Psi \circ f$, $f \in L$ and $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional.*

(i) *If $\phi : I \to \mathbb{R}$ is twice differentiable, $\phi \circ f \in L$ and $\phi''(t) \geq m\Psi''(t)$, $t \in \mathring{I}$ (where m is a given real number), then (4.3) holds.*

(ii) *If $\phi : I \to \mathbb{R}$ is twice differentiable, $\phi \circ f \in L$ and $\phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$ (where m is a given real number), then (4.4) holds.*

(iii) *If $m\Psi''(t) \leq \phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$, then both (4.3) and (4.4) hold.*

We now prove the following new result.

Theorem 4.5. *Let $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable convex function and $f : E \to I$ such that $\Psi \circ f$, $\Psi' \circ f$, $\Psi' \circ f \cdot f$, $f \in L$ and $A : L \to \mathbb{R}$ be an isotonic linear and normalised functional.*

(i) *If ϕ is differentiable, $\phi \in \mathcal{L}(\mathring{I}, m, \Psi)$ and $\phi \circ f$, $\phi' \circ f$, $\phi' \circ f \cdot f \in L$, then we have the inequality*

$$m[A(\Psi' \circ f \cdot f) + \Psi(A(f)) - A(f) \cdot A(\Psi' \circ f) - A(\Psi \circ f)] \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f). \tag{4.5}$$

(ii) *If ϕ is differentiable, $\phi \in \mathcal{U}(\mathring{I}, M, \Psi)$ and $\phi \circ f$, $\phi' \circ f$, $\phi' \circ f \cdot f \in L$, then we have the inequality*

$$A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f) \leq M[A(\Psi' \circ f \cdot f) + \Psi(A(f)) - A(f) \cdot A(\Psi' \circ f) - A(\Psi \circ f)]. \tag{4.6}$$

(iii) *If ϕ is differentiable, $\phi \in \mathcal{B}(\mathring{I}, m, M, \Psi)$ and $\phi \circ f$, $\phi' \circ f$, $\phi' \circ f \cdot f \in L$, then both (4.5) and (4.6) hold.*

Proof. The proof is as follows.

(i) As $\phi \in \mathcal{L}(I, m, \Psi)$, then $\phi - m\Psi$ is convex and we can apply the first part of the inequality (3.1) for $\phi - m\Psi$ getting

$$A[(\phi - m\Psi) \circ f] - (\phi - m\Psi)(A(f)) \leq A\left[(\phi - m\Psi)' \circ f \cdot f\right] - A(f)\,A\left((\phi - m\Psi)' \circ f\right). \tag{4.7}$$

However,

$$\begin{aligned} A[(\phi - m\Psi) \circ f] &= A(\phi \circ f) - mA(\Psi \circ f), \\ (\phi - m\Psi)(A(f)) &= \phi(A(f)) - m\Psi(A(f)), \\ A\left[(\phi - m\Psi)' \circ f \cdot f\right] &= A(\phi' \circ f \cdot f) - mA(\Psi' \circ f \cdot f) \end{aligned}$$

and

$$A\left((\phi - m\Psi)' \circ f\right) = A(\phi' \circ f) - mA(\Psi' \circ f)$$

and then, by (4.7), we deduce the desired inequality (4.5).

(ii) Goes likewise and we omit the details.

(iii) Follows by (i) and (ii).

□

The following corollary is useful in practice,

Corollary 4.6. *Let $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$, $f : E \to I$ such that $\Psi \circ f$, $\Psi' \circ f$, $\Psi' \circ f \cdot f$, $f \in L$ and $A : L \to \mathbb{R}$ be an isotonic linear and normalised functional.*

(i) *If $\phi : I \to \mathbb{R}$ is twice differentiable, $\phi \circ f$, $\phi' \circ f$, $\phi' \circ f \cdot f \in L$ and $\phi''(t) \geq m\Psi''(t)$, $t \in \mathring{I}$, (where m is a given real number), then the inequality (4.5) holds.*

(ii) *With the same assumptions, but if $\phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$, (where M is a given real number), then the inequality (4.6) holds.*

(iii) *If $m\Psi''(t) \leq \phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$, then both (4.5) and (4.6) hold.*

Some particular important cases of the above corollary are embodied in the following proposition.

Proposition 4.7. *Assume that the mapping $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is twice differentiable on $\mathring{I}$.*

(i) *If $\inf_{t\in\mathring{I}} \phi''(t) = k > -\infty$, then we have the inequality*

$$\frac{1}{2}k\left[A\left(f^2\right) - [A(f)]^2\right] \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f), \tag{4.8}$$

provided that $\phi \circ f$, $\phi' \circ f$, $\phi' \circ f \cdot f, f^2 \in L$.

(ii) *If $\sup_{t\in\mathring{I}} \phi''(t) = K < \infty$, then we have the inequality*

$$A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f) \leq \frac{1}{2}K\left[A\left(f^2\right) - [A(f)]^2\right]. \tag{4.9}$$

(iii) *If $-\infty < k \leq \phi''(t) \leq K < \infty$, $t \in \mathring{I}$, then both (4.8) and (4.9) hold.*

The proof follows by Corollary 4.6 applied for $\Psi(t) = \frac{1}{2}t^2$ and $m = k$, $M = K$.

Another result is the following one.

Proposition 4.8. *Assume that the mapping $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ is twice differentiable on $\mathring{I}$. Let $p \in (-\infty, 0) \cup (1, \infty)$ and define $g_p : I \to \mathbb{R}$, $g_p(t) = \phi''(t)\, t^{2-p}$.*

(i) *If $\inf_{t\in\mathring{I}} g_p(t) = \gamma > -\infty$, then we have the inequality*

$$\frac{\gamma}{p(p-1)}\left[(p-1)\left[A(f^p) - [A(f)]^p\right] - pA(f)\left[A\left(f^{p-1}\right) - [A(f)]^{p-1}\right]\right] \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f), \tag{4.10}$$

provided that $\phi \circ f$, $\phi' \circ f$, $\phi' \circ f \cdot f, f^p, f^{p-1} \in L$.

(ii) *If $\sup_{t\in\mathring{I}} g_p(t) = \Gamma < \infty$, then we have the inequality*

$$A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f) \leq \frac{\Gamma}{p(p-1)}\left[(p-1)\left[A(f^p) - [A(f)]^p\right] - pA(f)\left[A\left(f^{p-1}\right) - [A(f)]^{p-1}\right]\right]. \tag{4.11}$$

(iii) *If $-\infty < \gamma \leq g_p(t) \leq \Gamma < \infty$, $t \in \mathring{I}$, then both (4.10) and (4.11) hold.*

Proof. The proof is as follows.

(i) We have for the auxiliary mapping $h_p(t) = \phi(t) - \frac{\gamma}{p(p-1)} t^p$ that

$$\begin{aligned} h_p''(t) &= \phi''(t) - \gamma t^{p-2} = t^{p-2}\left(t^{2-p}\phi''(t) - \gamma\right) \\ &= t^{p-2}\left(g_p(t) - \gamma\right) \geq 0. \end{aligned}$$

That is, h_p is convex or, equivalently, $\phi \in \mathcal{L}\left(I, \frac{\gamma}{p(p-1)}, (\cdot)^p\right)$. Applying Corollary 4.6, we get

$$\frac{\gamma}{p(p-1)}\left[pA(f^p) + [A(f)]^p - pA(f)A\left(f^{p-1}\right) - A(f^p)\right] \\ \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f),$$

which is clearly equivalent to (4.10).

(ii) Goes similarly.

(iii) Follows by (i) and (ii).

□

The following proposition also holds.

Proposition 4.9. *Assume that the mapping $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ is twice differentiable on $\mathring{I}$. Define $l(t) = t^2\phi''(t)$, $t \in I$.*

(i) *If $\inf_{t\in\mathring{I}} l(t) = s > -\infty$, then we have the inequality*

$$(4.12)\quad s\left[A(f)A\left(\frac{1}{f}\right) - 1 - (\ln[A(f)] - A[\ln(f)])\right] \\ \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f),$$

provided that $\phi \circ f, \phi^{-1} \circ f, \phi^{-1} \circ f \cdot f, \frac{1}{f}, \ln f \in L$ and $A(f) > 0$.

(ii) *If $\sup_{t\in\mathring{I}} l(t) = S < \infty$, then we have the inequality*

$$(4.13)\quad A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f) \\ \leq S\left[A(f)A\left(\frac{1}{f}\right) - 1 - (\ln[A(f)] - A[\ln(f)])\right].$$

(iii) *If $-\infty < s \leq l(t) \leq S < \infty$ for $t \in \mathring{I}$, then both (4.12) and (4.13) hold.*

Proof. The proof is as follows.

(i) Define the auxiliary function $h(t) = \phi(t) + s\ln t$. Then

$$h''(t) = \phi''(t) - \frac{s}{t^2} = \frac{1}{t^2}\left(\phi''(t)\,t^2 - s\right) \geq 0$$

which shows that h is convex, or, equivalently, $\phi \in \mathcal{L}(I, s, -\ln(\cdot))$. Applying Corollary 4.6, we may write

$$s\left[-A(\mathbf{1}) - \ln A(f) + A(f)A\left(\frac{1}{f}\right) + A(\ln(f))\right] \\ \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f),$$

which is clearly equivalent to (4.12).

(ii) Goes similarly.

(iii) Follows by (i) and (ii).

□

Finally, the following result also holds.

Proposition 4.10. *Assume that the mapping $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ is twice differentiable on $\mathring{I}$. Define $\tilde{I}(t) = t\phi''(t)$, $t \in I$.*

(i) *If $\inf_{t\in\mathring{I}} \tilde{I}(t) = \delta > -\infty$, then we have the inequality*

$$\delta A(f)[\ln[A(f)] - A(\ln(f))] \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f), \tag{4.14}$$

provided that $\phi \circ f, \phi\prime \circ f, \phi\prime \circ f \cdot f, \ln f, f \in L$ and $A(f) > 0$.

(ii) *If $\sup_{t\in\mathring{I}} \tilde{I}(t) = \Delta < \infty$, then we have the inequality*

$$A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f) \leq \Delta A(f)[\ln[A(f)] - A(\ln(f))]. \tag{4.15}$$

(iii) *If $-\infty < \delta \leq \tilde{I}(t) \leq \Delta < \infty$ for $t \in\mathring{I}$, then both (4.14) and (4.15) hold.*

Proof. The proof is as follows.

(i) Define the auxiliary mapping $h(t) = \phi(t) - \delta t \ln t$, $t \in I$. Then

$$h''(t) = \phi''(t) - \frac{\delta}{t} = \frac{1}{t^2}[\phi''(t)t - \delta] = \frac{1}{t}\left[\tilde{I}(t) - \delta\right] \geq 0$$

which shows that h is convex or equivalently, $\phi \in \mathcal{L}(I, \delta, (\cdot)\ln(\cdot))$. Applying Corollary 4.6, we get

$$\delta[A[(\ln f + 1)f] + A(f)\ln A(f) - A(f)A(\ln f + 1) - A(f\ln f)] \leq A(\phi' \circ f \cdot f) + \phi(A(f)) - A(f) \cdot A(\phi' \circ f) - A(\phi \circ f)$$

which is equivalent with (4.14).

(ii) Goes similarly.

(iii) Follows by (i) and (ii).

□

5. Some Applications For Bullen's Inequality

The following inequality is well known in the literature as Bullen's inequality (see for example [7, p. 10])

$$\frac{1}{b-a}\int_a^b \phi(t)\,dt \leq \frac{1}{2}\left[\frac{\phi(a)+\phi(b)}{2} + \phi\left(\frac{a+b}{2}\right)\right], \tag{5.1}$$

provided that $\phi : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$. In other words, as (5.1) is equivalent to:

$$0 \leq \frac{1}{b-a}\int_a^b \phi(t)\,dt - \phi\left(\frac{a+b}{2}\right) \leq \frac{\phi(a)+\phi(b)}{2} - \frac{1}{b-a}\int_a^b \phi(t)\,dt \tag{5.2}$$

we can conclude that in the Hermite-Hadamard inequality

$$\frac{\phi(a)+\phi(b)}{2} \geq \frac{1}{b-a}\int_a^b \phi(t)\,dt \geq \phi\left(\frac{a+b}{2}\right) \tag{5.3}$$

the integral mean $\frac{1}{b-a}\int_a^b \phi(t)\,dt$ is closer to $\phi\left(\frac{a+b}{2}\right)$ than to $\frac{\phi(a)+\phi(b)}{2}$.

Using some of the results pointed out in the previous sections, we may upper and lower bound the *Bullen difference:*

$$B(\phi;a,b) := \frac{1}{2}\left[\frac{\phi(a)+\phi(b)}{2}+\phi\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b \phi(t)\,dt$$

(which is positive for convex functions) for different classes of twice differentiable functions ϕ.

Now, if we assume that $A(f) := \frac{1}{b-a}\int_a^b f(t)\,dt$, then for $f=e$, $e(x)=x$, $x\in[a,b]$, we have, for a differentiable function ϕ, that

$$\begin{aligned}
&A(\phi'\circ f\cdot f)+\phi(A(f))-A(f)\cdot A(\phi'\circ f)-A(\phi\circ f)\\
&\quad=\frac{1}{b-a}\int_a^b x\phi'(x)\,dx+\phi\left(\frac{a+b}{2}\right)\\
&\qquad\quad-\frac{a+b}{2}\cdot\frac{1}{b-a}\int_a^b\phi'(x)\,dx-\frac{1}{b-a}\int_a^b\phi(x)\,dx\\
&\quad=\frac{1}{b-a}\left[b\phi(b)-a\phi(a)-\int_a^b\phi(x)\,dx\right]+\phi\left(\frac{a+b}{2}\right)\\
&\qquad\quad-\frac{a+b}{2}\cdot\frac{\phi(b)-\phi(a)}{b-a}-\frac{1}{b-a}\int_a^b\phi(x)\,dx\\
&\quad=\frac{\phi(a)+\phi(b)}{2}+\phi\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b\phi(x)\,dx\\
&\quad=2B(\phi;a,b).
\end{aligned}$$

a) Assume that $\phi:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is a twice differentiable function satisfying the property that $-\infty<k\le\phi''(t)\le K<\infty$. Then by Proposition 4.7, we may state the inequality

$$\frac{1}{48}(b-a)^2 k\le B(\phi;a,b)\le\frac{1}{48}(b-a)^2 K. \tag{5.4}$$

This follows by Proposition 4.7 on taking into account that

$$\frac{1}{b-a}\int_a^b x^2dx-\left(\frac{1}{b-a}\int_a^b xdx\right)^2=\frac{(b-a)^2}{12}.$$

b) Now, assume that the twice differentiable function $\phi:[a,b]\subset(0,\infty)\to\mathbb{R}$ satisfies the property that $-\infty<\gamma\le t^{2-p}\phi''(t)\le\Gamma<\infty$, $t\in(a,b)$, $p\in(-\infty,0)\cup(1,\infty)$. Then by Proposition 4.8 and taking into account that

$$\begin{aligned}
A(f^p)-(A(f))^p &= \frac{1}{b-a}\int_a^b x^pdx-\left(\frac{1}{b-a}\int_a^b xdx\right)^p\\
&= L_p^p(a,b)-A^p(a,b),
\end{aligned}$$

and

$$A\left(f^{p-1}\right)-(A(f))^{p-1}=L_{p-1}^{p-1}(a,b)-A^{p-1}(a,b),$$

we may state the inequality

(5.5)
$$\begin{aligned} &\frac{\gamma}{p(p-1)}\left[(p-1)\left[L_p^p(a,b)-A^p(a,b)\right]-pA(a,b)\left[L_{p-1}^{p-1}(a,b)-A^{p-1}(a,b)\right]\right] \\ &\quad\leq B(\phi;a,b) \\ &\quad\leq \frac{\Gamma}{p(p-1)}\left[(p-1)\left[L_p^p(a,b)-A^p(a,b)\right]-pA(a,b)\left[L_{p-1}^{p-1}(a,b)-A^{p-1}(a,b)\right]\right]. \end{aligned}$$

c) Assume that the twice differentiable function $\phi:[a,b]\subset(0,\infty)\to\mathbb{R}$ satisfies the property that $-\infty<s\leq t^2\phi''(t)\leq S<\infty$, $t\in(a,b)$, then by Proposition 4.9, and taking into account that

$$\begin{aligned} A(f)A\left(f^{-1}\right)-1-\ln[A(f)]+A\ln(f) &= \frac{A(a,b)}{L(a,b)}-1-\ln A(a,b)+I(a,b) \\ &= \ln\left[\frac{I(a,b)}{A(a,b)}\cdot\exp\left(\frac{A(a,b)-L(a,b)}{L(a,b)}\right)\right], \end{aligned}$$

we get the inequality

(5.6)
$$\begin{aligned} \frac{s}{2}\ln&\left[\frac{I(a,b)}{A(a,b)}\cdot\exp\left(\frac{A(a,b)-L(a,b)}{L(a,b)}\right)\right] \\ &\leq B(\phi;a,b) \\ &\leq \frac{S}{2}\ln\left[\frac{I(a,b)}{A(a,b)}\cdot\exp\left(\frac{A(a,b)-L(a,b)}{L(a,b)}\right)\right]. \end{aligned}$$

d) Finally, if ϕ satisfies the condition $-\infty<\delta\leq t\phi''(t)\leq\Delta<\infty$, then by Proposition 4.10, we may state the inequality

(5.7)
$$\delta A(a,b)\ln\left[\frac{A(a,b)}{I(a,b)}\right]\leq B(\phi;a,b)\leq\Delta A(a,b)\ln\left[\frac{A(a,b)}{I(a,b)}\right].$$

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