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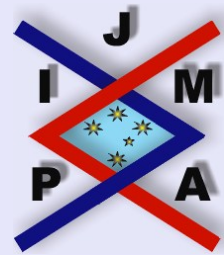
MATRIX AND OPERATOR INEQUALITIES

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Abstract

In this paper we prove certain inequalities involving matrices and operators on Hilbert spaces. In particular inequalities involving the trace and the determinant of the product of certain positive definite matrices.

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1.   Introduction

Inequalities have proved to be a powerful tool in mathematics , in particular in modeling error analysis for filtering and estimation problems, in adaptive stochastic control and for investigation of quantum mechanical Hamiltonians as it has been shown by Patel and Toda [10, 11, 12] and Lieb and Thirring [5].

It is the object of this paper to prove new interesting matrix and operator inequalities. We refer the reader to [4, 7, 8] for the basics of matrix and operator inequalities and for a survey of many other basic and important inequalities.

Through out the paper if A is an $n \times n$ matrix, we write $tr A$ to denote the trace of A and $\det A$ for the determinant of A. If A is positive definite we write $A > 0$. The adjoint of A (a matrix or operator) is denoted by A^*.
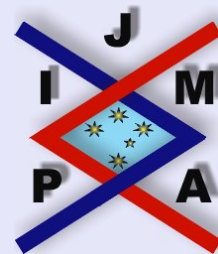
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2. Matrix Inequalities

Through out this section, we work with square matrices on a finite dimensional Hilbert space.

Theorem 2.1. *If $A > 0$ and $B > 0$, then*

$$(2.1) \qquad 0 < tr\,(AB)^m < (tr\,(AB))^m$$

for any integer $m > 0$.

Proof. The equality holds for $m = 1$. For $m > 1$, let $B = I$, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A. Since $\sum_{i=1}^{n} \lambda_i^m < \left(\sum_{i=1}^{n} \lambda_i\right)^m$, then

$$(2.2) \qquad 0 < tr\,(A^m) < (tr A)^m.$$

Since (2.2) is true for any $A > 0$, we let $D = B^{\frac{1}{2}} A B^{\frac{1}{2}}$. Then inequality (2.2) holds for D. Thus $0 < tr\,(D^m) < (tr D)^m$, from which the result follows. $\square$

Theorem 2.2. *Let A, B be positive definite matrices. Then*

$$(2.3) \qquad 0 < tr\,(AB)^m < \left[tr\,(AB)^s\right]^{\frac{m}{s}},$$

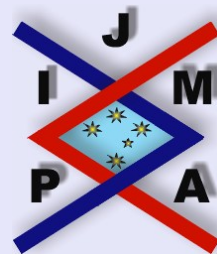*provided that m and s are positive integers and $m > s$.*

Proof. Clearly $tr\,(AB)^m = tr\left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right)^m > 0$. Let $l_1, l_2, \ldots, l_n$ be the eigenvalues of $A^{\frac{1}{2}} B A^{\frac{1}{2}}$. Then from Hardy's inequality [3] $(l_1^m + l_2^m + \cdots + l_n^m)^{\frac{1}{m}} < (l_1^s + l_2^s + \cdots + l_n^s)^{\frac{1}{s}}$ for $m > s > 0$, we get

$$\left[tr\left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right)^m\right]^{\frac{1}{m}} < \left[tr\left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right)^s\right]^{\frac{1}{s}}.$$

This implies (2.3). $\square$

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If $A_i > 0$ *and* $B_i > 0$ $(i = 1, 2, \ldots, k)$, *then*

$$(2.4) \qquad \left(tr \sum_{i=1}^{k} A_i B_i \right)^2 \leq \left(tr \sum_{i=1}^{k} A_i^2 \right) \cdot \left(tr \sum_{i=1}^{k} B_i^2 \right).$$

If $A_i B_i > 0 \ (i = 1, 2, \ldots, k)$, *then*

$$(2.5) \qquad \left(tr \sum_{i=1}^{k} A_i B_i \right)^2 < \left(tr \sum_{i=1}^{k} A_i^2 \right) \cdot \left(tr \sum_{i=1}^{k} B_i^2 \right).$$

Proof. Since

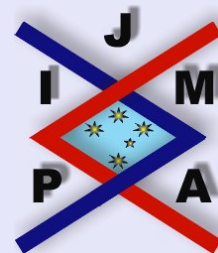$$0 \leq tr \sum_{i=1}^{k} (\theta A_i + B_i)^2 = \theta^2 tr \left(\sum_{i=1}^{k} A_i^2 \right) + 2\theta tr \left(\sum_{i=1}^{k} A_i B_i \right) + tr \left(\sum_{i=1}^{k} B_i^2 \right),$$

we conclude (2.4). To prove (2.5), it suffices to prove that

$$(2.6) \qquad tr \left(\sum_{i=1}^{k} A_i B_i \right)^2 < \left(tr \sum_{i=1}^{k} (A_i B_i) \right)^2.$$

Since $A_i B_i > 0$ for $i = 1, 2, \ldots, k$, then $U = \sum_{i=1}^{k} A_i B_i > 0$. Therefore the inequality $tr \, (U)^2 < (tr U)^2$ for positive definite U implies (2.6) and the proof is complete. □

Remark 2.1. *The condition* $A_i B_i > 0$ *in (2.5) is essential as the following example shows.*

Example 2.1. *Let*

$$A = \begin{pmatrix} 4 & -3 \\ -3 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix},$$

$$C = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -3 \\ -3 & 10 \end{pmatrix}.$$

It is clear that A, B, C, and D are positive definite matrices . Now

$$(AB + CD) = \begin{pmatrix} -10 & 10 \\ -9 & 66 \end{pmatrix}, \quad (AB + CD)^2 = \begin{pmatrix} 10 & 560 \\ -504 & 4266 \end{pmatrix}.$$

Thus

$$tr\,(AB + CD)^2 = 4276 > [tr\,(AB + CD)]^2 = 3136.$$

Remark 2.2. *R. Bellman [1] proved that $tr\,(AB)^2 \le tr\,(A^2 B^2)$ (*) for positive definite matrices A and B. Further he asked: "Does the above inequality (*) hold for higher powers?". Such a question had been solved by E.H.Lieb and W.E. Thirring [5], where they proved*

$$(2.7) \qquad\qquad tr\,(AB)^m < tr\,(A^m B^m)$$

for any positive integer m, and for A, B positive definite matrices. In 1995, Changqin Xu [2] proved a particular case of (2.7): that is when A and B are 2×2 positive definite matrices. Notice that $(tr\,AB)^m$ and $tr\,(A^m B^m)$ are upper bounds for $tr\,(AB)^m$ in (2.1) and (2.7) . One may ask what is $\max\{tr\,(A^m B^m), (tr\,AB)^m\}$. The following examples show that either $(tr\,AB)^m$ or $tr\,(A^m B^m)$ can be the least.

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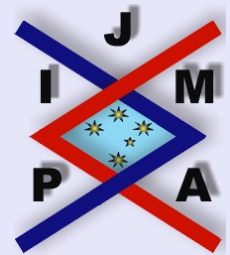
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Example 2.2. *Let*

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then $tr\,(AB)^2 = 144 < 204 = tr\,(A^2 B^2)$.

Example 2.3. *Let*

$$A = \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then $tr\,(A^2 B^2) = 25 < 36 = tr\,(AB)^2$.

Theorem 2.4. *If* $0 < A_1 \leq B_1$ *and* $0 < A_2 \leq B_2$, *then*

$$(2.8) \qquad 0 < tr\,(A_1 A_2) \leq tr\,(B_1 B_2).$$

Proof. Since $0 < A_1 \leq B_1$ and $0 < A_2 \leq B_2$, it follows that

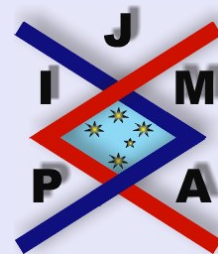$$0 < A_2^{\frac{1}{2}} A_1 A_2^{\frac{1}{2}} \leq A_2^{\frac{1}{2}} B_1 A_2^{\frac{1}{2}}$$

and

$$(2.9) \qquad 0 < B_1^{\frac{1}{2}} A_2 B_1^{\frac{1}{2}} \leq B_1^{\frac{1}{2}} B_2 B_1^{\frac{1}{2}}.$$

Since trace is a monotone function on the definite matrices, we get

$$(2.10) \qquad 0 < tr\,(A_1 A_2) \leq tr\,(B_1 A_2).$$

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and

(2.11) $$0 < tr\,(B_1 A_2) \le tr\,(B_1 B_2)\,.$$

This implies (2.8). □

Remark 2.3. *The conditions $A_1 > 0$ and $A_2 > 0$ in Theorem 2.4. are essential even if $A_1 A_2$ and $B_1 B_2$ are symmetric as the following example shows.*

Example 2.4. *Let*

$$A_1 = \begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

It is clear that $A_1 < B_1$ and $A_2 < B_2$. We have also

$$A_1 A_2 = \begin{pmatrix} \frac{1}{2} & -\frac{7}{2} \\ -\frac{7}{2} & \frac{15}{2} \end{pmatrix}, \qquad B_1 B_2 = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$$

and $tr\,(A_1 A_2) = 8 > 7 = tr\,(B_1 B_2)\,.$

Theorem 2.5. *If $A > 0$ and $B > 0$, then*

(2.12) $$n\,(\det A \cdot \det B)^{\frac{m}{n}} \le tr\,(A^m B^m)$$

for any positive integer m.

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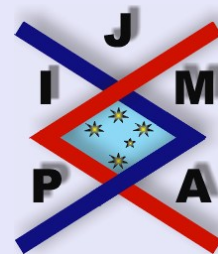
Proof. Since A is diagonalizable, there exists an orthogonal matrix P and a diagonal matrix Λ such that $\Lambda = P^T A P$. So if the eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\Lambda = diag\left(\lambda_1, \lambda_2, \ldots, \lambda_n\right)$.

Let $b_{11}\left(m\right), b_{22}\left(m\right), \ldots, b_{nn}\left(m\right)$ denote the elements of $\left(PBP^T\right)^m$. Then

$$(2.13) \quad \frac{1}{n} tr\left(A^m B^m\right) = \frac{1}{n} tr\left(P\Lambda^m P^T B^m\right)$$
$$= \frac{1}{n} tr\left(\Lambda^m P^T B^m P\right)$$
$$= \frac{1}{n} tr\left[\Lambda^m \left(P^T B P\right)^m\right]$$
$$= \frac{1}{n}\left[\lambda_1^m b_{11}\left(m\right) + \lambda_2^m b_{22}\left(m\right) + \cdots + \lambda_n^m b_{nn}\left(m\right)\right].$$

Using the arithmetic-mean geometric- mean inequality [9], we get

$$(2.14) \quad \frac{1}{n} tr\left(A^m B^m\right) \geq \left[\lambda_1^m \lambda_2^m \cdots \lambda_n^m\right]^{\frac{1}{n}} \left[b_{11}\left(m\right) b_{22}\left(m\right) \cdots b_{nn}\left(m\right)\right]^{\frac{1}{n}}.$$

Since $\det A \leq a_{11} a_{22} \cdots a_{nn}$ for any positive definite matrix A, [4] we conclude that

$$(2.15) \quad \det\left(P^T B P\right)^m \leq b_{11}\left(m\right) \cdot b_{22}\left(m\right) \cdot \cdots \cdot b_{nn}\left(m\right)$$

and

$$(2.16) \quad \det \Lambda^m \leq \lambda_1^m \lambda_2^m \cdots \lambda_n^m.$$
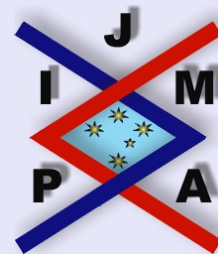
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Therefore from (2.14) it follows that

$$
\begin{aligned}
\frac{1}{n} tr\left(A^m B^m\right) & \geq \left[\det\left(\Lambda^m\right)\right]^{\frac{1}{n}} \cdot \left[\det\left(P^T BP\right)^m\right]^{\frac{1}{n}} \\
& = \left[\det\left(P^T AP\right)\right]^{\frac{m}{n}} \cdot \left[\det\left(P^T BP\right)\right]^{\frac{m}{n}} \\
& = \left(\det A \cdot \det B\right)^{\frac{m}{n}}.
\end{aligned}
$$

Here we used the fact that $A > 0$ and $B > 0$. The proof is complete. $\qquad\square$

Corollary 2.6. *[6] Let A and X be positive definite $n \times n$- matrices such that $\det X = 1$. Then*

$$
(2.17) \qquad\qquad n\left(\det A\right)^{\frac{1}{n}} \leq tr\left(AX\right).
$$

Proof. Take $B = X$ and $m = 1$ in Theorem 2.5. $\qquad\square$

Theorem 2.7. *If $A \geq 0$, $B \geq 0$ and $AB = BA$, then*

$$
(2.18) \qquad\qquad 2^{(m-1)n} \det\left(A^m + B^m\right) \geq \left[\det\left(A + B\right)\right]^m
$$

and

$$
(2.19) \qquad\qquad 2^{m-1} tr\left(A^m + B^m\right) \geq tr\left(A + B\right)^m
$$

for any positive integer m.

Proof. To prove inequality (2.18), it is enough to prove

$$
(2.20) \qquad\qquad \frac{A^m + B^m}{2} \geq \left(\frac{A + B}{2}\right)^m
$$

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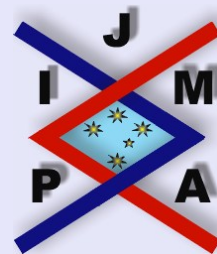
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for any pair of commuting positive definite matrices A and B. We use induction to prove (2.20). Clearly (2.20) holds true for $m = 2$. Assume that (2.20) is true for $m = k$. We have to prove (2.20) for $m = k + 1$. Indeed, since

$$\frac{A^k + B^k}{2} \cdot \frac{A + B}{2} = \frac{A + B}{2} \cdot \frac{A^k + B^k}{2},$$

it follows that

$$(2.21) \quad \left(\frac{A + B}{2}\right)^{k+1} \leq \frac{A + B}{2} \cdot \frac{A^k + B^k}{2}$$

$$= \frac{A^{k+1} + B^{k+1}}{2} - \frac{A^{k+1} + B^{k+1}}{4} + \frac{BA^k + AB^k}{4}$$

$$= \frac{A^{k+1} + B^{k+1}}{2} - \frac{A^{k+1} + B^{k+1} - BA^k - AB^k}{4}$$

$$= \frac{A^{k+1} + B^{k+1}}{2} - \frac{\left(A^k - B^k\right)\left(A - B\right)}{4}.$$

Now the equality

$$\left(A^k - B^k\right)\left(A - B\right) = \left(A^{k-1} + A^{k-2}B + \cdots + AB^{k-2} + B^{k-1}\right)\left(A - B\right)^2$$

for $A \geq 0$, $B \geq 0$ and $AB = BA$, implies $AB \geq 0$ [8]. Consequently

$$L = A^{k-1} + A^{k-2}B + \cdots + AB^{k-2} + B^{k-1} \geq 0.$$

Since $L \cdot (A - B)^2 = (A - B)^2 \cdot L$, then $\left(A^k - B^k\right)(A - B) \geq 0$. Therefore, from (2.21) we obtain

$$\left(\frac{A + B}{2}\right)^{k+1} \leq \frac{A^{k+1} + B^{k+1}}{2}.$$

The proof is complete. Inequality (2.19) follows directly from (2.20). $\quad\square$

Remark 2.4. *The condition $AB = BA$ in inequality (2.20) is essential as the following example shows.*

Example 2.5. *Let*

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}.$$
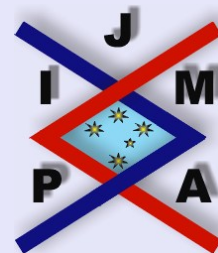
It is clear that $A > 0$, $B > 0$ and $AB \neq BA$. For $m = 3$ inequality (2.20) becomes

$$(2.22) \qquad\qquad 4\left(A^3 + B^3\right) \geq (A + B)^3.$$

Easily we find that

$$4\left(A^3 + B^3\right) = \begin{pmatrix} 256 & -216 \\ -216 & 196 \end{pmatrix}, \qquad (A + B)^3 = \begin{pmatrix} 84 & -45 \\ -45 & 24 \end{pmatrix}$$

and $\det C = -9$ which implies that $C < 0$.

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3. Operator Inequalities

In this section we consider inequalities involving operators on separable Hilbert space X. We start with the following simple well-known inequality.

Theorem 3.1. *Let S and T be self-adjoint bounded linear operators on the Hilbert space H. Then*

$$\frac{ST+TS}{2} \leq \left(\frac{S+T}{2}\right)^2. \tag{3.1}$$

Proof. Since

$$\left(\frac{S+T}{2}\right)^2 - \frac{ST+TS}{2} = \left(\frac{S+T}{2}\right)^2$$

and since the square of the self-adjoint operator is a non-negative operator, we get $\left(\frac{S+T}{2}\right)^2 \geq 0$. The claim of the theorem now follows. □

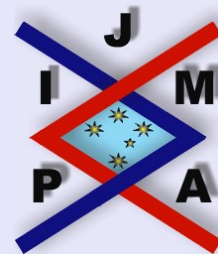Now we present a similar type result as Theorem 3.1 but for non-self-adjoint case . More precisely:

Theorem 3.2. *Let S and T be bounded linear operators on a Hilbert space X. Assume S to be self-adjoint . Then*

$$\frac{1}{4}K^2 + H_p \leq H_Q, \tag{3.2}$$

where

$$P = \frac{1}{2}(ST+TS), \qquad Q = \left(\frac{S+T}{2}\right)^2, \tag{3.3}$$

$$H_P = \frac{1}{2}(P+P^*), \qquad H_Q = \frac{1}{2}(Q+Q^*) \tag{3.4}$$

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and $K = \frac{1}{2}(T + T^*)$.

Proof. For any bounded linear operator T we have

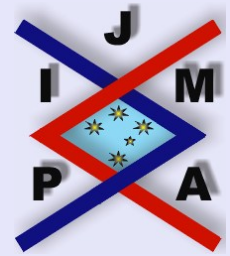$$T = \frac{1}{2}[(T + T^*) + (T - T^*)] = H_T + K,$$

where $H_T = \frac{1}{2}(T + T^*)$. Inequality (3.1) can be applied to the self-adjoint operators S and H_T, so we get

$$(3.5) \qquad \langle Ux, x \rangle \geq 0,$$

where $U = (S + H_T)^2 - 2(SH_T + H_T S)$. Now we have

$$
\begin{aligned}
(3.6) \qquad U &= (S + H_T)^2 - S(T + T^*) - (T + T^*)S \\
&= (S + H_T)^2 - 4H_P \\
&= \frac{1}{2}\left[2S^2 + \frac{T^2 + (T^*)^2}{2} + \frac{1}{2}(TT^* + T^*T) - 4H_P\right] \\
&= \frac{1}{2}\left(2S^2 + T^2 + (T^*)^2 - 4H_P - 2K^2\right) \\
&= \frac{1}{2}\left(8H_Q - 4H_P - 4H_P - 2K^2\right) \\
&= 4\left(H_Q - H_P - \frac{1}{4}K^2\right).
\end{aligned}
$$

Therefore the required inequality (3.2) follows from (3.6) and (3.5). $\qquad \square$

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Remark 3.1. *When both S and T are not self-adjoint operators, Theorem 3.2 does not hold. The following example illustrates this fact.*

Example 3.1. *Let S and T be defined on $\mathbb{R}^2 \to \mathbb{R}^2$ by the following matrices*

$$S = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 6 \\ -3 & 2 \end{pmatrix}.$$
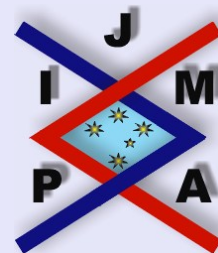
By computation we find that

$$
\begin{aligned}
P &= \frac{1}{2}(ST + TS) = \begin{pmatrix} 7 & 12 \\ -6 & 14 \end{pmatrix}, \\
Q &= \left(\frac{S+T}{2}\right)^2 = \begin{pmatrix} -1 & 8 \\ -4 & 7 \end{pmatrix}, \qquad K^2 = \begin{pmatrix} -\frac{81}{4} & 0 \\ 0 & -\frac{81}{4} \end{pmatrix}, \\
H_P &= \begin{pmatrix} 7 & 3 \\ 3 & 14 \end{pmatrix}, \qquad H_Q = \begin{pmatrix} -1 & 2 \\ 2 & 7 \end{pmatrix}, \\
H_Q - H_P - \frac{1}{4}K^2 &= \begin{pmatrix} -\frac{47}{16} & -1 \\ -1 & -\frac{31}{16} \end{pmatrix} < 0.
\end{aligned}
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References

[1] R. BELLMAN, Some inequalities for positive definite matrices, in *General Inequalities 2. Proceedings, 2nd Internat. Conf. on General Inequalities*", (E.F.Beckenbach, Ed.), p. 89–90, Birkhauser,1980.

[2] CHNG-QIN XU, Bellman's inequality, *Linear Algebra and its Appl.*, **229** (1995), 9–14.

[3] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge Univ. Press, Cambridge, 1952.

[4] R.A. HORN AND C.R. JOHNSON, *Matrix Analysis*, Cambridge Univ. Press, 1999.

[5] E.H. LIEB AND W.E. THIRRING, Inequalities for the moments of the eigenvalues of the Schrodinger Hamiltonian and their relation to Sobolev inequalities, *Studies in Mathematical Physics, Essays in Honor of Valentine*, (1976), Bartmann, Princeton, N.J., 269–303.

[6] J.R. MAGNUS AND H. NEUDECKER, *Matrix Differential calculus with Applications in Statistics and Econometrics*, John Wiley & Sons, 1990.

[7] M. MARCUS AND H. MINC, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Inc., Boston, 1964.

[8] M.L. MEHTA, Matrix Theory, Selected topics and useful results, *Les Editions de Physique*, 1989.

[9] D.S. MITRINOVIĆ, *Analytic Inequalities*, Springer -Verlag Berlin, Heidelberg, New York, 1970 .
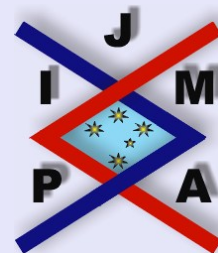
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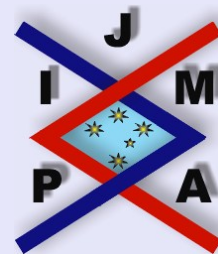
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[10] R. PATEL AND M. TODA, Modeling error analysis of stationary linear discrete time filters, *NASA* TM X-73, **225**(Feb) (1977).

[11] R. PATEL AND M. TODA, Trace inequalities involving Hermitian matrices, *Linear Algebra and its Appl.,* **23** (1979), 13–20.

[12] M. TODA AND R. PATEL, Algorithms for adaptive stochastic control for a class of linear systems, *NASA* TM X-73, **240**(Apr.) (1977).

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