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INEQUALITIES RELATED TO THE CHEBYCHEV FUNCTIONAL INVOLVING INTEGRALS OVER DIFFERENT INTERVALS

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Abstract

A generalised Chebychev functional involving integral means of functions over different intervals is investigated. Bounds are obtained for which the functions are assumed to be of Hölder type. A weighted generalised Chebychev functional is also introduced and bounds are obtained in terms of weighted Grüss, Chebychev and Lupaş inequalities.

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Contents

1. Introduction

For two measurable functions $f, g : [a, b] \to \mathbb{R}$, define the functional, which is known in the literature as Chebychev's functional

$$T(f, g; a, b) := \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{(b-a)^2} \int_a^b f(x) \, dx \cdot \int_a^b g(x) \, dx, \tag{1.1}$$

provided that the involved integrals exist.

The following inequality is well known as the Grüss inequality [9]

$$|T(f, g; a, b)| \le \frac{1}{4} (M - m)(N - n), \tag{1.2}$$

provided that $m \le f \le M$ and $n \le g \le N$ a.e. on $[a, b]$, where m, M, n, N are real numbers. The constant $\frac{1}{4}$ in (1.2) is the best possible.

Another inequality of this type is due to Chebychev (see for example [1, p. 207]). Namely, if f, g are absolutely continuous on $[a, b]$ and $f', g' \in L_\infty [a, b]$ and $\|f'\|_\infty := ess \sup_{t \in [a,b]} |f'(t)|$, then

$$|T(f, g; a, b)| \le \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b - a)^2 \tag{1.3}$$

and the constant $\frac{1}{12}$ is the best possible.

Finally, let us recall a result by Lupaş (see for example [1, p. 210]), which states that:

$$|T(f, g; a, b)| \le \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b - a), \tag{1.4}$$

provided f, g are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible here.

For other Grüss type inequalities, see the books [1] and [2], and the papers [3]-[10], where further references are given.

Recently, Cerone and Dragomir [11] have pointed out generalizations of the above results for integrals defined on two different intervals $[a,b]$ and $[c,d]$.

Define the functional (generalised Chebychev functional)

$$T(f,g;a,b,c,d) := M(fg;a,b) + M(fg;c,d) - M(f;a,b)\,M(g;c,d) - M(f;c,d)\,M(g;a,b), \tag{1.5}$$

where the integral mean is defined by

$$M(f;a,b) := \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.6}$$

Cerone and Dragomir [11] proved the following result.

Theorem 1.1. *Let $f, g : I \subseteq \mathbb{R} \to \mathbb{R}$ be measurable on I and the intervals $[a,b], [c,d] \subset I$. Assume that the integrals involved in (2.12) exist. Then we have the inequality*

$$\begin{aligned} |T(f,g;a,b,c,d)| &\leq \left[T(f;a,b) + T(f;c,d) + (M(f;a,b) - M(f;c,d))^2\right]^{\frac{1}{2}} \\ &\quad \times \left[T(g;a,b) + T(g;c,d) + (M(g;a,b) - M(g;c,d))^2\right]^{\frac{1}{2}}, \end{aligned} \tag{1.7}$$

where

$$T(f;a,b) := \frac{1}{b-a}\int_a^b f^2(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2 \tag{1.8}$$

and the integrals involved in the right membership of (2.3) exist.

They used a generalisation of the classical identity due to Korkine namely,

$$\begin{aligned}T(f,g;a,b,c,d) \\ = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (f(x)-f(y))(g(x)-g(y))\,dydx\end{aligned} \tag{1.9}$$

and the fact that

$$\begin{aligned}T(f,f;a,b,c,d) \\ = T(f;a,b)+T(f;c,d)+(M(f;a,b)-M(f;c,d))^2\,.\end{aligned} \tag{1.10}$$

In the current article, bounds are obtained for the generalised Chebychev functional (1.5) assuming that f and g are of Hölder type. The special case for which f and g are Lipschitzian is also investigated. A weighted generalised Chebychev functional is treated in Section 3 involving weighted means of functions over different intervals. Grüss, Chebychev and Lupaş results are utilised to obtain bounds for such a functional.

2. The Results for Functions of Hölder Type

The following lemma will prove to be useful in the subsequent work.

Lemma 2.1. *Let* $a, b, c, d \in \mathbb{R}$ *with* $a < b$ *and* $c < d$. *Define*

$$C_\theta(a,b,c,d) := \int_a^b \int_c^d |x-y|^\theta \, dydx, \quad \theta \geq 0, \tag{2.1}$$

then

$$\begin{aligned} (\theta+1)(\theta+2)\,C_\theta(a,b,c,d) \\ = |b-c|^{\theta+2} - |b-d|^{\theta+2} + |d-a|^{\theta+2} - |c-a|^{\theta+2}. \end{aligned} \tag{2.2}$$

Proof. Let $[\ \ ]$ denote the order in which a, b, c, d appear on the real number line. There are six possibilities to consider since we are given that $a < b$ and $c < d$.

Firstly, consider the situation $c = a$ and $d = b$. Then

$$\begin{aligned} D_\theta(a,b) &= C_\theta(a,b,a,b) \\ &= \int_a^b \int_a^b |x-y|^\theta \, dydx, \quad \theta \geq 0 \\ &= \int_a^b \left[\int_a^x (x-y)^\theta \, dy + \int_x^b (y-x)^\theta \, dy \right] dx \\ &= \frac{1}{\theta+1} \int_a^b \left[(x-a)^{\theta+1} + (b-x)^{\theta+1} \right] dx \end{aligned} \tag{2.3}$$

and so

$$(\theta+1)(\theta+2)\,D_\theta(a,b) = 2(b-a)^{\theta+2}. \tag{2.4}$$

Now, taking the six possibilities in turn, we have:

(i) For the ordering $[c,d,a,b]$, $y<x$ giving for $C_\theta(a,b,c,d)$

$$I_\theta(a,b,c,d) := \int_a^b\int_c^d (x-y)^\theta\,dydx \tag{2.5}$$
$$= \int_a^b\left[\int_c^x (x-y)^\theta\,dy + \int_x^d (y-x)^\theta\,dy\right]dx$$
$$= \frac{1}{\theta+1}\int_a^b\left[(x-c)^{\theta+1}-(x-d)^{\theta+1}\right]dx$$

and so

$$(\theta+1)(\theta+2)I_\theta(a,b,c,d) \tag{2.6}$$
$$=(b-c)^{\theta+2}-(a-c)^{\theta+2}+(a-d)^{\theta+2}-(b-d)^{\theta+2}$$
$$=(\theta+1)(\theta+2)C_\theta(a,b,c,d),\quad [c,d,a,b].$$

(ii) For the ordering $[c,a,d,b]$ we have

$$C_\theta(a,b,c,d)$$
$$=\int_a^b\int_c^a (x-y)^\theta\,dydx+\int_a^d\int_a^d |x-y|^\theta\,dydx+\int_b^d\int_a^d (x-y)^\theta\,dydx$$
$$=I_\theta(a,b,c,a)+D_\theta(a,d)+I_\theta(d,b,a,d),$$

where we have used (2.3) and (2.5). Further, utilising (2.4) and (2.6) gives

$$(\theta+1)(\theta+2)C_\theta(a,b,c,d) \tag{2.7}$$
$$=(b-c)^{\theta+2}-(b-d)^{\theta+2}+(d-a)^{\theta+2}-(a-c)^{\theta+2},\quad [c,a,d,b].$$

(iii) For the ordering $[a, c, d, b]$

$$C_\theta(a,b,c,d) = \int_a^c \int_c^d (y-x)^\theta \, dy dx + \int_c^d \int_c^d |y-x|^\theta \, dy dx + \int_d^b \int_c^d (x-y)^\theta \, dy dx = I_\theta(c,d,a,c) + D_\theta(c,d) + I_\theta(d,b,c,d),$$

giving, on using (2.4) and (2.6)

$$(\theta+1)(\theta+2) C_\theta(a,b,c,d) = (b-c)^{\theta+2} - (b-d)^{\theta+2} + (d-a)^{\theta+2} - (c-a)^{\theta+2}, \quad [a,c,d,b]. \tag{2.8}$$

(iv) For the ordering $[a, c, b, d]$

$$C_\theta(a,b,c,d) = \int_a^c \int_c^d (y-x)^\theta \, dy dx + \int_c^b \int_c^b |x-y|^\theta \, dy dx + \int_c^b \int_b^d (y-x)^\theta \, dy dx = I_\theta(c,d,a,c) + D_\theta(c,b) + I_\theta(b,d,c,b),$$

giving, from (2.4) and (2.6)

$$(\theta+1)(\theta+2) C_\theta(a,b,c,d) = (b-c)^{\theta+2} - (d-b)^{\theta+2} + (d-a)^{\theta+2} - (c-a)^{\theta+2}, \quad [a,c,b,d]. \tag{2.9}$$

(v) For the ordering $[a, b, c, d]$

$$(\theta+1)(\theta+2) C_\theta(a,b,c,d) = \theta(\theta+1) I_\theta(c,d,a,b)$$

and so from (2.6)

$$(2.10)\quad (\theta+1)(\theta+2)\,C_{\theta}(a,b,c,d) = (d-a)^{\theta+2}-(c-a)^{\theta+2}+(c-b)^{\theta+2}-(d-b)^{\theta+2},\quad [a,b,c,d]\,.$$

(vi) For the ordering $[c,a,d,b]$, interchanging a and c and b and d in case (iii) gives

$$(2.11)\quad (\theta+1)(\theta+2)\,C_{\theta}(a,b,c,d) = (d-a)^{\theta+2}-(d-b)^{\theta+2}+(b-c)^{\theta+2}-(a-c)^{\theta+2},\quad [c,a,b,d]\,.$$

Combining (2.6) – (2.11) produces the results (2.1) – (2.2) and the lemma is proved. □

Remark 2.1. It may be noticed from (2.1) – (2.2) that (2.4) is recaptured of $c=a$ and $d=b$. Further, the answer appears in terms of differences between a limit of one integral and the other integral. The differences between a top and bottom limit is associated with a positive sign while the difference between the two bottom limits or the two top limits is associated with a negative sign. The order of the differences depends on the order of the limits on the real number line and is taken in such a way that the difference is positive.

Theorem 2.2. *Let $f,g:I\subseteq\mathbb{R}\to\mathbb{R}$ be measurable on I and the intervals $[a,b]$, $[c,d]\subset I$. Further, suppose that f and g are of Hölder type so that for $x\in[a,b]$, $y\in[c,d]$*

$$(2.12)\quad |f(x)-f(y)|\le H_1|x-y|^{r}\ \text{ and }\ |g(x)-g(y)|\le H_2|x-y|^{s},$$

where $H_1, H_2 > 0$ and $r, s \in (0,1]$ are fixed. The following inequality then holds on the assumption that the integrals involved exist. Namely,

$$(2.13)\quad (\theta+1)(\theta+2)\left|T(f,g;a,b,c,d)\right| \leq \frac{H_1 H_2}{(b-a)(d-c)}\left[|b-c|^{\theta+2} - |b-d|^{\theta+2} + |d-a|^{\theta+2} - |c-a|^{\theta+2}\right],$$

where $\theta = r+s$ and $T(f,g;a,b,c,d)$ is as defined by (1.5) and (1.6).

Proof. From the Hölder assumption (2.12), we have

$$|(f(x)-f(y))(g(x)-g(y))| \leq H_1 H_2 |x-y|^{r+s}$$

for all $x \in [a,b]$, $y \in [c,d]$.

Hence,

$$\left|\int_a^b \int_c^d (f(x)-f(y))(g(x)-g(y))\,dydx\right| \leq \int_a^b \int_c^d |(f(x)-f(y))(g(x)-g(y))|\,dydx \leq H_1 H_2 \int_a^b \int_c^d |x-y|^{r+s}\,dydx = H_1 H_2 C_{r+s}(a,b,c,d),$$

where $C_\theta(a,b,c,d)$ is as given by (2.2).

Now, from identity (1.9) and the above inequality readily produces (2.13) and the theorem is thus proved. □

Corollary 2.3. *Let $f, g : I \subseteq \mathbb{R} \to \mathbb{R}$ be measurable on I and the intervals $[a, b], [c, d] \subset I$. Further, suppose that f and g are Lipschitzian mappings such that for $x \in [a, b]$ and $y \in [c, d]$*

$$|f(x) - f(y)| \leq L_1 |x - y| \quad \text{and} \quad |g(x) - g(y)| \leq L_2 |x - y|,$$

where $L_1, L_2 > 1$. Assuming that the integrals involved exist, then the following inequality holds. That is,

$$|T(f, g; a, b, c, d)| \\ \leq \frac{L_1 L_2}{12(b-a)(d-c)} \left[(b-c)^4 - (c-a)^4 + (d-a)^4 - (b-d)^4 \right].$$

Proof. Taking $r = s = 1$ in Theorem 2.2 and $L_1 = H_1$, $L_2 = H_2$, then from (2.13) we obtain the above inequality. □

Remark 2.2. The situation in which f is of Hölder type and g is Lipschitzian may be handled by taking $s = 1$ and $H_2 = L_2$. Further, taking $d = b$ and $c = a$ recaptures the results of Dragomir [7].

3. A Weighted Generalised Chebychev Functional

Define the weighted generalised Chebychev Functional by

$$\mathfrak{T}(f,g;a,b,c,d)=\mathfrak{M}(fg;a,b)+\mathfrak{M}(fg;c,d) - \mathfrak{M}(f;a,b)\,\mathfrak{M}(g;c,d)-\mathfrak{M}(f;c,d)\,\mathfrak{M}(g;a,b)\,, \tag{3.1}$$

where the $w-$weighted integral mean is given by

$$\mathfrak{M}(f;a,b)=\frac{1}{\int_a^b w(x)\,dx}\int_a^b w(x)\,f(x)\,dx \tag{3.2}$$

with $w:[a,b]\to[0,\infty)$ is integrable and $0<\int_a^b w(x)\,dx<\infty$.

Theorem 3.1. *Let $f,g,w:I\subseteq\mathbb{R}\to\mathbb{R}$ be measurable on I and the intervals $[a,b]\,,\,[c,d]\subset I$. Assuming that the integrals involved in (3.1) exist and $\int_I w(x)\,dx>0$, then we have*

$$\begin{aligned}|\mathfrak{T}(f,g;a,b,c,d)| &\le \left[\mathfrak{T}(f;a,b)+\mathfrak{T}(f;c,d)+(\mathfrak{M}(f;a,b)-\mathfrak{M}(f;c,d))^2\right]^{\frac{1}{2}}\\ &\quad\times\left[\mathfrak{T}(g;a,b)+\mathfrak{T}(g;c,d)+(\mathfrak{M}(g;a,b)-\mathfrak{M}(g;c,d))^2\right]^{\frac{1}{2}}\,,\end{aligned} \tag{3.3}$$

where

$$\mathfrak{T}(f;a,b):=\mathfrak{M}\left(f^2;a,b\right)-\mathfrak{M}^2(f;a,b) \tag{3.4}$$

and the integrals involved in the right hand side of (3.1) exist.

Proof. It is easily demonstrated that the identity

$$\mathfrak{T}(f,g;a,b,c,d) = \frac{1}{\int_a^b w(x)\,dx \int_c^d w(y)\,dy} \int_a^b \int_c^d w(x)\,w(y) \times (f(x)-f(y))(g(x)-g(y))\,dxdy \tag{3.5}$$

holds, which is a weighted generalised Korkine type identity involving integrals over different intervals.

Using the Cauchy-Buniakowski-Schwartz inequality for double integrals gives

$$|\mathfrak{T}(f,g;a,b,c,d)|^2 \le \mathfrak{T}(f,f;a,b,c,d)\,\mathfrak{T}(g,g;a,b,c,d), \tag{3.6}$$

where from (3.1)

$$\mathfrak{T}(f,f;a,b,c,d) = \mathfrak{M}\left(f^2;a,b\right) + \mathfrak{M}\left(f^2;c,d\right) - 2\mathfrak{M}(f;a,b)\,\mathfrak{M}(f;c,d)$$

and using (3.4) gives

$$\mathfrak{T}(f,f;a,b,c,d) = \mathfrak{T}(f;a,b) + \mathfrak{T}(f;c,d) + (\mathfrak{M}(f;a,b) - \mathfrak{M}(f;c,d))^2. \tag{3.7}$$

A similar identity to (3.7) holds for g and thus from (3.6) and (3.2), the result (3.3) is obtained and the theorem is thus proved. □

Corollary 3.2. *Let the conditions of Theorem 3.1 hold. Moreover, assume that*

$$m_1 \le f \le M_1, \text{ a.e. on } [a,b], \; m_2 \le f \le M_2, \text{ a.e. on } [c,d]$$

and

$$n_1 \le g \le N_1, \text{ a.e. on } [a,b], \; n_2 \le g \le N_2, \text{ a.e. on } [c,d].$$

The inequality

$$\begin{aligned} &\left|\mathfrak{T}(f,g;a,b,c,d)\right| \\ &\quad\le \frac{1}{4}\left[(M_1-m_1)^2+(M_2-m_2)^2+4\left(\mathfrak{M}(f;a,b)-\mathfrak{M}(f;c,d)\right)^2\right]^{\frac{1}{2}} \\ &\qquad\qquad \times\left[(N_1-n_1)^2+(N_2-n_2)^2+4\left(\mathfrak{M}(g;a,b)-\mathfrak{M}(g;c,d)\right)^2\right]^{\frac{1}{2}} \end{aligned}$$

holds.

Proof. From (3.3) and using the fact that for the Grüss inequality involving weighted means (see for example, Dragomir [7]), then

$$\mathfrak{T}(f;a,b) \le \left(\frac{M_1-m_1}{2}\right)^2, \quad \mathfrak{T}(f;c,d) \le \left(\frac{N_1-n_1}{2}\right)^2$$

and similar results for the mapping g readily produces the results as stated. □

Corollary 3.3. *Let f and g be absolutely continuous on $\mathring{I}$. In addition, assume that $f', g' \in L_\infty(\mathring{I})$ and $[a,b], [c,d] \subseteq \mathring{I}$ ($\mathring{I}$ is the closure of I). Then we have the inequality*

$$\begin{aligned} &\left|\mathfrak{T}(f,g;a,b,c,d)\right| \\ &\le \left[S(a,b)\left\|f'\right\|_{\infty,[a,b]}+S(c,d)\left\|f'\right\|_{\infty,[c,d]}+\left(\mathfrak{M}(f;a,b)-\mathfrak{M}(f;c,d)\right)^2\right]^{\frac{1}{2}} \\ &\times\left[S(a,b)\left\|g'\right\|_{\infty,[a,b]}+S(c,d)\left\|g'\right\|_{\infty,[c,d]}+12\left(\mathfrak{M}(g;a,b)-\mathfrak{M}(g;c,d)\right)^2\right]^{\frac{1}{2}}, \end{aligned}$$

where $\|f'\|_{\infty,[a,b]} := ess \sup_{x\in[a,b]} |f'(x)|$,

$$S(a,b) = \frac{W_2(a,b)}{W_0(a,b)} - \left(\frac{W_1(a,b)}{W_0(a,b)}\right)^2 \quad \text{and} \quad W_n(a,b) = \int_a^b x^n w(x)\,dx.$$

Proof. Using (3.3) and the fact that the weighted Chebychev inequality (see [7] for example) is such that

$$\mathfrak{T}(f;a,b) \le S(a,b)\,\|f'\|_{\infty,[a,b]}$$

then, the stated result is readily produced. □

Finally, using a weighted generalisation of the Lupaş inequality of G.V and I.Z. Milovanić [12], namely, for $w^{-\frac{1}{2}}f' \in L_2[a,b]$

$$\mathfrak{T}(f;a,b) \le \frac{W_0(a,b)}{\pi^2}\left\|w^{-\frac{1}{2}}f'\right\|_2^2$$

produces the following corollary.

Corollary 3.4. *Let f and g be absolutely continuous on $\mathring{I}$, $f', g' \in L_2(\mathring{I})$ and $[a,b]$, $[c,d] \subset \mathring{I}$. The following inequality then holds*

$$|\mathfrak{T}(f,g;a,b,c,d)| \le \frac{1}{\pi}\left[W_0^2(a,b)\left\|w^{-\frac{1}{2}}f'\right\|_{2,[a,b]}^2 + W_0^2(c,d)\left\|w^{-\frac{1}{2}}f'\right\|_{2,[c,d]}^2 \right.$$
$$\left. + \pi^2\left(\mathfrak{M}(f;a,b) - \mathfrak{M}(f;c,d)\right)^2\right]^{\frac{1}{2}}$$

$$\times \frac{1}{\pi}\left[W_0^2(a,b)\left\|w^{-\frac{1}{2}}g'\right\|_{2,[a,b]}^2 + W_0^2(c,d)\left\|w^{-\frac{1}{2}}g'\right\|_{2,[c,d]}^2 \right.$$
$$\left. + \pi^2\left(\mathfrak{M}(g;a,b) - \mathfrak{M}(g;c,d)\right)^2\right]^{\frac{1}{2}},$$

where

$$\left\|w^{-\frac{1}{2}}f'\right\|_{2,[a,b]} := \left(\int_a^b w^{-\frac{1}{2}}\left|f'(x)\right|^2 dx\right)^{\frac{1}{2}}$$

and $W_0(a,b)$ *is the zeroth moment of* $w(\cdot)$ *over* (a,b).

Remark 3.1. If $c = a$ and $d = b$ then prior results are recaptured.

Remark 3.2. If f and g are assumed to be of Hölder type, then bounds along similar lines to those obtained in Section 2 could also be obtained for the weighted Chebychev functional utilising identity (3.5). This will however not be pursued further.

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