REVERSE WEIGHTED L_p–NORM INEQUALITIES IN CONVOLUTIONS

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ABSTRACT. Various weighted L_p $(p > 1)$–norm inequalities in convolutions were derived by
using Hölder's inequality. Therefore, by using reverse Hölder inequalities one can obtain reverse
weighted L_p–norm inequalities. These inequalities are important in studying stability of some
inverse problems.

1. INTRODUCTION

For the Fourier convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - \xi)g(\xi) \, d\xi,$$

the Young's inequality

(1.1) $\quad \|f * g\|_r \le \|f\|_p \|g\|_q, \quad f \in L_p(\mathbb{R}), g \in L_q(\mathbb{R}), r^{-1} = p^{-1} + q^{-1} - 1 \quad (p, q, r > 0),$

is fundamental. Note, however, that for the typical case of $f, g \in L_2(\mathbb{R}^n)$, the inequality (1.1)
does not hold. In a series of papers [4, 5, 6, 7] (see also [1]) the first author obtained the
following weighted L_p $(p > 1)$ inequality for convolution.

Proposition 1.1. *([7]). For two nonvanishing functions $\rho_j \in L_1(\mathbb{R})\,(j = 1, 2)$ the following $L_p\,(p > 1)$ weighted convolution inequality*

$$(1.2) \qquad \left\| \left((F_1 \rho_1) * (F_2 \rho_2) \right) (\rho_1 * \rho_2)^{\frac{1}{p}-1} \right\|_p \leq \|F_1\|_{L_p(\mathbb{R}, |\rho_1|)} \|F_2\|_{L_p(\mathbb{R}, |\rho_2|)}$$

holds for $F_j \in L_p(\mathbb{R}, |\rho_j|) \quad (j = 1, 2)$. Equality holds if and only if

$$(1.3) \qquad\qquad\qquad F_j(x) = C_j e^{\alpha x},$$

where α is a constant such that $e^{\alpha x} \in L_p(\mathbb{R}, |\rho_j|) \quad (j = 1, 2)$.

Here

$$\|F\|_{L_p(\mathbb{R}; \rho)} = \left\{ \int_{-\infty}^{\infty} |F(x)|^p \rho(x)\, dx \right\}^{\frac{1}{p}}.$$

Unlike the Young's inequality, the inequality (1.2) holds also in case $p = 2$.

In many cases of interest, the convolution is given in the form

$$(1.4) \qquad\qquad \rho_2(x) \equiv 1, \quad F_2(x) = G(x),$$

where $G(x - \xi)$ is some Green's function. Then the inequality (1.2) takes the form

$$(1.5) \qquad\qquad \| (F\rho) * G \|_p \leq \|\rho\|_p^{1-\frac{1}{p}} \|G\|_p \|F\|_{L_p(\mathbb{R}, |\rho|)},$$

where ρ, F, and G are such that the right hand side of (1.5) is finite.

The inequality (1.5) enables us to estimate the output function

$$(1.6) \qquad\qquad \int_{-\infty}^{\infty} F(\xi) \rho(\xi) G(x - \xi)\, d\xi$$

in terms of the input function F. In this paper we are interested in the reverse type inequality for (1.5), namely, we wish to estimate the input function F by means of the output (1.6). This kind of estimate is important in inverse problems. Our estimate is based on the following version of the reverse Hölder inequality

Proposition 1.2. *([2], see also [3], pages 125-126). For two positive functions f and g satisfying*

$$(1.7) \qquad\qquad 0 < m \leq \frac{f}{g} \leq M < \infty$$

on the set X, and for $p, q > 0,\ p^{-1} + q^{-1} = 1$,

$$(1.8) \qquad \left(\int_X f\, d\mu \right)^{\frac{1}{p}} \left(\int_X g\, d\mu \right)^{\frac{1}{q}} \leq A_{p,q} \left(\frac{m}{M} \right) \int_X f^{\frac{1}{p}} g^{\frac{1}{q}}\, d\mu,$$

if the right hand side integral converges. Here

$$A_{p,q}(t) = p^{-\frac{1}{p}} q^{-\frac{1}{q}} t^{-\frac{1}{pq}} (1 - t) \left(1 - t^{\frac{1}{p}} \right)^{-\frac{1}{p}} \left(1 - t^{\frac{1}{q}} \right)^{-\frac{1}{q}}.$$

2. A GENERAL REVERSE WEIGHTED L_p CONVOLUTION INEQUALITY

Our main result is the following

Theorem 2.1. *Let F_1 and F_2 be positive functions satisfying*

$$(2.1) \quad 0 < m_1^{\frac{1}{p}} \le F_1(x) \le M_1^{\frac{1}{p}} < \infty, \quad 0 < m_2^{\frac{1}{p}} \le F_2(x) \le M_2^{\frac{1}{p}} < \infty, \quad p > 1, \quad x \in \mathbb{R}.$$

Then for any positive functions ρ_1 and ρ_2 we have the reverse L_p–weighted convolution inequality

$$(2.2) \quad \left\| \left((F_1\rho_1) * (F_2\rho_2) \right) (\rho_1 * \rho_2)^{\frac{1}{p}-1} \right\|_p \ge \left\{ A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \right\}^{-1} \|F_1\|_{L_p(\mathbb{R}, \rho_1)} \|F_2\|_{L_p(\mathbb{R}, \rho_2)}.$$

Inequality (2.2) and others should be understood in the sense that if the left hand side is finite, then so is the right hand side, and in this case the inequality holds.

Proof. Let

$$f(\xi) = F_1^p(\xi) F_2^p(x-\xi) \rho_1(\xi) \rho_2(x-\xi), \qquad g(\xi) = \rho_1(\xi)\rho_2(x-\xi).$$

Then condition (2.1) implies

$$m_1 m_2 \le \frac{f(\xi)}{g(\xi)} \le M_1 M_2, \qquad \xi \in \mathbb{R}.$$

Hence, one can apply the reverse Hölder inequality (1.8) for f and g to get

$$A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \int_{-\infty}^{\infty} F_1(\xi)\rho_1(\xi) F_2(x-\xi)\rho_2(x-\xi)\, d\xi$$

$$\ge \left\{ \int_{-\infty}^{\infty} F_1^p(\xi) F_2^p(x-\xi)\rho_1(\xi)\rho_2(x-\xi) d\xi \right\}^{\frac{1}{p}} \left\{ \int_{-\infty}^{\infty} \rho_1(\xi)\rho_2(x-\xi) d\xi \right\}^{1-\frac{1}{p}}.$$

Hence,

$$(2.3) \quad \left\{ \int_{-\infty}^{\infty} F_1(\xi)\rho_1(\xi) F_2(x-\xi)\rho_2(x-\xi)\, d\xi \right\}^p \left\{ \int_{-\infty}^{\infty} \rho_1(\xi)\rho_2(x-\xi) d\xi \right\}^{1-p}$$

$$\ge \left\{ A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \right\}^{-p} \int_{-\infty}^{\infty} F_1^p(\xi) F_2^p(x-\xi)\rho_1(\xi)\rho_2(x-\xi) d\xi.$$

Taking integration of both sides of (2.3) with respect to x from $-\infty$ to ∞ we obtain the inequality

$$(2.4) \quad \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} F_1(\xi)\rho_1(\xi) F_2(x-\xi)\rho_2(x-\xi)\, d\xi \right\}^p \left\{ \int_{-\infty}^{\infty} \rho_1(\xi)\rho_2(x-\xi) d\xi \right\}^{1-p} dx$$

$$\ge \left\{ A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \right\}^{-p} \int_{-\infty}^{\infty} F_1^p(\xi)\rho_1(\xi) d\xi \int_{-\infty}^{\infty} F_2^p(x)\rho_2(x) dx.$$

Raising both sides of the inequality (2.4) to power $\frac{1}{p}$ yields the inequality (2.2). $\qquad \square$

Inequality (1.8) reverses the sign if $0 < p < 1$. Hence, inequality (2.2) reverses the sign if $0 < p < 1$.

In formula (2.3) replacing ρ_2 by 1, and $F_2(x - \xi)$ by $G(x - \xi)$, and taking integration with respect to x from c to d we arrive at the following inequality

$$(2.5) \quad \int_c^d \left(\int_{-\infty}^\infty F(\xi)\, \rho(\xi)\, G(x - \xi)\, d\xi \right)^p dx$$

$$\geq \left\{ A_{p,q} \left(\frac{m}{M} \right) \right\}^{-p} \left(\int_{-\infty}^\infty \rho(\xi)\, d\xi \right)^{p-1} \int_{-\infty}^\infty F^p(\xi)\, \rho(\xi)\, d\xi \int_{c-\xi}^{d-\xi} G^p(x)\, dx,$$

valid if positive continuous functions ρ, F, and G satisfy

$$(2.6) \qquad\qquad 0 < m^{\frac{1}{p}} \leq F(\xi) G(x - \xi) \leq M^{\frac{1}{p}}, \quad x \in [c, d], \quad \xi \in \mathbb{R}.$$

Inequality (2.5) is especially important when $G(x - \xi)$ is a Green's function. See examples in the next section.

3. Examples

3.1. The first order differential equation.
The solution $y(x)$ of the first order differential equation

$$y'(x) + \lambda\, y(x) = F(x), \quad y(0) = 0,$$

is represented in the form

$$y(x) = \int_0^x F(t) e^{-\lambda(x-t)}\, dt.$$

So we shall consider the integral transform

$$f(x) = \int_0^x F(t)\, \rho(t) e^{-\lambda(x-t)}\, dt, \qquad \lambda > 0.$$

Take

$$G(x) = \begin{cases} e^{-\lambda x}, & x > 0, \\ 0, & x < 0. \end{cases}$$

The condition (2.6) reads

$$(3.1) \qquad\qquad 0 < m^{\frac{1}{p}} \leq F(t) e^{-\lambda(x-t)} \leq M^{\frac{1}{p}}.$$

It will be satisfied for $0 \leq t \leq x \leq d < \infty$, if we have

$$(3.2) \qquad 0 < m^{\frac{1}{p}} e^{\lambda d - \lambda t} \leq F(t) \leq M^{\frac{1}{p}}, \qquad 0 < d < \frac{1}{p\lambda} \log \frac{M}{m}.$$

Notice that

$$\int_{c-\xi}^{d-\xi} G^p(x)\, dx = \begin{cases} \frac{e^{-\lambda pc} - e^{-\lambda pd}}{\lambda p}\, e^{\lambda p \xi}, & \xi < c, \\ \frac{1 - e^{\lambda p \xi - \lambda p d}}{\lambda p}, & c < \xi < d. \end{cases}$$

Thus the inequality (2.5) yields

$$(3.3) \quad \int_c^d f^p(x) \left(\int_0^x \rho(t)\, dt \right)^{1-p} dx$$

$$\geq \left\{ A_{p,q} \left(\frac{m}{M} \right) \right\}^{-p} \frac{1}{\lambda p} \left[(e^{-\lambda pc} - e^{-\lambda pd}) \int_0^c F^p(\xi)\rho(\xi) e^{\lambda p \xi}\, d\xi \right.$$

$$\left. + \int_c^d F^p(\xi)\rho(\xi)(1 - e^{-\lambda pd} e^{\lambda p \xi})\, d\xi \right].$$

Here we assume that ρ is a positive continuous function on $[0, d]$, and F satisfies (3.2).

3.2. **Picard transform.**

Note that $\frac{1}{2}e^{-|x-t|}$ is the Green's function for the boundary value problem

$$y'' - y = 0, \qquad \lim_{x \to \pm\infty} y(x) = 0.$$

So, we shall consider the Picard transform

$$f(x) = \frac{1}{2} \int_{-\infty}^{\infty} F(t)\rho(t)e^{-|x-t|}dt.$$

Take $G(x) = e^{-|x|}$. Since

$$e^{-a}e^{|t|} \leq e^{|x-t|} \leq e^{a}e^{|t|}, \quad |x| \leq a,$$

we see that the condition (2.6)

$$(3.4) \qquad\qquad 0 < m^{\frac{1}{p}} \leq F(t)e^{-|x-t|} \leq M^{\frac{1}{p}},$$

holds if

$$(3.5) \qquad 0 < m^{\frac{1}{p}}e^{a}e^{|t|} \leq F(t) \leq M^{\frac{1}{p}}e^{-a}e^{|t|}, \quad t \in \mathbb{R}, \qquad 0 < a < \frac{1}{2p}\log\frac{M}{m}.$$

We have

$$\int_{c-t}^{d-t} G^p(x)\,dx = \int_{c-t}^{d-t} e^{-p|x|}\,dx = \begin{cases} \frac{e^{pt}}{p}\left[e^{-pc} - e^{-pd}\right], & t < c, \\ \frac{e^{-pt}}{p}\left[e^{pd} - e^{pc}\right], & t > d, \\ \frac{1}{p}\left(2 - e^{pc-pt} - e^{pt-pd}\right), & c < t < d. \end{cases}$$

Thus, for $-a \leq c, d \leq a$ the inequality (2.5) yields

$$(3.6) \quad \int_{c}^{d} f^p(x)\,dx \geq \frac{1}{2^p p}\left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-p}\left(\int_{-\infty}^{\infty}\rho(t)dt\right)^{p-1}$$

$$\left[\left(e^{-pc} - e^{-pd}\right)\int_{-\infty}^{c} F^p(t)\rho(t)e^{pt}dt + \left(e^{pd} - e^{pc}\right)\int_{d}^{\infty} F^p(t)\rho(t)e^{-pt}dt\right.$$

$$\left. + \int_{c}^{d} F^p(t)\rho(t)\left(2 - e^{pc-pt} - e^{pt-pd}\right)dt\right],$$

if ρ is positive continuous, and F satisfies (3.5).

3.3. **Poisson integrals.**

Consider the Poisson integral

$$(3.7) \qquad\qquad u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(\xi)\rho(\xi)\frac{y}{(x-\xi)^2 + y^2}d\xi.$$

Take

$$G(x) = \frac{y}{x^2 + y^2}.$$

Let

$$\xi \in [a, b], \quad x \in [c, d].$$

Denote

$$\alpha = \max\{|a - c|, |a - d|, |b - c|, |b - d|\}.$$

We have

$$\frac{y}{\alpha^2 + y^2} \leq \frac{y}{(x - \xi)^2 + y^2} \leq \frac{1}{y}.$$

Thus,

$$\int_{c-\xi}^{d-\xi} G^p(x)\,dx = \int_{c-\xi}^{d-\xi}\left(\frac{y}{x^2 + y^2}\right)^p dx \geq (d - c)\left(\frac{y}{\alpha^2 + y^2}\right)^p.$$

Hence, for a function F satisfying

$$\frac{\alpha^2 + y^2}{y} \, m^{\frac{1}{p}} \leq F(\xi) \leq y \, M^{\frac{1}{p}},$$

and for a positive continuous function ρ on $[a, b]$ we obtain

$$(3.8) \quad \int_c^d u^p(x, y)\, dx \geq \frac{(d-c)}{\pi^p} \left(\frac{y}{\alpha^2 + y^2}\right)^p \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-p}$$

$$\left(\int_a^b \rho(\xi)\, d\xi\right)^{p-1} \int_a^b F^p(\xi)\rho(\xi) d\xi.$$

Consider now the conjugate Poisson integral

$$(3.9) \quad v(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(\xi)\rho(\xi) \frac{x-\xi}{(x-\xi)^2 + y^2} d\xi.$$

Take

$$G(x) = \frac{x}{x^2 + y^2}.$$

For

$$\xi \in [a, b], \quad x \in [c, d], \qquad (b < c),$$

we have

$$\frac{c-b}{(d-a)^2 + y^2} \leq \frac{x-\xi}{(x-\xi)^2 + y^2} \leq \frac{d-a}{(c-b)^2 + y^2}.$$

Thus,

$$\int_{c-\xi}^{d-\xi} G^p(x)\, dx = \int_{c-\xi}^{d-\xi} \left(\frac{x}{x^2 + y^2}\right)^p dx \geq (d-c)\left(\frac{c-b}{(d-a)^2 + y^2}\right)^p.$$

Hence, for a function F satisfying

$$\frac{(d-a)^2 + y^2}{c-b} \, m^{\frac{1}{p}} \leq F(\xi) \leq \frac{(c-b)^2 + y^2}{d-a} \, M^{\frac{1}{p}},$$

and for a positive continuous function ρ on $[a, b]$ we obtain

$$(3.10) \quad \int_c^d v^p(x, y)\, dx \geq \frac{(d-c)}{\pi^p} \left(\frac{c-b}{(d-a)^2 + y^2}\right)^p \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-p}$$

$$\left(\int_a^b \rho(\xi)\, d\xi\right)^{p-1} \int_a^b F^p(\xi)\rho(\xi) d\xi.$$

3.4. **Heat equation.** We consider the Weierstrass transform

$$(3.11) \quad u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} F(\xi)\rho(\xi) \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi,$$

which gives the formal solution $u(x, t)$ of the heat equation

$$u_t = \Delta u \quad \text{on} \quad \mathbb{R}_+ \times \mathbb{R},$$

subject to the initial condition

$$u(x, 0) = F(x)\rho(x) \quad \text{on} \quad \mathbb{R}.$$

Take

$$G(x) = e^{-\frac{x^2}{4t}}.$$

Let

$$x \in [-a,a], \quad \xi \in [-b,b], \quad a+b \le \sqrt{\frac{4t}{p} \log \frac{M}{m}}.$$

From

$$1 \le \exp\left(\frac{(x-\xi)^2}{4t}\right) \le \exp\left(\frac{(a+b)^2}{4t}\right),$$

we have

$$0 < m^{\frac{1}{p}} \le F(\xi) \exp\left(-\frac{(x-\xi)^2}{4t}\right) \le M^{\frac{1}{p}},$$

if

(3.12) $$m^{\frac{1}{p}} \exp\left(\frac{(a+b)^2}{4t}\right) \le F(\xi) \le M^{\frac{1}{p}}, \qquad \xi \in [-b,b].$$

It is easy to see that

$$\int_{c-\xi}^{d-\xi} e^{-\frac{px^2}{4t}} \, dx = \sqrt{\frac{\pi t}{p}} \left[\text{erf}\left(\frac{\sqrt{p}\,(d-\xi)}{2\sqrt{t}}\right) - \text{erf}\left(\frac{\sqrt{p}\,(c-\xi)}{2\sqrt{t}}\right)\right],$$

where

$$\text{erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$$

is the error function. Therefore, for $-a \le c < d \le a$, the inequality (2.5) yields

(3.13) $$\int_c^d u(x,t)^p dx \ge \frac{1}{2^p (\pi t)^{(p-1)/2} \sqrt{p}} \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-p} \left(\int_{-b}^b \rho(\xi) d\xi\right)^{p-1}$$

$$\int_{-b}^b F^p(\xi)\rho(\xi) \left[\text{erf}\left(\frac{\sqrt{p}\,(d-\xi)}{2\sqrt{t}}\right) - \text{erf}\left(\frac{\sqrt{p}\,(c-\xi)}{2\sqrt{t}}\right)\right] d\xi,$$

where ρ is a positive continuous function on $[-b,b]$, and F satisfies (3.12).

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