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AN INEQUALITY FOR LINEAR POSITIVE FUNCTIONALS

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Abstract

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Abstract

Using P_0 -simple functionals, we generalise the result from Theorem 1.1 obtained by Professor F. Qi (F. Qi, An algebraic inequality, *RGMA Res. Rep. Coll.*, 2(1) (1999), article 8).

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1. Introduction

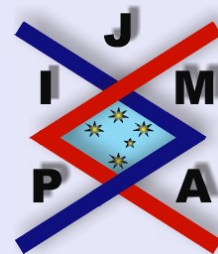
In [4] Professor Dr. F. Qi proved the following algebraic inequality

Theorem 1.1. *Let $b > a > 0$ and $\delta > 0$ be real numbers, then for any given positive $r \in \mathbb{R}$, we have*

$$(1.1) \quad \left(\frac{b + \delta - a}{b - a} \cdot \frac{b^{r+1} - a^{r+1}}{(b + \delta)^{r+1} - a^{r+1}} \right)^{1/r} > \frac{b}{b + \delta}.$$

The lower bound in (1.1) is the best possible.

In this paper we will present a generalization of the inequality (1.1).



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2. Some Lemmas

It is well-known that

$$C[a, b] = \{f : [a, b] \rightarrow \mathbb{R}; f \text{ is continuous on } [a, b]\},$$

and let

$$\omega(f; t) = \sup\{|f(x + h) - f(x)|; 0 \leq h \leq t, x, x + h \in [a, b]\}.$$

The least concave majorant of this modulus with respect to the variable t is given by

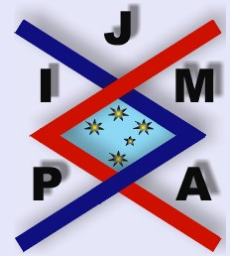
$$\tilde{\omega}(f; t) = \begin{cases} \sup_{0 \leq x \leq t \leq y} \frac{(t-x)\omega(f; y) + (y-t)\omega(f; x)}{y-x}, & \text{for } 0 \leq t \leq b-a, \\ \omega(f; b-a), & \text{for } t > b-a. \end{cases}$$

Let $I = [a, b]$ be a compact interval of the real axis, S a subspace of $C(I)$ and A a linear functional defined on S . The following definition was given by T. Popoviciu in [3].

Definition 2.1 ([3]). A linear functional A defined on the subspace S which contains all polynomials is called P_n -simple for $n \geq -1$ if

- (i) $A(e_{n+1}) \neq 0$;
- (ii) For every $f \in S$ there exist $n + 2$ distinct points t_1, t_2, \dots, t_{n+2} in $[a, b]$ such that

$$A(f) = A(e_{n+1})[t_1, t_2, \dots, t_{n+2}; f],$$



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where $[t_1, t_2, \dots, t_{n+2}; f]$ is the divided difference of the function f on the points t_1, t_2, \dots, t_{n+2} , and e_{n+1} denotes the monomial of degree $n + 1$.

Lemma 2.1 ([2]). *Let A be a linear bounded functional, $A : C(I) \rightarrow \mathbb{R}$. If A is P_0 -simple, then for all $f \in C(I)$ we have*

$$(2.1) \quad |A(f)| \leq \frac{\|A\|}{2} \tilde{\omega} \left(f; \frac{2A(e_1)}{\|A\|} \right).$$

Lemma 2.2 ([2]). *Let A be a linear bounded functional, $A : C(I) \rightarrow \mathbb{R}$. If $A(e_1) \neq 0$ and the inequality (2.1) holds for all $f \in C(I)$, then A is P_0 -simple.*

A function $f \in C^{(k)}[a, b]$ is called P_n -nonconcave if the inequality

$$[t_1, t_2, \dots, t_{n+2}; f] \geq 0$$

holds for any given $n + 2$ points $t_1, t_2, \dots, t_{n+2} \in [a, b]$.

The following result was proved by I. Raşa in [5]:

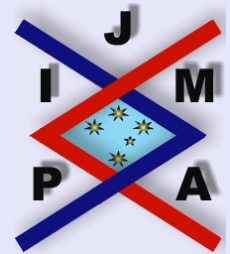
Lemma 2.3 ([5]). *Let k be a natural number such that $0 \leq k \leq n$ and $A : C^{(k)}[a, b] \rightarrow \mathbb{R}$ a linear bounded functional, $A \neq 0$, $A(e_i) = 0$ for $i = 0, 1, \dots, n$ such that $A(f) \geq 0$ for every f which belongs to $C^{(k)}[a, b]$ and is P_0 -nonconcave. Then A is P_0 -simple.*

In [1], S. G. Gal gave the exact formula for the usual modulus of continuity of the nonconcave continuous functions on $[a, b]$. He proved the following result:

Lemma 2.4 ([1]). *Let $f \in C[a, b]$ be nonconcave and monotone on $[a, b]$. For any given $t \in (0, b - a)$ we have*

(i) $\omega(f; t) = f(b) - f(b - t)$ if f is nondecreasing on $[a, b]$;

(ii) $\omega(f; t) = f(a) - f(a + t)$ if f is nonincreasing on $[a, b]$.



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3. Main results

Let a, b, d be real numbers such that $a < b < d$. Consider the functions u_b and u_b^* defined on $[a, d]$ by

$$u_b(t) = \begin{cases} 1, & t \in [a, b]; \\ 0, & t \in (b, d], \end{cases}$$

and

$$u_b^*(t) = \begin{cases} 0, & t \in [a, b]; \\ 1, & t \in (b, d]. \end{cases}$$

It is clear that

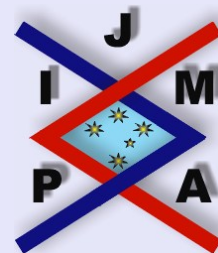
$$(3.1) \quad u_b(t) + u_b^*(t) = 1, \quad t \in [a, d].$$

Let A be a linear positive functional defined on the subspace S containing the functions u_b and u_b^* , which satisfies

1. $0 < A(u_b) \leq A(e_0)$, $0 < A(u_b^*) \leq A(e_0)$;
2. The functionals A_1 and A_2 defined by $A_1(f) = A(u_b f)$ and $A_2(f) = A(u_b^* f)$ are well defined for every $f \in C[a, b]$;
3. $A(e_1)A(u_b) - A(e_0)A(u_b e_1) \neq 0$.

Theorem 3.1. *Let A be a linear positive functional which satisfies conditions 1, 2 and 3 above. Then the functional $B : C[a, d] \rightarrow \mathbb{R}$ defined by*

$$(3.2) \quad B(f) = \frac{A(f)}{A(e_0)} - \frac{A(u_b f)}{A(u_b)}$$



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is P_0 -simple, and

$$(3.3) \quad \left| \frac{A(f)}{A(e_0)} - \frac{A(u_b f)}{A(u_b)} \right| \leq \frac{A(u_b^*)}{A(e_0)} \tilde{\omega}(f; t_b),$$

where

$$t_b = \frac{A(e_1 u_b^*)}{A(u_b^*)} - \frac{A(e_1 u_b)}{A(u_b)}.$$

Proof. In order to prove that the functional B is P_0 -simple, from Lemma 2.3, it is sufficient to verify $B(f) \geq 0$ for every nondecreasing function f on $[a, d]$.

It is easy to see that

$$(3.4) \quad \begin{aligned} B(f) &= \frac{(A(f u_b) + A(f u_b^*))A(u_b) - A(f u_b)(A(u_b) + A(u_b^*))}{A(e_0)A(u_b)} \\ &= \frac{A(u_b)A(f u_b^*) - A(f u_b)A(u_b^*)}{A(e_0)A(u_b)}. \end{aligned}$$

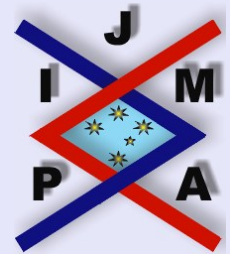
From the definitions of functions u_b and u_b^* and f being nondecreasing, we have

$$(3.5) \quad \begin{aligned} f u_b^* &\geq f(b) u_b^* \\ -f u_b &\geq -f(b) u_b. \end{aligned}$$

Substitution of inequality (3.5) into (3.4) yields $B(f) \geq 0$ for every nondecreasing function $f \in C[a, d]$.

From the equality (3.4) we get

$$(3.6) \quad \|B\| = \frac{2A(u_b^*)}{A(e_0)}$$



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and

$$(3.7) \quad B(e_1) = \frac{A(u_b)A(e_1u_b^*) - A(e_1u_b)A(u_b^*)}{A(e_0)A(u_b)}.$$

Since the functional B is P_0 -simple, from Lemma 2.1, the inequality (3.3) follows. \square

Corollary 3.1. *Let $f \in C[a, b]$ be nonconcave and monotone on $[a, b]$ and A a functional defined as in Theorem 3.1, then*

$$(3.8) \quad \frac{A(f)}{A(e_0)} - \frac{A(u_b f)}{A(u_b)} \leq \frac{A(u_b^*)}{A(e_0)}(f(d) - f(d - t_b))$$

if f is nondecreasing on $[a, d]$, and

$$(3.9) \quad -\frac{A(f)}{A(e_0)} + \frac{A(u_b f)}{A(u_b)} \leq \frac{A(u_b^*)}{A(e_0)}(f(a) - f(a + t_b))$$

if f is nonincreasing on $[a, d]$.

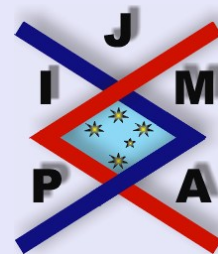
Proof. From Lemma 2.3 we have

$$(3.10) \quad \omega(f; t) = f(d) - f(d - t)$$

if f is nondecreasing on $[a, d]$, and

$$(3.11) \quad \omega(f; t) = f(a) - f(a + t)$$

if the function f is nonincreasing on $[a, d]$.



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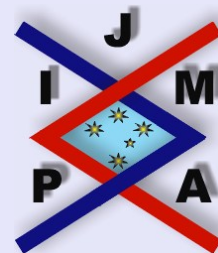
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The functions $f(d) - f(d - \cdot)$ and $f(a) - f(a + \cdot)$ are concave on $[0, d - a]$ if the function f is a convex function. Since $\tilde{\omega}(f; \cdot)$ is the least concave majorant of the function ω under above conditions, then we get $\tilde{\omega}(f; \cdot) = \omega(f; \cdot)$.

Combining (3.10) and (3.11) with Theorem 3.1 leads to inequalities (3.8) and (3.9). □



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4. Applications

Let a, b and d be positive numbers such that $0 < a < b < d$. Consider the functional $A : C[a, d] \rightarrow \mathbb{R}$ defined by

$$(4.1) \quad A(f) = \int_a^d w(t)f(t)dt,$$

where $w : (a, d) \rightarrow \mathbb{R}$ is a positive weight function.

It is easy to verify that the functional A defined by (4.1) satisfies conditions in Theorem 3.1 and the functional B can be expressed as

$$B(f) = \frac{\int_a^d w(t)f(t)dt}{\int_a^d w(t)f(t)dt} - \frac{\int_a^b w(t)f(t)dt}{\int_a^b w(t)f(t)dt}.$$

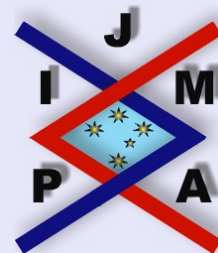
Then, from Theorem 3.1, we obtain

Theorem 4.1. For every $f \in C[a, b]$,

$$(4.2) \quad \left| \frac{\int_a^d w(t)f(t)dt}{\int_a^d w(t)f(t)dt} - \frac{\int_a^b w(t)f(t)dt}{\int_a^b w(t)f(t)dt} \right| \leq \frac{\int_b^d w(t)dt}{\int_a^d w(t)dt} \tilde{\omega}(f; t_b),$$

where

$$t_b = \frac{\int_b^d tw(t)dt}{\int_b^d w(t)dt} - \frac{\int_a^b tw(t)dt}{\int_a^b w(t)dt}.$$



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Corollary 4.1. Let a, b and c be positive numbers such that $0 < a < b < d$. Then we have the following inequalities:

$$(4.3) \quad 0 < \frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt - \frac{da}{d-a} \int_a^d \frac{f(t)}{t^2} dt \leq \frac{d-b}{d-a} \cdot \frac{a}{b} (f(a) - f(a+t_b))$$

for every convex and nonincreasing function f on $[a, d]$, where

$$t_b = \frac{bd \ln \frac{d}{b}}{d-b} - \frac{ab \ln \frac{b}{a}}{b-a}.$$

Proof. Taking $w(t) = \frac{1}{t^2}$, $t \in [a, d]$ in Theorem 4.1 produces inequality (4.3). \square

Remark 1. Letting $f(t) = \frac{1}{t^r}$, $r > 0$ in inequality (4.3) gives us

$$(4.4) \quad \frac{b^{r+1} - a^{r+1}}{d^{r+1} - a^{r+1}} \cdot \frac{d-a}{b-a} > \frac{b^r}{d^r}$$

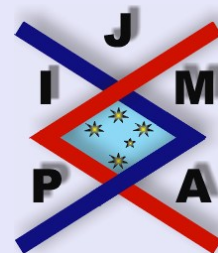
and

$$(4.5) \quad \frac{b^{r+1} - a^{r+1}}{d^{r+1} - a^{r+1}} \cdot \frac{d-a}{b-a} < \frac{b^r}{d^r} + (r+1)(d-b) \left(\frac{b}{a+t_b} \right)^r \frac{(a+t_b)^r - a^r}{d^{r+1} - a^{r+1}} \cdot \frac{a}{b}.$$

If we let $d = b + \delta$ in inequality (4.4), inequality (1.1) follows. Thus Theorem 1.1 by Professor Dr. F. Qi in [4] is generalized.

Remark 2. We can obtain some discrete inequalities if we select the functional A of the form

$$A(f) = \sum_{k=1}^{n+m} \lambda_k f(x_k),$$



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where $x_k, k = 1, 2, \dots, n + m$, are $n + m$ distinct points such that

$$x_1 < x_2 < \dots < x_n < x_{n+1} < \dots < x_{n+m},$$

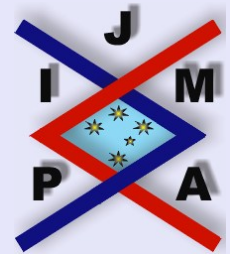
and $\lambda_k, k = 1, 2, \dots, n + m$, are $n + m$ positive numbers.

Choose the point $b = x_n$, then from Theorem 3.1, we obtain the discrete analogue of Theorem 4.1:

$$\left| \frac{\sum_{k=1}^{n+m} \lambda_k f(x_k)}{\sum_{k=1}^{n+m} \lambda_k} - \frac{\sum_{k=n+1}^{n+m} \lambda_k f(x_k)}{\sum_{k=n+1}^{n+m} \lambda_k} \right| \leq \frac{\sum_{k=n+1}^{n+m} \lambda_k}{\sum_{k=1}^{n+m} \lambda_k} \tilde{\omega}(f; t_b),$$

where

$$t_b = \frac{\sum_{k=n+1}^{n+m} \lambda_k x_k}{\sum_{k=n+1}^{n+m} \lambda_k} - \frac{\sum_{k=1}^n \lambda_k x_k}{\sum_{k=n+1}^{n+m} \lambda_k}.$$



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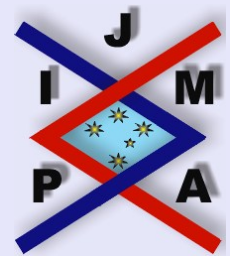
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