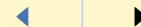
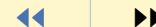


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HEISENBERG UNCERTAINTY PRINCIPLES FOR SOME q^2 -ANALOGUE FOURIER TRANSFORMS

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Abstract: The aim of this paper is to state q -analogues of the Heisenberg uncertainty principles for some q^2 -analogue Fourier transforms introduced and studied in [7, 8].

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q -Heisenberg Uncertainty Principles

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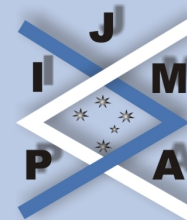
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1. Introduction

One of the most famous uncertainty principles is the so-called Heisenberg uncertainty principle. With the use of an inequality involving a function and its Fourier transform, it states that in classical Fourier analysis it is impossible to find a function f that is arbitrarily well localized together with its Fourier transform \widehat{f} .

In this paper, we will prove that similar to the classical theory, a non-zero function and its q^2 -analogue Fourier transform (see [7, 8]) cannot both be sharply localized. For this purpose we will prove a q -analogue of the Heisenberg uncertainty principle. This paper is organized as follows: in Section 2, some notations, results and definitions from the theory of the q^2 -analogue Fourier transform are presented. All of these results can be found in [7] and [8]. In Section 3, q -analogues of the Heisenberg uncertainty principle are stated.



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2. Notations and Preliminaries

Throughout this paper, we will follow the notations of [7, 8]. We fix $q \in]0, 1[$ such that $\frac{\text{Log}(1-q)}{\text{Log}(q)} \in 2\mathbb{Z}$. For the definitions, notations and properties of the q -shifted factorials and the q -hypergeometric functions, refer to the book by G. Gasper and M. Rahman [3].

Define

$$\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\} \quad \text{and} \quad \mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}.$$

We also denote

$$(2.1) \quad [x]_q = \frac{1 - q^x}{1 - q}, \quad x \in \mathbb{C}$$

and

$$(2.2) \quad [n]_q! = \frac{(q; q)_n}{(1 - q)^n}, \quad n \in \mathbb{N}.$$

The q^2 -analogue differential operator (see [8]) is

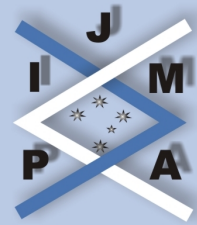
$$(2.3) \quad \partial_q(f)(z) = \frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1 - q)z}.$$

We remark that if f is differentiable at z , then $\lim_{q \rightarrow 1} \partial_q(f)(z) = f'(z)$.

∂_q is closely related to the classical q -derivative operators studied in [3, 5].

The q -trigonometric functions q -cosine and q -sine are defined by (see [7, 8]):

$$(2.4) \quad \cos(x; q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n]_q!}$$



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and

$$(2.5) \quad \sin(x; q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n+1}}{[2n+1]_q!}.$$

These functions induce a ∂_q -adapted q^2 -analogue exponential function by

$$(2.6) \quad e(z; q^2) = \cos(-iz; q^2) + i \sin(-iz; q^2).$$

$e(z; q^2)$ is absolutely convergent for all z in the plane since both of its component functions are absolutely convergent. $\lim_{q \rightarrow 1^-} e(z; q^2) = e^z$ (exponential function) pointwise and uniformly on compacta.

The q -Jackson integrals are defined by (see [4])

$$(2.7) \quad \int_{-\infty}^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} \{f(q^n) + f(-q^n)\} q^n$$

and

$$(2.8) \quad \int_0^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n),$$

provided that the sums converge absolutely. Using these q -integrals, we define for $p > 0$,

$$(2.9) \quad L_q^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty \right\},$$

$$(2.10) \quad L_q^p(\mathbb{R}_{q,+}) = \left\{ f : \left(\int_0^{\infty} |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty \right\}$$

and

$$(2.11) \quad L_q^\infty(\mathbb{R}_q) = \left\{ f : \|f\|_{\infty,q} = \sup_{x \in \mathbb{R}_q} |f(x)| < \infty \right\}.$$

The following result can be verified by direct computation.

Lemma 2.1. *If $\int_{-\infty}^{\infty} f(t)d_q t$ exists, then*

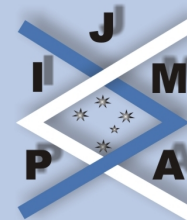
1. *for all integers n , $\int_{-\infty}^{\infty} f(q^n t)d_q t = q^{-n} \int_{-\infty}^{\infty} f(t)d_q t$;*
2. *f odd implies that $\int_{-\infty}^{\infty} f(t)d_q t = 0$;*
3. *f even implies that $\int_{-\infty}^{\infty} f(t)d_q t = 2 \int_0^{\infty} f(t)d_q t$.*

The following lemma lists some useful computational properties of ∂_q , and reflects the sensitivity of this operator to the parity of its argument. The proof is straightforward.

Lemma 2.2.

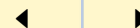
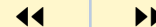
1. *If f is odd $\partial_q f(z) = \frac{f(z)-f(qz)}{(1-q)z}$ and if f is even $\partial_q f(z) = \frac{f(q^{-1}z)-f(z)}{(1-q)z}$.*
2. *We have $\partial_q \sin(x; q^2) = \cos(x; q^2)$, $\partial_q \cos(x; q^2) = -\sin(x; q^2)$ and $\partial_q e(x; q^2) = e(x; q^2)$.*
3. *If f and g are both odd, then*

$$\partial_q(fg)(z) = q^{-1}(\partial_q f) \left(\frac{z}{q} \right) g(z) + q^{-1}f \left(\frac{z}{q} \right) (\partial_q g) \left(\frac{z}{q} \right).$$



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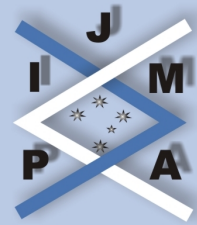


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4. If f is odd and g is even, then

$$\partial_q(fg)(z) = (\partial_q f)(z)g(z) + qf(qz)(\partial_q g)(qz).$$

5. If f and g are both even, then

$$\partial_q(fg)(z) = (\partial_q f)(z)g\left(\frac{z}{q}\right) + f(z)(\partial_q g)(z).$$

The following simple result, giving a q -analogue of the integration by parts theorem, can be verified by direct calculation.

Lemma 2.3. If $\int_{-\infty}^{\infty} (\partial_q f)(x)g(x)d_q x$ exists, then

$$(2.12) \quad \int_{-\infty}^{\infty} (\partial_q f)(x)g(x)d_q x = - \int_{-\infty}^{\infty} f(x)(\partial_q g)(x)d_q x.$$

With the use of the q -Gamma function

$$\Gamma_q(x) = \frac{(q; q)_{\infty}}{(q^x; q)_{\infty}} (1 - q)^{1-x},$$

R.L. Rubin defined in [8] the q^2 -analogue Fourier transform as

$$(2.13) \quad \widehat{f}(x; q^2) = K \int_{-\infty}^{\infty} f(t)e(-itx; q^2)d_q t,$$

where $K = \frac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}(\frac{1}{2})}$.

We define the q^2 -analogue Fourier-cosine and Fourier-sine transform as (see [2] and [6])

$$(2.14) \quad \mathcal{F}_q(f)(x) = 2K \int_0^{\infty} f(t) \cos(xt; q^2)d_q t$$

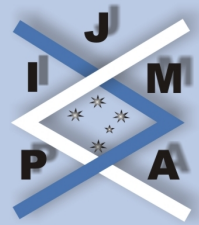
and

$$(2.15) \quad {}_q\mathcal{F}(f)(x) = 2K \int_0^\infty f(t) \sin(xt; q^2) d_q t.$$

Observe that if f is even then $\widehat{f}(\cdot; q^2) = \mathcal{F}_q$ and if f is odd then $\widehat{f}(\cdot; q^2) = {}_q\mathcal{F}$.
It was shown in [8] that we have the following theorem.

Theorem 2.4.

1. If $f(u)$, $uf(u) \in L_q^1(\mathbb{R}_q)$, then $\partial_q \left(\widehat{f} \right) (x; q^2) = (-iuf(u)) \widehat{\cdot}(x; q^2)$.
2. If f , $\partial_q f \in L_q^1(\mathbb{R}_q)$, then $(\partial_q f) \widehat{\cdot}(x; q^2) = ix \widehat{f}(x; q^2)$
3. For $f \in L_q^2(\mathbb{R}_q)$, $\|\widehat{f}(\cdot; q^2)\|_{2,q} = \|f\|_{2,q}$.



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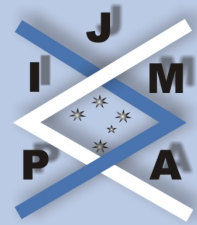


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3. *q*-Analogue of the Heisenberg Uncertainty Principle

For a function f defined on \mathbb{R}_q , we denote by f_o and f_e its odd and even parts respectively. Let us begin with the following theorem.

Theorem 3.1. *If f , xf and $xf(x; q^2)$ are in $L^2_q(\mathbb{R}_q)$, then*

$$(3.1) \quad \|f\|_{2,q}^2 \leq \|xf(x; q^2)\|_{2,q} \left[q \left(1 + q^{-\frac{3}{2}} \right) \|xf_o\|_{2,q} + \left(1 + q^{\frac{3}{2}} \right) \|xf_e\|_{2,q} \right].$$

Proof. Using the properties of the q^2 -analogue differential operator ∂_q , the properties of the q -integrals, the Hölder inequality and Theorem 2.4, we can see that

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} x \partial_q (f \bar{f})(x) d_q x \right| \\ &= \left| \int_{-\infty}^{\infty} x (q \bar{f}_o(x) + \bar{f}_e(q^{-1}x)) (\partial_q f)(x) d_q x \right. \\ & \quad \left. + \int_{-\infty}^{\infty} x (q f_o(qx) + f_e(x)) (\partial_q \bar{f})(x) d_q x \right| \\ &\leq q \int_{-\infty}^{\infty} |x f_o(x)| |\partial_q f(x)| d_q x + \int_{-\infty}^{\infty} |x f_e(q^{-1}x)| |\partial_q f(x)| d_q x \\ & \quad + \int_{-\infty}^{\infty} |x f_e(x)| |\partial_q f(x)| d_q x + q \int_{-\infty}^{\infty} |x f_o(x)| |\partial_q f(x)| d_q x \\ &\leq \|\partial_q f\|_{2,q} \left[q \left(\int_{-\infty}^{\infty} |x f_o(x)|^2 d_q x \right)^{\frac{1}{2}} + \left(\int_{-\infty}^{\infty} |x f_e(q^{-1}x)|^2 d_q x \right)^{\frac{1}{2}} \right. \\ & \quad \left. + \left(\int_{-\infty}^{\infty} |x f_e(x)|^2 d_q x \right)^{\frac{1}{2}} + q \left(\int_{-\infty}^{\infty} |x f_o(qx)|^2 d_q x \right)^{\frac{1}{2}} \right] \end{aligned}$$

$$= \|x\widehat{f}\|_{2,q} \left[q \left(1 + q^{-\frac{3}{2}} \right) \|xf_o\|_{2,q} + \left(1 + q^{\frac{3}{2}} \right) \|xf_e\|_{2,q} \right].$$

On the other hand, using the q -integration by parts theorem, we obtain

$$\int_{-\infty}^{\infty} x \partial_q (f \bar{f})(x) d_q x = - \int_{-\infty}^{\infty} |f(x)|^2 d_q x = -\|f\|_{2,q}^2,$$

which completes the proof. □

Corollary 3.2. *If f , xf and $x\widehat{f}$ are in $L_q^2(\mathbb{R}_q)$, then*

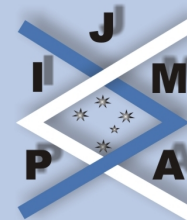
$$(3.2) \quad \|xf\|_{2,q} \|x\widehat{f}(x; q^2)\|_{2,q} \geq \frac{1}{q^{-\frac{1}{2}} + 1 + q + q^{\frac{3}{2}}} \|f\|_{2,q}^2.$$

Proof. The properties of the q -integral imply

$$\begin{aligned} \|xf\|_{2,q}^2 &= \int_{-\infty}^{\infty} x^2 (f_o(x) + f_e(x)) (\bar{f}_o(x) + \bar{f}_e(x)) d_q x \\ &= \int_{-\infty}^{\infty} x^2 f_o(x) \bar{f}_o(x) d_q x + \int_{-\infty}^{\infty} x^2 f_e(x) \bar{f}_e(x) d_q x \\ &= \|xf_o\|_{2,q}^2 + \|xf_e\|_{2,q}^2. \end{aligned}$$

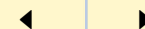
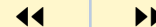
So, $\|xf_o\|_{2,q} \leq \|xf\|_{2,q}$ and $\|xf_e\|_{2,q} \leq \|xf\|_{2,q}$.

These inequalities together with the previous theorem give the desired result. □



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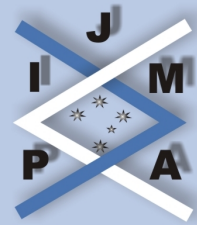


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Corollary 3.3.

1. If f , xf and $x\mathcal{F}_q$ are in $L_q^2(\mathbb{R}_{q,+})$, then

$$(3.3) \quad \left(\int_0^\infty x^2 |f(x)|^2 d_q x \right)^{\frac{1}{2}} \left(\int_0^\infty x^2 |\mathcal{F}_q(x)|^2 d_q x \right)^{\frac{1}{2}} \geq \frac{1}{1 + q^{\frac{3}{2}}} \int_0^\infty |f(x)|^2 d_q x.$$

2. If f , xf and $x {}_q\mathcal{F}$ are in $L_q^2(\mathbb{R}_{q,+})$, then

$$(3.4) \quad \left(\int_0^\infty x^2 |f(x)|^2 d_q x \right)^{\frac{1}{2}} \left(\int_0^\infty x^2 |{}_q\mathcal{F}(x)|^2 d_q x \right)^{\frac{1}{2}} \geq \frac{1}{q \left(1 + q^{-\frac{3}{2}} \right)} \int_0^\infty |f(x)|^2 d_q x.$$

Proof. The proof is a simple application of the previous theorem on taking $g(x) = f(x)$ if x is positive and $g(x) = f(-x)$ (resp. $g(x) = -f(-x)$) if not in the first case (resp. second case). \square

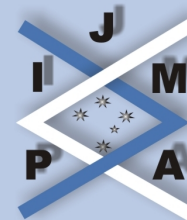
Remark 1. Corollary 3.2 gives a q -analogue of the Heisenberg uncertainty principle for the q^2 -analogue Fourier transform $\widehat{f}(\cdot; q^2)$.

Remark 2. Corollary 3.3 gives a q -analogue of the Heisenberg uncertainty principles for the q^2 -analogue Fourier-cosine and Fourier-sine transforms. These inequalities are slightly different from those given in [1]. This is due to the related q -analogue of special functions used.

Remark 3. Note that when q tends to 1, these inequalities tend at least formally to the corresponding classical ones.

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