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A DYNAMIC PROBLEM WITH ADHESION AND DAMAGE IN ELECTRO-VISCOELASTICITY WITH LONG-TERM MEMORY

SELMANI MOHAMED
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SETIF
19000 SETIF ALGERIA
s_elmanih@yahoo.fr

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ABSTRACT. We consider a dynamic frictionless contact problem for an electro-viscoelastic body with long-term memory and damage. The contact is modelled with normal compliance. The adhesion of the contact surfaces is taken into account and modelled by a surface variable, the bonding field. We derive variational formulation for the model which is formulated as a system involving the displacement field, the electric potential field, the damage field and the adhesion field. We prove the existence of a unique weak solution to the problem. The proof is based on arguments of evolution equations with monotone operators, parabolic inequalities, differential equations and fixed point.

Key words and phrases: Dynamic process, electro-viscoelastic material with long-term memory, frictionless contact, normal compliance, adhesion, damage, existence and uniqueness, monotone operator, fixed point, weak solution.

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1. INTRODUCTION

The piezoelectric effect is the apparition of electric charges on surfaces of particular crystals after deformation. Its reverse effect consists of the generation of stress and strain in crystals under the action of the electric field on the boundary. Materials undergoing piezoelectric materials effects are called piezoelectric materials, and their study requires techniques and results from electromagnetic theory and continuum mechanics. Piezoelectric materials are used extensively as switches and, actually, in many engineering systems in radioelectronics, electroacoustics and measuring equipment. However, there are very few mathematical results concerning contact problems involving piezoelectric materials and therefore there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in [12, 13, 14, 22, 23] and more recently in [1, 21]. The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has also recently received increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [3, 4, 6, 7, 16, 17, 18] and recently in the monographs [19, 20].

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The novelty in all these papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by α, which describes the pointwise fractional density of adhesion of active bonds on the contact surface, and is sometimes referred to as the intensity of adhesion. Following [6, 7], the bonding field satisfies the restriction $0 \leq \alpha \leq 1$. When $\alpha = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\alpha = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \alpha < 1$ the adhesion is partial and only a fraction α of the bonds is active. The importance of the paper lies in the coupling of the electric effect and the mechanical damage of the material. We study a dynamic problem of frictionless adhesive contact. We model the material with an electro-viscoelastic constitutive law with long-term memory and damage. The contact is modelled with normal compliance. We derive a variational formulation and prove the existence and uniqueness of the weak solution.

The paper is structured as follows. In Section 2 we present notation and some preliminaries. The model is described in Section 3 where the variational formulation is given. In Section 4, we present our main result stated in Theorem 4.1 and its proof which is based on arguments of evolution equations with monotone operators, parabolic inequalities, differential equations and fixed points.

2. NOTATION AND PRELIMINARIES

In this short section, we present the notation we shall use and some preliminary material. For more details, we refer the reader to [2, 5, 15]. We denote by S^d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while " $\cdot$ " and $|\cdot|$ represent the inner product and the Euclidean norm on S^d and $\mathbb{R}^d$, respectively. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let ν denote the unit outer normal on Γ. We shall use the notation

$$\begin{gathered} H = L^2(\Omega)^d = \left\{ \mathbf{u} = (u_i) \ / \ u_i \in L^2(\Omega) \ \right\}, \\ H^1(\Omega)^d = \left\{ \mathbf{u} = (u_i) \ / \ u_i \in H^1(\Omega) \ \right\}, \\ \mathcal{H} = \left\{ \sigma = (\sigma_{ij}) \ / \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \ \right\}, \\ \mathcal{H}_1 = \left\{ \sigma \in \mathcal{H} \ / \ Div\, \sigma \in H \right\}, \end{gathered}$$

where $\varepsilon : H^1(\Omega)^d \to \mathcal{H}$ and $Div : \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad Div\, \sigma = (\sigma_{ij,j}).$$

Here and below, the indices i and j run between 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces H, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{gathered} (\mathbf{u}, \mathbf{v})_H = \int_\Omega \mathbf{u} \cdot \mathbf{v} \, dx \qquad \forall \mathbf{u}, \mathbf{v} \in H, \\ (\mathbf{u}, \mathbf{v})_{H^1(\Omega)^d} = \int_\Omega \mathbf{u} \cdot \mathbf{v} \, dx + \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, dx \qquad \forall \mathbf{u}, \mathbf{v} \in H^1(\Omega)^d, \end{gathered}$$

where

$$\begin{gathered} \nabla \mathbf{v} = (v_{i,j}) \quad \forall \mathbf{v} \in H^1(\Omega)^d, \\ (\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma \cdot \tau \, dx \qquad \forall \sigma, \tau \in \mathcal{H}, \end{gathered}$$

$$(\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (Div\ \sigma, Div\ \tau)_H \quad \forall \sigma,\tau \in \mathcal{H}_1.$$

The associated norms on the spaces H, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{H^1(\Omega)^d}$, $|\cdot|_{\mathcal{H}}$ and $|\cdot|_{\mathcal{H}_1}$respectively. Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and let $\gamma : H^1(\Omega)^d \to H_\Gamma$ be the trace map. For every element $\mathbf{v} \in H^1(\Omega)^d$, we also use the notation $\mathbf{v}$ to denote the trace $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ and we denote by v_ν and $\mathbf{v}_\tau$ the normal and the tangential components of $\mathbf{v}$ on the boundary Γ given by

(2.1) $$v_\nu = \mathbf{v}\cdot\nu, \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu\nu.$$

Similarly, for a regular (say C^1) tensor field $\sigma : \Omega \to S^d$ we define its normal and tangential components by

(2.2) $$\sigma_\nu = (\sigma\nu)\cdot\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu\nu,$$

and we recall that the following Green's formula holds:

(2.3) $$(\sigma, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (Div\ \sigma, \mathbf{v})_H = \int_\Gamma \sigma\nu\cdot\mathbf{v}\, da \qquad \forall\, \mathbf{v} \in H^1(\Omega)^d.$$

(2.4) $$(\mathbf{D}, \nabla\phi)_H + (div\ \mathbf{D}, \phi)_{L^2(\Omega)} = \int_\Gamma \mathbf{D}\cdot\nu\,\phi\, da \qquad \forall\,\phi \in H^1(\Omega).$$

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \le p \le +\infty$ and $k \ge 1$. We denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$|\mathbf{f}|_{C(0,T;X)} = \max_{t\in[0,T]} |\mathbf{f}(t)|_X\,,$$
$$|\mathbf{f}|_{C^1(0,T;X)} = \max_{t\in[0,T]} |\mathbf{f}(t)|_X + \max_{t\in[0,T]} \left|\dot{\mathbf{f}}(t)\right|_X,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable and, for a real number r, we use r_+ to represent its positive part, that is $r_+ = \max\{0, r\}$. For the convenience of the reader, we recall the following version of the classical theorem of Cauchy-Lipschitz (see, e.g., [20, p. 48]).

Theorem 2.1. *Assume that $(X, |\cdot|_X)$ is a real Banach space and $T > 0$. Let $F(t,\cdot) : X \to X$ be an operator defined a.e. on $(0,T)$ satisfying the following conditions:*

(1) *There exists a constant $L_F > 0$ such that*

$$|F(t,x) - F(t,y)|_X \le L_F\,|x-y|_X \quad \forall\, x,y \in X, \text{ a.e. } t \in (0,T)\,.$$

(2) *There exists $p \ge 1$ such that $t \longmapsto F(t,x) \in L^p(0,T;X) \quad \forall\, x \in X$.*

Then for any $x_0 \in X$, there exists a unique function $x \in W^{1,p}(0,T;X)$ such that

$$\dot{x}(t) = F(t, x(t)) \qquad \text{a.e. } t \in (0,T)\,,$$
$$x(0) = x_0.$$

Theorem 2.1 will be used in Section 4 to prove the unique solvability of the intermediate problem involving the bonding field. Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1\times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(\cdot,\cdot)_{X_1\times X_2}$.

3. MECHANICAL AND VARIATIONAL FORMULATIONS

We describe the model for the process and present its variational formulation. The physical setting is the following. An electro-viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with outer Lipschitz surface Γ. The body is submitted to the action of body forces of density $\mathbf{f}_0$ and volume electric charges of density q_0. It is also submitted to mechanical and electric constraints on the boundary. We consider partitioning Γ into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, on one hand, and into two measurable parts Γ_a and Γ_b, on the other hand, such that $meas\ (\Gamma_1) > 0$, $meas\ (\Gamma_a) > 0$ and $\Gamma_3 \subset \Gamma_b$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$, so the displacement field vanishes there. A surface traction of density $\mathbf{f}_2$ acts on $\Gamma_2 \times (0, T)$ and a body force of density $\mathbf{f}_0$ acts in $\Omega \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0, T)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (0, T)$. The body is in adhesive contact with an obstacle, or foundation, over the contact surface Γ_3. Moreover, the process is dynamic, and thus the inertial terms are included in the equation of motion. We denote by $\mathbf{u}$ the displacement field, by σ the stress tensor field and by $\varepsilon(\mathbf{u})$ the linearized strain tensor. We use an electro-viscoelastic constitutive law with long-term memory given by

$$\sigma = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{G}(\varepsilon(\mathbf{u}), \beta) + \int_0^t M(t-s)\varepsilon(\mathbf{u}(s))\, ds - \mathcal{E}^* E(\varphi),$$
$$\mathbf{D} = \mathcal{E}\varepsilon(\mathbf{u}) + BE(\varphi),$$

where $\mathcal{A}$ is a given nonlinear function, M is the relaxation tensor, and $\mathcal{G}$ represents the elasticity operator where β is an internal variable describing the damage of the material caused by elastic deformations. $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represents the third order piezoelectric tensor, $\mathcal{E}^*$ is its transposition and B denotes the electric permittivity tensor. The inclusion used for the evolution of the damage field is

$$\dot{\beta} - k \triangle \beta + \partial\varphi_K(\beta) \ni S(\varepsilon(\mathbf{u}), \beta),$$

where K denotes the set of admissible damage functions defined by

$$K = \{\xi \in H^1(\Omega)\ /\ 0 \le \xi \le 1 \text{ a.e. in } \Omega\},$$

k is a positive coefficient, $\partial\varphi_K$ denotes the subdifferential of the indicator function φ_K and S is a given constitutive function which describes the sources of the damage in the system. When $\beta = 1$ the material is undamaged, when $\beta = 0$ the material is completely damaged, and for $0 < \beta < 1$ there is partial damage. General models of mechanical damage, which were derived from thermodynamical considerations and the principle of virtual work, can be found in [8] and [9] and references therein. The models describe the evolution of the material damage which results from the excess tension or compression in the body as a result of applied forces and tractions. Mathematical analysis of one-dimensional damage models can be found in [10].

To simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\mathbf{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$. Then, the classical formulation of the mechanical problem of electro-viscoelastic material, frictionless, adhesive contact may be stated as follows.

Problem P. *Find a displacement field* $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$*, a stress field* $\sigma : \Omega \times [0, T] \to S^d$*, an electric potential field* $\varphi : \Omega \times [0, T] \to \mathbb{R}$*, an electric displacement field* $\mathbf{D} : \Omega \times [0, T] \to \mathbb{R}^d$*, a damage field* $\beta : \Omega \times [0, T] \to \mathbb{R}$ *and a bonding field* $\alpha : \Gamma_3 \times [0, T] \to \mathbb{R}$ *such that*

$$\sigma = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{G}(\varepsilon(\mathbf{u}), \beta) + \int_0^t M(t-s)\varepsilon(\mathbf{u}(s))\, ds + \mathcal{E}^*\nabla\varphi \text{ in } \Omega \times (0, T)\,, \tag{3.1}$$

$$\mathbf{D} = \mathcal{E}\varepsilon(\mathbf{u}) - B\nabla\varphi \quad \text{in } \Omega \times (0, T)\,, \tag{3.2}$$

$$\dot{\beta} - k \triangle \beta + \partial\varphi_K(\beta) \ni S(\varepsilon(\mathbf{u}), \beta) \quad \text{in } \Omega \times (0,T)\,, \tag{3.3}$$

$$\rho \ddot{\mathbf{u}} = Div\, \sigma + \mathbf{f}_0 \qquad \text{in } \Omega \times (0,T)\,, \tag{3.4}$$

$$div\, \mathbf{D} = q_0 \quad \text{in } \Omega \times (0,T)\,, \tag{3.5}$$

$$\mathbf{u} = 0 \qquad \text{on } \Gamma_1 \times (0,T)\,, \tag{3.6}$$

$$\sigma\nu = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0,T)\,, \tag{3.7}$$

$$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu \alpha^2 R_\nu(u_\nu) \qquad \text{on } \Gamma_3 \times (0,T)\,, \tag{3.8}$$

$$-\sigma_\tau = p_\tau(\alpha)\mathbf{R}_\tau(\mathbf{u}_\tau) \quad \text{on } \Gamma_3 \times (0,T)\,, \tag{3.9}$$

$$\dot{\alpha} = -\left(\alpha\left(\gamma_\nu \left(R_\nu(u_\nu)\right)^2 + \gamma_\tau \left|\mathbf{R}_\tau(\mathbf{u}_\tau)\right|^2\right) - \varepsilon_a\right)_+ \text{ on } \Gamma_3 \times (0,T)\,, \tag{3.10}$$

$$\frac{\partial\beta}{\partial\nu} = 0 \text{ on } \Gamma_3 \times (0,T)\,, \tag{3.11}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T)\,, \tag{3.12}$$

$$\mathbf{D}\cdot\nu = q_2 \quad \text{on } \Gamma_b \times (0,T)\,, \tag{3.13}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \dot{\mathbf{u}}(0) = \mathbf{v}_0, \beta(0) = \beta_0 \quad \text{in } \Omega, \tag{3.14}$$

$$\alpha(0) = \alpha_0 \text{ on } \Gamma_3. \tag{3.15}$$

First, (3.1) and (3.2) represent the electro-viscoelastic constitutive law with long term-memory and damage, the evolution of the damage field is governed by the inclusion of parabolic type given by the relation (3.3), where S is the mechanical source of the damage, and $\partial\varphi_K$ is the subdifferential of the indicator function of the admissible damage functions set K. Equations (3.4) and (3.5) represent the equation of motion for the stress field and the equilibrium equation for the electric-displacement field while (3.6) and (3.7) are the displacement and traction boundary condition, respectively. Condition (3.8) represents the normal compliance condition with adhesion where γ_ν is a given adhesion coefficient and p_ν is a given positive function which will be described below. In this condition the interpenetrability between the body and the foundation is allowed, that is u_ν can be positive on Γ_3. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu \alpha^2 R_\nu(u_\nu)$, the adhesive traction is tensile and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but only as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu L$. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \leq s \leq 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of the operator R_ν, together with the operator $\mathbf{R}_\tau$ defined below, is motivated by mathematical arguments but it is not restrictive from the physical point of view, since no restriction on the size of the parameter L is made in what follows. Condition (3.9) represents the adhesive contact condition on the tangential plane, in which p_τ is a given function and $\mathbf{R}_\tau$ is the truncation operator given by

$$\mathbf{R}_\tau(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{if} \qquad |\mathbf{v}| \leq L, \\ L\frac{\mathbf{v}}{|\mathbf{v}|} & \text{if} \qquad |\mathbf{v}| > L. \end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but only as long as it does not exceed the bond length L. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted.

Next, the equation (3.10) is an ordinary differential equation which describes the evolution of the bonding field and it has already been used in [3], see also [19, 20] for more details. Here, besides γ_ν, two new adhesion coefficients are involved, γ_τ and ε_a. Notice that in this model, once debonding occurs bonding cannot be re-established since, from (3.10), $\dot{\alpha} \leq 0$. The relation (3.11) represents a homogeneous Neumann boundary condition where $\frac{\partial\beta}{\partial\nu}$ represents the normal derivative of β. (3.12) and (3.13) represent the electric boundary conditions. (3.14) represents the initial displacement field, the initial velocity and the initial damage field. Finally (3.15) represents the initial condition in which α_0 is the given initial bonding field. To obtain the variational formulation of the problems (3.1) – (3.15), we introduce for the bonding field the set

$$Z = \left\{\theta \in L^\infty(0,T;L^2(\Gamma_3)) \;/\; 0 \leq \theta(t) \leq 1 \;\forall t \in [0,T], \text{a.e. on } \Gamma_3\right\},$$

and for the displacement field we need the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \left\{\mathbf{v} \in H^1(\Omega)^d \;/\; \mathbf{v} = 0 \text{ on } \Gamma_1\right\}.$$

Since $meas\,(\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_k > 0$, that depends only on Ω and Γ_1, such that

$$|\varepsilon(\mathbf{v})|_{\mathcal{H}} \geq C_k\,|\mathbf{v}|_{H^1(\Omega)^d} \qquad \forall\mathbf{v} \in V.$$

A proof of Korn's inequality may be found in [15, p. 79]. On the space V we consider the inner product and the associated norm given by

$$(\mathbf{u},\mathbf{v})_V = (\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathcal{H}}, \qquad |\mathbf{v}|_V = |\varepsilon(\mathbf{v})|_{\mathcal{H}} \quad \forall\mathbf{u},\mathbf{v} \in V. \tag{3.16}$$

It follows that $|\cdot|_{H^1(\Omega)^d}$ and $|\cdot|_V$ are equivalent norms on V and therefore $(V, |\cdot|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace Theorem and (3.16), there exists a constant $C_0 > 0$, depending only on Ω, Γ_1 and Γ_3 such that

$$|\mathbf{v}|_{L^2(\Gamma_3)^d} \leq C_0\,|\mathbf{v}|_V \qquad \forall\mathbf{v} \in V. \tag{3.17}$$

We also introduce the spaces

$$W = \left\{\phi \in H^1(\Omega) \,/\, \phi = 0 \text{ on } \Gamma_a\right\},$$
$$\mathcal{W} = \left\{\mathbf{D} = (D_i) \,/\, D_i \in L^2(\Omega),\ div\,\mathbf{D} \in L^2(\Omega)\right\},$$

where $div\,\mathbf{D} = (D_{i,i})$. The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products given by

$$(\varphi, \phi)_W = \int_\Omega \nabla\varphi \cdot \nabla\phi\, dx,$$
$$(\mathbf{D}, \mathbf{E})_{\mathcal{W}} = \int_\Omega \mathbf{D} \cdot \mathbf{E}\, dx + \int_\Omega div\,\mathbf{D} \cdot div\,\mathbf{E}\, dx.$$

The associated norms will be denoted by $|\cdot|_W$ and $|\cdot|_{\mathcal{W}}$, respectively. Notice also that, since $meas(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

(3.18) $$|\nabla\phi|_H \geq C_F\, |\phi|_{H^1(\Omega)} \qquad \forall \phi \in W,$$

where $C_F > 0$ is a constant which depends only on Ω and Γ_a. In the study of the mechanical problems (3.1) – (3.15), we assume that the viscosity function $\mathcal{A} : \Omega \times S^d \to S^d$ satisfies

(3.19) $$\begin{cases} (a) & \text{There exists constants } C_1^{\mathcal{A}}, C_2^{\mathcal{A}} > 0 \text{ such that} \\ & |\mathcal{A}(\mathbf{x}, \varepsilon)| \leq C_1^{\mathcal{A}}\, |\varepsilon| + C_2^{\mathcal{A}} \quad \forall \varepsilon \in S^d,\ \text{a.e. } \mathbf{x} \in \Omega. \\ (b) & \text{There exists a constant } m_{\mathcal{A}} > 0 \text{ Such that} \\ & (\mathcal{A}(\mathbf{x}, \varepsilon_1) - \mathcal{A}(\mathbf{x}, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}}\, |\varepsilon_1 - \varepsilon_2|^2 \\ & \forall \varepsilon_1, \varepsilon_2 \in S^d,\ \text{a.e. } \mathbf{x} \in \Omega. \\ (c) & \text{The mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x}, \varepsilon) \text{ is Lebesgue measurable on } \Omega \text{ for any } \varepsilon \in S^d. \\ (d) & \text{The mapping } \varepsilon \to \mathcal{A}(\mathbf{x}, \varepsilon) \text{ is continuous on } S^d,\ \text{a.e. } \mathbf{x} \in \Omega. \end{cases}$$

The elasticity Operator $\mathcal{G} : \Omega \times S^d \times \mathbb{R} \to S^d$ satisfies

(3.20) $$\begin{cases} (a) & \text{There exists a constant } L_{\mathcal{G}} > 0 \text{ Such that} \\ & |\mathcal{G}(\mathbf{x}, \varepsilon_1, \alpha_1) - \mathcal{G}(\mathbf{x}, \varepsilon_2, \alpha_2)| \leq L_{\mathcal{G}}(|\varepsilon_1 - \varepsilon_2| + |\alpha_1 - \alpha_2|) \\ & \forall \varepsilon_1, \varepsilon_2 \in S^d,\ \forall \alpha_1, \alpha_2 \in \mathbb{R} \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) & \text{The mapping } \mathbf{x} \to \mathcal{G}(\mathbf{x}, \varepsilon, \alpha) \text{ is Lebesgue measurable on } \Omega \\ & \text{for any } \varepsilon \in S^d \text{ and } \alpha \in \mathbb{R}. \\ (c) & \text{The mapping } \mathbf{x} \to \mathcal{G}(\mathbf{x}, \mathbf{0}, 0) \text{ belongs to } \mathcal{H}. \end{cases}$$

The damage source function $S : \Omega \times S^d \times \mathbb{R} \to \mathbb{R}$ satisfies

(3.21) $$\begin{cases} (a) & \text{There exists a constant } L_S > 0 \text{ such that} \\ & |S(\mathbf{x}, \varepsilon_1, \alpha_1) - S(\mathbf{x}, \varepsilon_2, \alpha_2)| \leq L_S(|\varepsilon_1 - \varepsilon_2| + |\alpha_1 - \alpha_2|) \\ & \forall \varepsilon_1, \varepsilon_2 \in S^d,\ \forall \alpha_1, \alpha_2 \in \mathbb{R} \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) & \text{For any } \varepsilon \in S^d \text{ and } \alpha \in \mathbb{R},\ \mathbf{x} \to S(\mathbf{x}, \varepsilon, \alpha) \text{ is Lebesgue measurable on } \Omega. \\ (c) & \text{The mapping } \mathbf{x} \to S(\mathbf{x}, \mathbf{0}, 0) \text{ belongs to } L^2(\Omega). \end{cases}$$

The electric permittivity operator $B = (b_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$(3.22)\quad \begin{cases} (a) & B(\mathbf{x}, \mathbf{E}) = (b_{ij}(\mathbf{x})E_j)\ \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) & b_{ij} = b_{ji},\ b_{ij} \in L^\infty(\Omega), \quad 1 \le i, j \le d. \\ (c) & \text{There exists a constant } m_B > 0 \quad \text{such that} \\ & B\mathbf{E}.\mathbf{E} \ge m_B |\mathbf{E}|^2 \quad \forall\, \mathbf{E} = (E_i) \in \mathbb{R}^d, \quad \text{a.e. in } \Omega. \end{cases}$$

The piezoelectric operator $\mathcal{E} : \Omega \times S^d \to \mathbb{R}^d$ satisfies

$$(3.23)\quad \begin{cases} (a) & \mathcal{E}(\mathbf{x}, \tau) = (e_{ijk}(\mathbf{x})\tau_{jk}) \quad \forall \tau = (\tau_{ij}) \in S^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) & e_{ijk} = e_{ikj} \in L^\infty(\Omega), \quad 1 \le i, j, k \le d. \end{cases}$$

The normal compliance function $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$(3.24)\quad \begin{cases} (a) & \text{There exists a constant } L_\nu > 0 \text{ such that} \\ & |p_\nu(\mathbf{x}, r_1) - p_\nu(\mathbf{x}, r_2)| \le L_\nu |r_1 - r_2|\ \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ (b) & \text{The mapping } \mathbf{x} \to p_\nu(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \text{ for any } r \in \mathbb{R}. \\ (c) & p_\nu(\mathbf{x}, r) = 0 \text{ for all } r \le 0, \text{ a.e. } \mathbf{x} \in \Gamma_3. \end{cases}$$

The tangential contact function $p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$(3.25)\quad \begin{cases} (a) & \text{There exists a constant } L_\tau > 0 \text{ such that} \\ & |p_\tau(\mathbf{x}, d_1) - p_\tau(\mathbf{x}, d_2)| \le L_\tau |d_1 - d_2|\ \forall d_1, d_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ (b) & \text{There exists } M_\tau > 0 \text{ such that } |p_\tau(\mathbf{x}, d)| \le M_\tau\ \forall d \in \mathbb{R}, \text{a.e. } \mathbf{x} \in \Gamma_3. \\ (c) & \text{The mapping } \mathbf{x} \to p_\tau(\mathbf{x}, d) \text{ is measurable on } \Gamma_3, \text{ for any } d \in \mathbb{R}. \\ (d) & \text{The mapping } \mathbf{x} \to p_\tau(\mathbf{x}, 0) \in L^2(\Gamma_3). \end{cases}$$

The relaxation tensor M satisfies

$$(3.26)\quad M \in C(0, T; \mathcal{H}).$$

We suppose that the mass density satisfies

$$(3.27)\quad \rho \in L^\infty(\Omega), \text{ there exists } \rho^* > 0 \text{ such that } \rho(\mathbf{x}) \ge \rho^* \text{ a.e. } \mathbf{x} \in \Omega.$$

We also suppose that the body forces and surface tractions have the regularity

$$(3.28)\quad \mathbf{f}_0 \in L^2(0, T; H), \quad \mathbf{f}_2 \in L^2(0, T; L^2(\Gamma_2)^d),$$

$$(3.29)\quad q_0 \in C(0, T; L^2(\Omega)), \quad q_2 \in C(0, T; L^2(\Gamma_b)).$$

$$(3.30)\quad q_2(t) = 0 \text{ on } \Gamma_3 \quad \forall t \in [0, T].$$

Note that we need to impose assumption (3.30) for physical reasons. Indeed the foundation is assumed to be insulator and therefore the electric charges (which are prescribed on $\Gamma_b \supset \Gamma_3$) have to vanish on the potential contact surface. The adhesion coefficients satisfy

$$(3.31)\quad \gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \quad \varepsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu, \gamma_\tau, \varepsilon_a \ge 0 \quad \text{a.e. on } \Gamma_3.$$

The initial displacement field satisfies

$$(3.32)\quad \mathbf{u}_0 \in V, \quad \mathbf{v}_0 \in H,$$

the initial bonding field satisfies

$$\alpha_0 \in L^2(\Gamma_3), \quad 0 \leq \alpha_0 \leq 1 \text{ a.e. on } \Gamma_3, \tag{3.33}$$

and the initial damage field satisfies

$$\beta_0 \in K. \tag{3.34}$$

We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(\xi, \varphi) = k \int_\Omega \nabla\xi \cdot \nabla\varphi \, dx. \tag{3.35}$$

We will use a modified inner product on $H = L^2(\Omega)^d$, given by

$$((\mathbf{u}, \mathbf{v}))_H = (\rho\mathbf{u}, \mathbf{v})_H \quad \forall \mathbf{u}, \mathbf{v} \in H,$$

that is, it is weighted with ρ, and we let $\|\cdot\|_H$ be the associated norm, i.e.,

$$\|\mathbf{v}\|_H = (\rho\mathbf{v}, \mathbf{v})_H^{\frac{1}{2}} \quad \forall \mathbf{v} \in H.$$

It follows from assumption (3.27) that $\|\cdot\|_H$ and $|\cdot|_H$ are equivalent norms on H, and the inclusion mapping of $(V, |\cdot|_V)$ into $(H, \|\cdot\|_H)$ is continuous and dense. We denote by V' the dual of V. Identifying H with its own dual, we can write the Gelfand triple

$$V \subset H \subset V'.$$

Using the notation $(\cdot, \cdot)_{V' \times V}$ to represent the duality pairing between V' and V, we have

$$(\mathbf{u}, \mathbf{v})_{V' \times V} = ((\mathbf{u}, \mathbf{v}))_H \quad \forall \mathbf{u} \in H, \forall \mathbf{v} \in V.$$

Finally, we denote by $|\cdot|_{V'}$ the norm on V'. Assumption (3.28) allows us, for a.e. $t \in (0, T)$, to define $\mathbf{f}(t) \in V'$ by

$$(\mathbf{f}(t), \mathbf{v})_{V' \times V} = \int_\Omega \mathbf{f}_0(t) \cdot \mathbf{v} \, dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v} \, da \quad \forall \mathbf{v} \in V. \tag{3.36}$$

We denote by $q : [0, T] \to W$ the function defined by

$$(q(t), \phi)_W = \int_\Omega q_0(t) \cdot \phi \, dx - \int_{\Gamma_b} q_2(t) \cdot \phi \, da \quad \forall \phi \in W, \ t \in [0, T]. \tag{3.37}$$

Next, we denote by $j : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ the adhesion functional defined by

$$j(\alpha, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\nu(u_\nu) v_\nu \, da + \int_{\Gamma_3} (-\gamma_\nu \alpha^2 R_\nu(u_\nu) \, v_\nu + p_\tau(\alpha)\mathbf{R}_\tau(\mathbf{u}_\tau) \cdot \mathbf{v}_\tau) \, da. \tag{3.38}$$

Keeping in mind (3.24) – (3.25), we observe that the integrals (3.38) are well defined and we note that conditions (3.28) – (3.29) imply

$$\mathbf{f} \in L^2(0, T; V'), \quad q \in C(0, T; W). \tag{3.39}$$

Using standard arguments we obtain the variational formulation of the mechanical problem (3.1) – (3.15).

Problem PV. *Find a displacement field* $\mathbf{u} : [0,T] \to V$*, an electric potential field* $\varphi : [0,T] \to W$*, a damage field* $\beta : [0,T] \to H^1(\Omega)$ *and a bonding field* $\alpha : [0,T] \to L^\infty(\Gamma_3)$ *such that*

$$(3.40)\quad \begin{aligned}&(\ddot{\mathbf{u}}, \mathbf{v})_{V'\times V} + (\mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)), \varepsilon(\mathbf{v}))_{\mathcal{H}}\\ &\quad + (\mathcal{G}(\varepsilon(\mathbf{u}(t)), \beta(t)), \varepsilon(\mathbf{v}))_{\mathcal{H}} + \left(\int_0^t M(t-s)\varepsilon(\mathbf{u}(s))ds, \varepsilon(\mathbf{v})\right)_{\mathcal{H}}\\ &\quad + (\mathcal{E}^*\nabla\varphi(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\alpha(t), \mathbf{u}(t), \mathbf{v})\\ &\quad = (\mathbf{f}(t), \mathbf{v})_{V'\times V} \quad \forall \mathbf{v} \in V, t \in (0,T)\,,\end{aligned}$$

$$(3.41)\quad \begin{aligned}&\beta(t) \in K \text{ for all } t \in [0,T]\,, \quad \left(\dot{\beta}(t), \xi - \beta(t)\right)_{L^2(\Omega)} + a(\beta(t), \xi - \beta(t))\\ &\qquad \geq (S(\varepsilon(\mathbf{u}(t)), \beta(t)), \xi - \beta(t))_{L^2(\Omega)} \quad \forall \xi \in K,\end{aligned}$$

$$(3.42)\quad (B\nabla\varphi(t), \nabla\phi)_H - (\mathcal{E}\varepsilon(\mathbf{u}(t)), \nabla\phi)_H = (q(t), \phi)_W \quad \forall \phi \in W, t \in (0,T)\,,$$

$$(3.43)\quad \dot{\alpha}(t) = -\left(\alpha(t)\left[\gamma_\nu(R_\nu(u_\nu(t)))^2 + \gamma_\tau\left|\mathbf{R}_\tau(\mathbf{u}_\tau(t))\right|^2\right] - \varepsilon_a\right)_+ \quad \text{a.e. } t \in (0,T)\,,$$

$$(3.44)\quad \mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad \beta(0) = \beta_0, \quad \alpha(0) = \alpha_0.$$

We notice that the variational problem PV is formulated in terms of a displacement field, an electrical potential field, a damage field and a bonding field. The existence of the unique solution of problem PV is stated and proved in the next section. To this end, we consider the following remark which is used in different places of the paper.

Remark 1. We note that, in the problem P and in the problem PV we do not need to impose explicitly the restriction $0 \leq \alpha \leq 1$. Indeed, equations (3.43) guarantee that $\alpha(\mathbf{x}, t) \leq \alpha_0(\mathbf{x})$ and, therefore, assumption (3.33) shows that $\alpha(\mathbf{x}, t) \leq 1$ for $t \geq 0$, a.e. $\mathbf{x} \in \Gamma_3$. On the other hand, if $\alpha(\mathbf{x}, t_0) = 0$ at time t_0, then it follows from (3.43) that $\dot{\alpha}(\mathbf{x}, t) = 0$ for all $t \geq t_0$ and therefore, $\alpha(\mathbf{x}, t) = 0$ for all $t \geq t_0$, a.e. $\mathbf{x} \in \Gamma_3$. We conclude that $0 \leq \alpha(\mathbf{x}, t) \leq 1$ for all $t \in [0,T]$, a.e. $\mathbf{x} \in \Gamma_3$.

4. AN EXISTENCE AND UNIQUENESS RESULT

Now, we propose our existence and uniqueness result.

Theorem 4.1. *Assume that (3.19) – (3.34) hold. Then there exists a unique solution* $\{\mathbf{u}, \varphi, \beta, \alpha\}$ *to problem PV. Moreover, the solution satisfies*

$$(4.1)\quad \mathbf{u} \in H^1(0,T;V) \cap C^1(0,T;H), \quad \ddot{\mathbf{u}} \in L^2(0,T;V'),$$

$$(4.2)\quad \varphi \in C(0,T;W),$$

$$(4.3)\quad \beta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$

$$(4.4)\quad \alpha \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap Z.$$

The functions $\mathbf{u}$,$\varphi, \sigma, \mathbf{D}$,$\beta$ and α which satisfy (3.1) – (3.2) and (3.40) – (3.44) are called weak solutions of the contact problem P. We conclude that, under the assumptions (3.19) – (3.34), the mechanical problem (3.1) – (3.15) has a unique weak solution satisfying (4.1) – (4.4). The regularity of the weak solution is given by (4.1) – (4.4) and, in term of stresses,

$$\sigma \in L^2(0,T;\mathcal{H}), \quad Div\, \sigma \in L^2(0,T;V'), \tag{4.5}$$

$$\mathbf{D} \in C(0,T;\mathcal{W}). \tag{4.6}$$

Indeed, it follows from (3.40) and (3.42) that $\rho\ddot{\mathbf{u}} = Div\ \sigma(t) + \mathbf{f}_0(t)$, $div\ \mathbf{D} = q_0(t)$ for all $t \in [0,T]$. Therefore the regularity (4.1) and (4.2) of $\mathbf{u}$ and φ, combined with (3.19) – (3.29) implies (4.5) and (4.6).

The proof of Theorem 4.1 is carried out in several steps that we prove in what follows. Everywhere in this section we suppose that the assumptions of Theorem 4.1 hold, and we assume that C is a generic positive constant which depends on $\Omega, \Gamma_1, \Gamma_3, p_\nu, p_\tau, \gamma_\nu, \gamma_\tau$ and L and may change from place to place. Let $\eta \in L^2(0,T;V')$ be given, in the first step we consider the following variational problem.

Problem $\mathbf{PV}_\eta$. *Find a displacement field $\mathbf{u}_\eta : [0,T] \to V$ such that*

$$\begin{aligned}(\ddot{\mathbf{u}}_\eta(t), \mathbf{v})_{V'\times V} + (\mathcal{A}\varepsilon(\dot{\mathbf{u}}_\eta(t)), \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\eta(t), \mathbf{v})_{V'\times V} \\ = (\mathbf{f}(t), \mathbf{v})_{V'\times V} \quad \forall \mathbf{v} \in V \text{ a.e. } t \in (0,T)\,,\end{aligned} \tag{4.7}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0, \quad {}_\eta(0) = \mathbf{v}_0. \tag{4.8}$$

To solve problem PV_η, we apply an abstract existence and uniqueness result which we recall now, for the convenience of the reader. Let V and H denote real Hilbert spaces such that V is dense in H and the inclusion map is continuous, H is identified with its dual and with a subspace of the dual V' of V, i.e., $V \subset H \subset V'$, and we say that the inclusions above define a Gelfand triple. The notations $|\cdot|_V$, $|\cdot|_{V'}$ and $(\cdot,\cdot)_{V'\times V}$ represent the norms on V and on V' and the duality pairing between them, respectively. The following abstract result may be found in [20, p. 48].

Theorem 4.2. *Let V, H be as above, and let $A : V \to V'$ be a hemicontinuous and monotone operator which satisfies*

$$(A\mathbf{v}, \mathbf{v})_{V'\times V} \geq \omega\, |\mathbf{v}|_V^2 + \lambda \quad \forall \mathbf{v} \in V, \tag{4.9}$$

$$|A\mathbf{v}|_{V'} \leq C(|\mathbf{v}|_V + 1) \quad \forall \mathbf{v} \in V, \tag{4.10}$$

for some constants $\omega > 0$, $C > 0$ and $\lambda \in \mathbb{R}$. Then, given $\mathbf{u}_0 \in H$ and $\mathbf{f} \in L^2(0,T;V')$, there exists a unique function $\mathbf{u}$ which satisfies

$$\mathbf{u} \in L^2(0,T;V') \cap C(0,T;H), \quad \dot{\mathbf{u}} \in L^2(0,T;V'),$$

$$\dot{\mathbf{u}}(t) + A\mathbf{u}(t) = \mathbf{f}(t) \quad a.e.\ t \in (0,T)\,,$$

$$\mathbf{u}(0) = \mathbf{u}_0.$$

We apply it to problem PV_η.

Lemma 4.3. *There exists a unique solution to problem PV_η and it has its regularity expressed in (4.1).*

Proof. We define the operator $A : V \to V'$ by

$$(A\mathbf{u}, \mathbf{v})_{V' \times V} = (\mathcal{A}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in V. \tag{4.11}$$

Using (4.11), (3.19) and (3.16) it follows that

$$|A\mathbf{u} - A\mathbf{v}|_{V'} \leq |\mathcal{A}\varepsilon(\mathbf{u}) - \mathcal{A}\varepsilon(\mathbf{v})|_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in V,$$

and keeping in mind the Krasnoselski Theorem (see for instance [11, p. 60]), we deduce that $A : V \to V'$ is a continuous operator. Now, by (4.11), (3.19) and (3.16) we find

$$(A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v})_{V' \times V} \geq m_{\mathcal{A}} |\mathbf{u} - \mathbf{v}|_V^2 \quad \forall \mathbf{u}, \mathbf{v} \in V, \tag{4.12}$$

i.e., that $A : V \to V'$ is a monotone operator. Choosing $\mathbf{v} = \mathbf{0}_V$ in (4.12) we obtain

$$\begin{aligned}(A\mathbf{u}, \mathbf{u})_{V' \times V} &\geq m_{\mathcal{A}} |\mathbf{u}|_V^2 - |A\mathbf{0}_V|_{V'} |\mathbf{u}|_V \\ &\geq \frac{1}{2} m_{\mathcal{A}} |\mathbf{u}|_V^2 - \frac{1}{2m_{\mathcal{A}}} |A\mathbf{0}_V|_{V'}^2 \quad \forall \mathbf{u} \in V,\end{aligned}$$

which implies that A satisfies condition (4.9) with $\omega = \frac{m_{\mathcal{A}}}{2}$ and $\lambda = \frac{-|A\mathbf{0}_V|_{V'}^2}{2m_{\mathcal{A}}}$. Moreover, by (4.11) and (3.19) we find

$$|A\mathbf{u}|_{V'} \leq |\mathcal{A}\varepsilon(\mathbf{u})|_{\mathcal{H}} \leq C_1^{\mathcal{A}} |\mathbf{u}|_V + C_2^{\mathcal{A}} \quad \forall \mathbf{u} \in V.$$

This inequality and (3.16) imply that A satisfies condition (4.10). Finally, we recall that by (3.28) and (3.32) we have $\mathbf{f} - \eta \in L^2(0, T; V')$ and $\mathbf{v}_0 \in H$.

It follows now from Theorem 4.2 that there exists a unique function $\mathbf{v}_\eta$ which satisfies

$$\mathbf{v}_\eta \in L^2(0, T; V) \cap C(0, T; H), \quad \dot{\mathbf{v}}_\eta \in L^2(0, T; V'), \tag{4.13}$$

$$\dot{\mathbf{v}}_\eta(t) + A\mathbf{v}_\eta(t) + \eta(t) = \mathbf{f}(t) \quad \text{a.e. } t \in (0, T), \tag{4.14}$$

$$\mathbf{v}_\eta(0) = \mathbf{v}_0. \tag{4.15}$$

Let $\mathbf{u}_\eta : [0, T] \to V$ be the function defined by

$$\mathbf{u}_\eta(t) = \int_0^t \mathbf{v}_\eta(s)\, ds + \mathbf{u}_0 \quad \forall t \in [0, T]. \tag{4.16}$$

It follows from (4.11) and (4.13) – (4.16) that $\mathbf{u}_\eta$ is a unique solution of the variational problem PV_η and it satisfies the regularity expressed in (4.1). □

In the second step, let $\eta \in L^2(0, T; V')$, we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.3 and we consider the following variational problem.

Problem $\mathbf{QV}_\eta$. *Find the electric potential field $\varphi_\eta : [0, T] \to W$ such that*

$$(B\nabla\varphi_\eta(t), \nabla\phi)_H - (\mathcal{E}\varepsilon(\mathbf{u}_\eta(t)), \nabla\phi)_H = (q(t), \phi)_W \quad \forall \phi \in W,\ t \in (0, T). \tag{4.17}$$

We have the following result.

Lemma 4.4. *QV_η has a unique solution φ_η which satisfies the regularity (4.2).*

Proof. We define a bilinear form: $b(\cdot,\cdot) : W \times W \to \mathbb{R}$ such that

$$b(\varphi,\phi) = (B\nabla\varphi, \nabla\phi)_H \quad \forall \varphi,\phi \in W. \tag{4.18}$$

We use (4.18), (3.18) and (3.22) to show that the bilinear form b is continuous, symmetric and coercive on W, moreover using the Riesz Representation Theorem we may define an element $q_\eta : [0,T] \to W$ such that

$$(q_\eta(t),\phi)_W = (q(t),\phi)_W + (\mathcal{E}\varepsilon(\mathbf{u}_\eta(t)), \nabla\phi)_H \quad \forall \phi \in W,\ t \in (0,T)\,.$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\varphi_\eta(t) \in W$ such that

$$b(\varphi_\eta(t),\phi) = (q_\eta(t),\phi)_W \quad \forall \phi \in W. \tag{4.19}$$

We conclude that $\varphi_\eta(t)$ is a solution of QV_η. Let $t_1, t_2 \in [0,T]$, it follows from (4.17) that

$$|\varphi_\eta(t_1) - \varphi_\eta(t_2)|_W \le C\left(|\mathbf{u}_\eta(t_1) - \mathbf{u}_\eta(t_2)|_V + |q(t_1) - q(t_2)|_W\right),$$

and the previous inequality, the regularity of $\mathbf{u}_\eta$ and q imply that $\varphi_\eta \in C(0,T;W)$. □

In the third step, we let $\theta \in L^2(0,T;L^2(\Omega))$ be given and consider the following variational problem for the damage field.

Problem $\mathbf{PV}_\theta$. *Find a damage field $\beta_\theta : [0,T] \to H^1(\Omega)$ such that*

$$\beta_\theta(t) \in K, \quad (\dot\beta_\theta(t), \xi - \beta_\theta(t))_{L^2(\Omega)} + a(\beta_\theta(t), \xi - \beta_\theta(t)) \ge (\theta(t), \xi - \beta_\theta(t))_{L^2(\Omega)}\ \forall \xi \in K \text{ a.e. } t \in (0,T)\,, \tag{4.20}$$

$$\beta_\theta(0) = \beta_0. \tag{4.21}$$

To solve PV_θ, we recall the following standard result for parabolic variational inequalities (see, e.g., [20, p. 47]).

Theorem 4.5. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty closed, and convex set of V. Assume that $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\zeta > 0$ and c_0,*

$$a(v,v) + c_0\,|v|_H^2 \ge \zeta\,|v|_V^2 \quad \forall v \in V.$$

Then, for every $u_0 \in K$ and $f \in L^2(0,T;H)$, there exists a unique function $u \in H^1(0,T;H) \cap L^2(0,T;V)$ such that $u(0) = u_0$, $u(t) \in K$ for all $t \in [0,T]$, and for almost all $t \in (0,T)$,

$$(\dot u(t), v - u(t))_{V'\times V} + a(u(t), v - u(t)) \ge (f(t), v - u(t))_H \quad \forall v \in K.$$

We apply this theorem to problem PV_θ.

Lemma 4.6. *Problem PV_θ has a unique solution β_θ such that*

$$\beta_\theta \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{4.22}$$

Proof. The inclusion mapping of $\left(H^1(\Omega), |\cdot|_{H^1(\Omega)}\right)$ into $\left(L^2(\Omega), |\cdot|_{L^2(\Omega)}\right)$ is continuous and its range is dense. We denote by $(H^1(\Omega))'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $L^2(\Omega)$ with itself, we can write the Gelfand triple

$$H^1(\Omega) \subset L^2(\Omega) \subset (H^1(\Omega))'.$$

We use the notation $(\cdot,\cdot)_{(H^1(\Omega))'\times H^1(\Omega)}$ to represent the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$. We have

$$(\beta,\xi)_{(H^1(\Omega))'\times H^1(\Omega)} = (\beta,\xi)_{L^2(\Omega)} \quad \forall \beta \in L^2(\Omega), \xi \in H^1(\Omega),$$

and we note that K is a closed convex set in $H^1(\Omega)$. Then, using the definition (3.35) of the bilinear form a, and the fact that $\beta_0 \in K$ in (3.34), it is easy to see that Lemma 4.6 is a straightforward consequence of Theorem 4.5. □

In the fourth step, we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.3 and we consider the following initial-value problem.

Problem $\mathbf{PV}_\alpha$. *Find the adhesion field $\alpha_\eta : [0,T] \to L^2(\Gamma_3)$ such that for a.e. $t \in (0,T)$*

$$\dot\alpha_\eta(t) = -\left(\alpha_\eta(t)\left[\gamma_\nu(R_\nu(u_{\eta\nu}(t)))^2 + \gamma_\tau\left|\mathbf{R}_\tau(\mathbf{u}_{\eta\tau}(t))\right|^2\right] - \varepsilon_a\right)_+, \tag{4.23}$$

$$\alpha_\eta(0) = \alpha_0. \tag{4.24}$$

We have the following result.

Lemma 4.7. *There exists a unique solution $\alpha_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap Z$ to Problem PV_α.*

Proof. For simplicity, we suppress the dependence of various functions on Γ_3, and note that the equalities and inequalities below are valid a.e. on Γ_3. Consider the mapping $F_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_\eta(t,\alpha) = -\left(\alpha\left[\gamma_\nu(R_\nu(u_{\eta\nu}(t)))^2 + \gamma_\tau\left|\mathbf{R}_\tau(\mathbf{u}_{\eta\tau}(t))\right|^2\right] - \varepsilon_a\right)_+,$$

for all $t \in [0,T]$ and $\alpha \in L^2(\Gamma_3)$. It follows from the properties of the truncation operators R_ν and $\mathbf{R}_\tau$ that F_η is Lipschitz continuous with respect to the second argument. Moreover, for all $\alpha \in L^2(\Gamma_3)$, the mapping $t \to F_\eta(t,\alpha)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using a version of the Cauchy-Lipschitz Theorem given in Theorem 2.1, we deduce that there exists a unique function $\alpha_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution which satisfies (4.23)- (4.24). Also, the arguments used in Remark 1 show that $0 \le \alpha_\eta(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set Z, we find that $\alpha_\eta \in Z$, which concludes the proof of the lemma. □

Finally as a consequence of these results and using the properties of the operator $\mathcal{G}$, the operator $\mathcal{E}$, the functional j and the function S, for $t \in [0,T]$, we consider the operator

$$\Lambda : L^2(0,T;V' \times L^2(\Omega)) \to L^2(0,T;V' \times L^2(\Omega))$$

which maps every element $(\eta,\theta) \in L^2(0,T;V' \times L^2(\Omega))$ to the element $\Lambda(\eta,\theta) \in L^2(0,T;V' \times L^2(\Omega))$ defined by

$$\Lambda(\eta,\theta)(t) = (\Lambda^1(\eta,\theta)(t), \Lambda^2(\eta,\theta)(t)) \in V' \times L^2(\Omega), \tag{4.25}$$

defined by the equalities

$$\begin{aligned}(\Lambda^1(\eta,\theta)(t),\mathbf{v})_{V'\times V} &= (\mathcal{G}(\varepsilon(\mathbf{u}_\eta(t)),\beta_\theta(t)),\varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_\eta(t),\varepsilon(\mathbf{v}))_{\mathcal{H}} \\ &\quad + \left(\int_0^t M(t-s)\varepsilon(\mathbf{u}_\eta(s))ds, \varepsilon(\mathbf{v})\right)_{\mathcal{H}} + j(\alpha_\eta(t),\mathbf{u}_\eta(t),\mathbf{v}) \quad \forall \mathbf{v} \in V,\end{aligned} \tag{4.26}$$

$$\Lambda^2(\eta,\theta)(t) = S(\varepsilon(\mathbf{u}_\eta(t)),\beta_\theta(t)). \tag{4.27}$$

Here, for every $(\eta,\theta) \in L^2(0,T;V' \times L^2(\Omega))$, $\mathbf{u}_\eta, \varphi_\eta, \beta_\theta$ and α_η represent the displacement field, the potential electric field, the damage field and the bonding field obtained in Lemmas 4.3, 4.4, 4.6 and 4.7 respectively. We have the following result.

Lemma 4.8. *The operator Λ has a unique fixed point $(\eta^*,\theta^*) \in L^2(0,T;V' \times L^2(\Omega))$ such that $\Lambda(\eta^*,\theta^*) = (\eta^*,\theta^*)$.*

Proof. Let $(\eta,\theta) \in L^2(0,T;V' \times L^2(\Omega))$ and $(\eta_1,\theta_1),(\eta_2,\theta_2) \in L^2(0,T;V' \times L^2(\Omega))$. We use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\dot{\mathbf{u}}_{\eta_i} = \mathbf{v}_{\eta_i} = \mathbf{v}_i$, $\varphi_{\eta_i} = \varphi_i$, $\beta_{\theta_i} = \beta_i$ and $\alpha_{\eta_i} = \alpha_i$ for $i = 1,2$. Using (3.20), (3.23), (3.24), (3.25), (3.26), the definition of R_ν, $\mathbf{R}_\tau$ and Remark 1, we have

$$\begin{aligned}
(4.28)\qquad & \left|\Lambda^1(\eta_1,\theta_1)(t) - \Lambda^1(\eta_2,\theta_2)(t)\right|^2_{V'} \\
&\le |\mathcal{G}(\varepsilon(\mathbf{u}_1(t)),\beta_1(t)) - \mathcal{G}(\varepsilon(\mathbf{u}_2(t)),\beta_2(t))|^2_{\mathcal{H}} \\
&\quad + \int_0^t |M(t-s)\varepsilon(\mathbf{u}_1(s) - \mathbf{u}_2(s))|^2_{\mathcal{H}}\, ds + |\mathcal{E}^*\nabla\varphi_1(t) - (\mathcal{E}^*\nabla\varphi_2(t)|^2_{\mathcal{H}} \\
&\quad + C\,|p_\nu(u_{1\eta\nu}(t)) - p_\nu(u_{2\eta\nu}(t))|^2_{L^2(\Gamma_3)} \\
&\quad + C\left|\alpha_1^2(t)R_\nu(u_{1\eta\nu}(t)) - \alpha_2^2(t)R_\nu(u_{1\eta\nu}(t))\right|^2_{L^2(\Gamma_3)} \\
&\quad + C\,|p_\tau(\alpha_1(t))\mathbf{R}_\tau(\mathbf{u}_{1\eta\tau}(t)) - p_\tau(\alpha_2(t))\mathbf{R}_\tau(\mathbf{u}_{1\eta\tau}(t))|^2_{L^2(\Gamma_3)} \\
&\le C\Bigg(|\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2_V + \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|^2_V\, ds + |\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)} \\
&\quad + |\varphi_1(t) - \varphi_2(t)|^2_W + |\alpha_1(t) - \alpha_2(t)|^2_{L^2(\Gamma_3)}\Bigg).
\end{aligned}$$

Recall that $u_{\eta\nu}$ and $\mathbf{u}_{\eta\tau}$ denote the normal and the tangential component of the function $\mathbf{u}_\eta$ respectively. By a similar argument, from (4.27) and (3.21) it follows that

$$(4.29)\qquad \left|\Lambda^2(\eta_1,\theta_1)(t) - \Lambda^2(\eta_2,\theta_2)(t)\right|^2_{L^2(\Omega)} \le C\left(|\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2_V + |\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)}\right).$$

Therefore,

$$\begin{aligned}
(4.30)\qquad & |\Lambda(\eta_1,\theta_1)(t) - \Lambda(\eta_2,\theta_2)(t)|^2_{V'\times L^2(\Omega)} \\
&\le C(\Bigg(|\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2_V + \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|^2_V\, ds + |\varphi_1(t) - \varphi_2(t)|^2_W \\
&\qquad\qquad + |\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)} + |\alpha_1(t) - \alpha_2(t)|^2_{L^2(\Gamma_3)}\Bigg).
\end{aligned}$$

Moreover, from (4.7) we obtain

$$\begin{aligned}
(\dot{\mathbf{v}}_1 - \dot{\mathbf{v}}_2, \mathbf{v}_1 - \mathbf{v}_2)_{V'\times V} + (\mathcal{A}\varepsilon(\mathbf{v}_1) - \mathcal{A}\varepsilon(\mathbf{v}_2), \varepsilon(\mathbf{v}_1 - \mathbf{v}_2))_{\mathcal{H}} \\
+ (\eta_1 - \eta_2, \mathbf{v}_1 - \mathbf{v}_2)_{V'\times V} = 0.
\end{aligned}$$

We integrate this equality with respect to time, use the initial conditions $\mathbf{v}_1(0) = \mathbf{v}_2(0) = \mathbf{v}_0$ and condition (3.19) to find

$$m_{\mathcal{A}} \int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|^2_V\, ds \le -\int_0^t (\eta_1(s) - \eta_2(s), \mathbf{v}_1(s) - \mathbf{v}_2(s))_{V'\times V}\, ds,$$

for all $t \in [0,T]$. Then, using the inequality $2ab \le \frac{a^2}{m_{\mathcal{A}}} + m_{\mathcal{A}} b^2$ we obtain

$$(4.31)\qquad \int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|^2_V\, ds \le C\int_0^t |\eta_1(s) - \eta_2(s)|^2_{V'}\, ds \quad \forall t \in [0,T]\,.$$

On the other hand, from the Cauchy problem (4.23) – (4.24) we can write

$$\alpha_i(t) = \alpha_0 - \int_0^t \left(\alpha_i(s)\left[\{\gamma_\nu(R_\nu(u_{i\nu}(s))\}^2 + \gamma_\tau\,|\mathbf{R}_\tau(\mathbf{u}_{i\tau}(s))|^2\right] - \varepsilon_a\right)_+\, ds,$$

and then

$$|\alpha_1(t)-\alpha_2(t)|_{L^2(\Gamma_3)} \le C\int_0^t \left|\alpha_1(s)\left[R_\nu(u_{1\nu}(s))\right]^2-\alpha_2(s)\left[R_\nu(u_{2\nu}(s))\right]^2\right|_{L^2(\Gamma_3)} ds$$
$$+\,C\int_0^t \left|\alpha_1(s)\left|\mathbf{R}_\tau(\mathbf{u}_{1\tau}(s))\right|^2-\alpha_2(s)\left|\mathbf{R}_\tau(\mathbf{u}_{2\tau}(s))\right|^2\right|_{L^2(\Gamma_3)} ds.$$

Using the definition of R_ν and $\mathbf{R}_\tau$ and writing $\alpha_1=\alpha_1-\alpha_2+\alpha_2$, we get

$$|\alpha_1(t)-\alpha_2(t)|_{L^2(\Gamma_3)} \le C\left(\int_0^t |\alpha_1(s)-\alpha_2(s)|_{L^2(\Gamma_3)}\,ds+\int_0^t |\mathbf{u}_1(s)-\mathbf{u}_2(s)|_{L^2(\Gamma_3)^d}\,ds\right). \tag{4.32}$$

Next, we apply Gronwall's inequality to deduce

$$|\alpha_1(t)-\alpha_2(t)|_{L^2(\Gamma_3)} \le C\int_0^t |\mathbf{u}_1(s)-\mathbf{u}_2(s)|_{L^2(\Gamma_3)^d}\,ds,$$

and from the relation (3.17) we obtain

$$|\alpha_1(t)-\alpha_2(t)|^2_{L^2(\Gamma_3)} \le C\int_0^t |\mathbf{u}_1(s)-\mathbf{u}_2(s)|^2_V\,ds. \tag{4.33}$$

We use now (4.17), (3.22), (3.23) and (3.18) to find

$$|\varphi_1(t)-\varphi_2(t)|^2_W \le C\,|\mathbf{u}_1(t)-\mathbf{u}_2(t)|^2_V\,. \tag{4.34}$$

From (4.20) we deduce that

$$(\dot\beta_1-\dot\beta_2,\beta_1-\beta_2)_{L^2(\Omega)}+a(\beta_1-\beta_2,\beta_1-\beta_2)\le(\theta_1-\theta_2,\beta_1-\beta_2)_{L^2(\Omega)}\quad \text{a.e. } t\in(0,T)\,.$$

Integrating the previous inequality with respect to time, using the initial conditions $\beta_1(0)=\beta_2(0)=\beta_0$ and the inequality $a(\beta_1-\beta_2,\beta_1-\beta_2)\ge 0$, we have

$$\frac{1}{2}\,|\beta_1(t)-\beta_2(t)|^2_{L^2(\Omega)}\le\int_0^t(\theta_1(s)-\theta_2(s),\beta_1(s)-\beta_2(s))_{L^2(\Omega)}\,ds,$$

which implies that

$$|\beta_1(t)-\beta_2(t)|^2_{L^2(\Omega)}\le\int_0^t|\theta_1(s)-\theta_2(s)|^2_{L^2(\Omega)}\,ds+\int_0^t|\beta_1(s)-\beta_2(s)|^2_{L^2(\Omega)}\,ds.$$

This inequality combined with Gronwall's inequality leads to

$$|\beta_1(t)-\beta_2(t)|^2_{L^2(\Omega)}\le C\int_0^t|\theta_1(s)-\theta_2(s)|^2_{L^2(\Omega)}\,ds\quad\forall t\in[0,T]\,. \tag{4.35}$$

We substitute (4.33) and (4.34) in (4.30) to obtain

$$\begin{aligned}&|\Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_2)(t)|^2_{V'\times L^2(\Omega)}\\ &\le C\left(|\mathbf{u}_1(t)-\mathbf{u}_2(t)|^2_V+\int_0^t|\mathbf{u}_1(s)-\mathbf{u}_2(s)|^2_V\,ds+|\beta_1(t)-\beta_2(t)|^2_{L^2(\Omega)}\right)\\ &\le C\left(\int_0^t|\mathbf{v}_1(s)-\mathbf{v}_2(s)|^2_V\,ds+|\beta_1(t)-\beta_2(t)|^2_{L^2(\Omega)}\right).\end{aligned}$$

It follows now from the previous inequality, the estimates (4.31) and (4.35) that

$$|\Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_2)(t)|^2_{V'\times L^2(\Omega)}\le C\int_0^t|(\eta_1,\theta_1)(s)-(\eta_2,\theta_2)(s)|^2_{V'\times L^2(\Omega)}\,ds.$$

Reiterating this inequality m times leads to

$$|\Lambda^m(\eta_1,\theta_1) - \Lambda^m(\eta_2,\theta_2)|^2_{L^2(0,T;V'\times L^2(\Omega))} \leq \frac{C^m T^m}{m!} |(\eta_1,\theta_1) - (\eta_2,\theta_2)|^2_{L^2(0,T;V'\times L^2(\Omega))}.$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $L^2(0,T;V' \times L^2(\Omega))$, and so Λ has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 4.1.

Proof. Existence. Let $(\eta^*,\theta^*) \in L^2(0,T;V' \times L^2(\Omega))$ be the fixed point of Λ defined by (4.25) – (4.27) and denote

$$\mathbf{u}_* = \mathbf{u}_{\eta^*}, \varphi_* = \varphi_{\eta^*}, \beta_* = \beta_{\theta^*}, \alpha_* = \alpha_{\eta^*}. \tag{4.36}$$

$$\sigma_*(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}_*(t)) + \mathcal{G}(\varepsilon(\mathbf{u}_*(t)),\beta_*(t)) + \int_0^t M(t-s)\varepsilon(\mathbf{u}_*(s))ds + \mathcal{E}^*\nabla\varphi_*(t) \quad \forall t \in [0,T], \tag{4.37}$$

$$\mathbf{D}_*(t) = -B\nabla\varphi_*(t) + \mathcal{E}\varepsilon(\mathbf{u}_*(t)) \quad \forall t \in [0,T]. \tag{4.38}$$

We prove that the quadruplet $(\mathbf{u}_*,\varphi_*,\beta_*,\alpha_*)$ satisfies (3.40) – (3.44) and the regularites (4.1) – (4.4). Indeed, we write (4.7) for $\eta = \eta^*$ and use (4.36) to find

$$(\ddot{\mathbf{u}}_*(t),\mathbf{v}))_{V'\times V} + (\mathcal{A}\varepsilon(\dot{\mathbf{u}}_*(t)),\varepsilon(\mathbf{v}))_{\mathcal{H}} + (\eta^*(t),\mathbf{v})_{V'\times V} = (\mathbf{f}(t),\mathbf{v})_{V'\times V} \quad \forall \mathbf{v} \in V, t \in [0,T]. \tag{4.39}$$

We write (4.20) for $\theta = \theta^*$ and use (4.36) to obtain

$$\beta_*(t) \in K, \quad \left(\dot{\beta}_*(t),\xi - \beta_*(t)\right)_{L^2(\Omega)} + a(\beta_*(t),\xi - \beta_*(t)) \geq (\theta^*(t),\xi - \beta_*(t))_{L^2(\Omega)} \ \forall \xi \in K \text{ a.e. } t \in (0,T). \tag{4.40}$$

Equalities $\Lambda^1(\eta^*,\theta^*) = \eta^*$ and $\Lambda^2(\eta^*,\theta^*) = \theta^*$ combined with (4.26) and (4.27) show that

$$(\eta^*(t),\mathbf{v})_{V'\times V} = (\mathcal{G}(\varepsilon(\mathbf{u}_*(t)),\beta_*(t)),\varepsilon(\mathbf{v}))_{\mathcal{H}} + \left(\int_0^t M(t-s)\varepsilon(\mathbf{u}_*(s))ds,\varepsilon(\mathbf{v})\right)_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_*(t),\varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\alpha_*(t),\mathbf{u}_*(t),\mathbf{v}) \quad \forall \mathbf{v} \in V, \tag{4.41}$$

$$\theta^*(t) = S(\varepsilon(\mathbf{u}_*(t)),\beta_*(t)). \tag{4.42}$$

We now substitute (4.41) in (4.39) to obtain

$$\begin{aligned}(\mathbf{u}_*(t),\mathbf{v})_{V'\times V} &+ (\mathcal{A}\varepsilon(\dot{\mathbf{u}}_*(t)),\varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathcal{G}(\varepsilon(\mathbf{u}_*(t)),\beta_*(t)),\varepsilon(\mathbf{v}))_{\mathcal{H}} + \left(\int_0^t M(t-s)\varepsilon(\mathbf{u}_*(s))ds,\varepsilon(\mathbf{v})\right)_{\mathcal{H}} \\ &+ (\mathcal{E}^*\nabla\varphi_*(t),\varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\alpha_*(t),\mathbf{u}_*(t),\mathbf{v}) = (\mathbf{f}(t),\mathbf{v})_{V'\times V} \quad \forall \mathbf{v} \in V, t \in (0,T),\end{aligned} \tag{4.43}$$

and (4.42) in (4.40) to get

$$\beta_*(t) \in K \text{ for all } t \in [0,T], \quad \left(\dot{\beta}_*(t),\xi - \beta_*(t)\right)_{L^2(\Omega)} + a(\beta_*(t),\xi - \beta_*(t)) \geq (S(\varepsilon(\mathbf{u}_*(t)),\beta_*(t)),\xi - \beta_*(t))_{L^2(\Omega)} \quad \forall \xi \in K. \tag{4.44}$$

Using $\mathbf{u}_{\eta^*}$ in (4.17), by (4.36) we have:

$$(B\nabla\varphi_*(t), \nabla\phi)_H - (\mathcal{E}\varepsilon(\mathbf{u}_*(t)), \nabla\phi)_H = (q(t), \phi)_W \quad \forall\phi \in W, t \in (0,T)\,. \tag{4.45}$$

Additionally, we use $\mathbf{u}_{\eta^*}$ in (4.23) and (4.36) to find

$$\dot{\alpha}_*(t) = -\left(\alpha_*(t)(\gamma_\nu(R_\nu(u_{*\nu}(t)))^2 + \gamma_\tau\left|\mathbf{R}_\tau(\mathbf{u}_{*\tau}(t))\right|^2) - \varepsilon_a\right)_+ \quad \text{a.e. } t \in (0,T)\,. \tag{4.46}$$

The relations (4.43), (4.44), (4.45) and (4.46) allow us to conclude now that $(\mathbf{u}_*, \varphi_*, \beta_*, \alpha_*)$ satisfies (3.40) – (3.43). Next, (3.44) and the regularity (4.1) – (4.4) follow from Lemmas 4.3, 4.4, 4.6 and 4.7. Since $\mathbf{u}_*$ and φ_* satisfy (4.1) and (4.2), it follows from (4.37) that

$$\sigma_* \in L^2(0,T;\mathcal{H}). \tag{4.47}$$

We choose $\mathbf{v} = \omega \in D(\Omega)^d$ in (4.43) and by (4.37) and (3.36):

$$\rho\ddot{\mathbf{u}}_*(t) = Div\sigma_*(t) + \mathbf{f}_0(t) \quad \text{in } V' \ \forall t \in [0,T]\,.$$

Also, by (3.27), (3.28) and (4.47) we have:

$$Div\sigma_* \in L^2(0,T;V').$$

Let $t_1, t_2 \in [0,T]$. By (3.22), (3.23), (3.18) and (4.38), we deduce that

$$|\mathbf{D}_*(t_1) - \mathbf{D}_*(t_2)|_H \leq C(|\varphi_*(t_1) - \varphi_*(t_2)|_W + |\mathbf{u}_*(t_1) - \mathbf{u}_*(t_2)|_V).$$

The regularity of $\mathbf{u}_*$ and φ_* given by (4.1) and (4.2) implies

$$\mathbf{D}_* \in C(0,T;H). \tag{4.48}$$

We choose $\phi \in D(\Omega)$ in (4.45) and using (3.37) we find

$$div\mathbf{D}_*(t) = q_0(t) \quad \forall t \in [0,T]\,.$$

By (3.29) and (4.48) we obtain

$$\mathbf{D}_* \in C(0,T;\mathcal{W}).$$

Finally we conclude that the weak solution $(\mathbf{u}_*, \sigma_*, \varphi_*, \mathbf{D}_*, \beta_*, \alpha_*)$ of the piezoelectric contact problem P has the regularity (4.1) – (4.6), which concludes the existence part of Theorem 4.1.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.25) – (4.27). □

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