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ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING THE AL-OBOUDI DIFFERENTIAL OPERATOR

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ABSTRACT. In this paper we introduce a new subclass of normalized analytic functions in the open unit disc which is defined by the Al-Oboudi differential operator. A coefficient inequality, extreme points and integral mean inequalities of a differential operator for this class are given. We investigate various subordination results for the subclass of analytic functions and obtain sufficient conditions for univalent close-to-starlikeness. We also discuss the boundedness properties associated with partial sums of functions in the class. Several interesting connections with the class of close-to-starlike and close-to-convex functions are also pointed out.

Key words and phrases: Close-to-convex function, Close-to-starlike function, Ruscheweyh derivative operator, Al-Oboudi differential operator and subordination relationship.

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1. INTRODUCTION AND PRELIMINARIES

Let A denote the class of normalized functions f defined by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. For $f \in A$, [1] has introduced the following differential operator.

$$D^0 f(z) = f(z) \tag{1.2}$$

$$D^1 f(z) = (1-\delta)f(z) + \delta z f'(z) = D_\delta f(z), \qquad \delta \geq 0 \tag{1.3}$$

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$$D^n f(z) = D_\delta(D^{n-1} f(z)), \qquad (n \in \mathbb{N}). \tag{1.4}$$

For $f(z)$ given by (1.1), we notice from (1.3) and (1.4) that

$$D^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.5}$$

For $\delta = 1$ we obtain the Sălăgean operator [11].

Definition 1.1. A function f in A is said to be starlike of order α $(0 \leq \alpha < 1)$ in U, that is, $f \in S^*(\alpha)$, if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in U). \tag{1.6}$$

Definition 1.2. A function f in A is said to be convex of order α $(0 \leq \alpha < 1)$ in U, that is, $f \in K(\alpha)$, if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U). \tag{1.7}$$

Definition 1.3. A function f in A is said to be close-to-convex in U, of order α, that is, $f \in C(\alpha)$, if and only if

$$\operatorname{Re}\{f'(z)\} > \alpha \qquad (z \in U). \tag{1.8}$$

Definition 1.4. A function f in A is said to be close-to-starlike of order α $(0 \leq \alpha < 1)$ in U, that is, $f \in CS^*(\alpha)$, if and only if

$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\} > \alpha \qquad (z \in U \setminus \{0\}). \tag{1.9}$$

We note that the classes S, $S^*(0) = S^*$, $K(0) = K$, $C(0) = C$, $CS^*(0) = CS^*$ are the well known classes of univalent, starlike, convex, close-to-convex and close-to-starlike functions in U, respectively. It is also clear that

(i) $f \in K(\alpha)$ if and only if $zf' \in S^*(\alpha)$;
(ii) $K(\alpha) \subset S^*(\alpha) \subset C(\alpha) \subset S$.

Definition 1.5. For two functions f and g analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U, and write

$$f(z) \prec g(z) \qquad (z \in U) \tag{1.10}$$

if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)) \qquad (z \in U). \tag{1.11}$$

In particular, if the function g is univalent in U, the above subordination is equivalent to

$$f(0) = g(0), \qquad f(U) \subset g(U). \tag{1.12}$$

Littlewood [7] in 1925 has proved the following subordination theorem which we state as a lemma.

Lemma 1.1. *Let f and g be analytic in the unit disc, and suppose $g \prec f$. Then for $0 < p < \infty$,*

$$\int_0^{2\pi} |g(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \qquad (0 \leq r < 1, p > 0). \tag{1.13}$$

Strict inequality holds for $0 < r < 1$ unless f is constant or $w(z) = \alpha z,\ |\alpha| = 1$.

Definition 1.6. Let $n \in \mathbb{N} \cup \{0\}$ and $\lambda \geq 0$. Let $D_\lambda^n f$ denote the operator defined by $D_\lambda^n : A \to A$ such that

$$D_\lambda^n f(z) = (1-\lambda)S^n f(z) + \lambda R^n f(z) \quad z \in U, \tag{1.14}$$

where $S^n f$ is the Sălăgean differential operator and $R^n f$ is the Ruscheweyh differential operator defined by $R^n : A \to A$ such that

$$R^0 f(z) = f(z), R^1 f(z) = zf'(z),$$

with recurrence relation given by

$$(n+1)R^{n+1} f(z) = z[R^n f(z)]' + nR^n f(z) \ \ (z \in U). \tag{1.15}$$

For $f \in A$ given by (1.1)

$$R^n f(z) = z + \sum_{k=2}^{\infty} {}^nC_{n+k-1} a_k z^k \qquad (z \in U). \tag{1.16}$$

Notice that D_λ^n is a linear operator and for $f \in A$ defined by (1.1), we have

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [(1-\lambda)k^n + \lambda \, {}^nC_{n+k-1}] a_k z^k. \tag{1.17}$$

It is observed that for $n = 0$,

$$D_\lambda^0 f(z) = (1-\lambda)S^0 f(z) + \lambda R^0 f(z) = f(z) = S^0 f(z) = R^0 f(z),$$

and for $n = 1$

$$D_\lambda^1 f(z) = (1-\lambda)S^1 f(z) + \lambda R^1 f(z) = zf'(z) = S^1 f(z) = R^1 f(z).$$

Definition 1.7. Let $K(\gamma, \mu, m, \beta)$ denote the subclass of A consisting of functions f which satisfy the inequality

$$\left| \frac{1}{\gamma} \left((1-\mu) \frac{D^m f}{z} + \mu (D^m f)' - 1 \right) \right| < \beta, \tag{1.18}$$

where $z \in U$, $\gamma \in \mathbb{C} \setminus \{0\}$, $0 < \beta \leq 1$, $0 \leq \mu \leq 1$, $m \in \mathbb{N}_0$ and D^m is as defined in (1.5).

Remark 1. For $\gamma = 1$, $\mu = 1$, $m = 0$, we obtain the class of close-to-convex functions of order $(1-\beta)$. For the values $\gamma = 1$, $\mu = 0$, $m = 0$, we obtain the class of close-to-starlike functions of order $(1-\beta)$.

Let

$$T(\eta, f) = (1-\eta)\frac{f(z)}{z} + \eta \ \ f'(z) \quad (z \in U \setminus \{0\})$$

for η real and $f \in A$. Define

$$T_\eta := \{f \in A : \mathrm{Re}\{T(\eta, f)\} > 0\}.$$

We note that T_η can be derived from the class $K(\gamma, \mu, m, \beta)$ by replacing μ by η and $D^m f$ by f.

2. COEFFICIENT INEQUALITIES, GROWTH AND DISTORTION THEOREMS

Here we first give a sufficient condition for $f \in A$ to belong to the class $K(\gamma, \mu, m, \beta)$.

Theorem 2.1. *Let $f(z) \in A$ satisfy*

$$\sum_{k=2}^{\infty}(1+(k-1)\mu)(1+(k-1)\delta)^m|a_k| \leq |\gamma|\beta, \tag{2.1}$$

where $\gamma \in \mathbb{C} \setminus \{0\}$, $0 < \beta \leq 1$, $0 \leq \mu \leq 1$, $m \in \mathbb{N}_0$, $\delta \geq 0$. Then $f(z) \in K(\gamma, \mu, m, \beta)$.

Proof. Suppose that (2.1) is true for γ $(\gamma \in \mathbb{C} \setminus \{0\})$, β $(0 < \beta \leq 1)$, μ $(0 \leq \mu \leq 1)$, $m \in \mathbb{N}_0$, and δ $(\delta \geq 0)$ for $f(z) \in A$.

Using (1.5) for $|z| = 1$, we have

$$\begin{aligned}\left|(1-\mu)\frac{D^m f}{z} + \mu(D^m f)' - 1\right| &\leq \sum_{k=2}^{\infty}(1+(k-1)\mu)(1+(k-1)\delta)^m|a_k| \\ &\leq |\gamma|\beta.\end{aligned}$$

Thus by Definition 1.7 $f(z) \in K(\gamma, \mu, m, \beta)$.

Notice that the function given by

$$f(z) = z + \sum_{k=2}^{\infty}\frac{|\gamma|\beta}{(1+(k-1)\mu)(1+(k-1)\delta)^m}\, z^k \tag{2.2}$$

belongs to the class $K(\gamma, \mu, m, \beta)$ and plays the role of extremal function for the result (2.1). □

We denote by $\tilde{K}(\gamma, \mu, m, \beta) \subseteq K(\gamma, \mu, m, \beta)$ the functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

where the Taylor-Maclaurin coefficients satisfy inequality (2.1).

Next we state the growth and distortion theorems for the class $\tilde{K}(\gamma, \mu, m, \beta)$. The results follow easily on applying Theorem 2.1, therefore, we omit the proof.

Theorem 2.2. *Let the function $f(z)$ defined by (1.1) be in the class $\tilde{K}(\gamma, \mu, m, \beta)$. Then*

$$|z| - \frac{|\gamma|\beta}{(1+\mu)(1+\delta)^m}|z|^2 \leq |f(z)| \leq |z| + \frac{|\gamma|\beta}{(1+\mu)(1+\delta)^m}|z|^2. \tag{2.3}$$

The equality in (2.3) is attained for the function $f(z)$ given by

$$f(z) = z + \frac{|\gamma|\beta}{(1+\mu)(1+\delta)^m}\, z^2. \tag{2.4}$$

Theorem 2.3. *Let the function $f(z)$ defined by (1.1) be in the class $\tilde{K}(\gamma, \mu, m, \beta)$. Then*

$$1 - \frac{2|\gamma|\beta}{(1+\mu)(1+\delta)^m}|z| \leq |f'(z)| \leq 1 + \frac{2|\gamma|\beta}{(1+\mu)(1+\delta)^m}|z|. \tag{2.5}$$

The equality in (2.5) is attained for the function $f(z)$ given by (2.4).

In view of Remark 1, Theorem 2.2 and Theorem 2.3 would yield the corresponding distortion properties for the class of close-to-convex and close-to-starlike functions.

3. EXTREME POINTS

Now we determine the extreme points of the class $\tilde{K}(\gamma, \mu, m, \beta)$.

Remark 2. For $\gamma \in \mathbb{C}\setminus\{0\}$, $0 < \beta \leq 1$, $0 \leq \mu \leq 1$, $m \in \mathbb{N}_0$ and $\delta \geq 0$ the following functions are in the class $\tilde{K}(\gamma, \mu, m, \beta)$

$$f_1(z) = z + \frac{\beta|\gamma|}{(1+\mu)(1+\delta)^m} z^2 \qquad (z \in U);$$

$$f_2(z) = z + \frac{\beta|\gamma|}{(1+2\mu)(1+2\delta)^m} z^3 \qquad (z \in U);$$

$$f_3(z) = z + \frac{1}{(1+\mu)(1+\delta)^m} z^2 + \frac{(|\gamma|\beta - 1)}{(1+2\mu)(1+2\delta)^m}\, z^3 \qquad (z \in U).$$

Theorem 3.1. *Let* $f_1(z) = z$ *and*

$$f_k(z) = z + \frac{|\gamma|\beta}{(1+(k-1)\mu)(1+(k-1)\delta)^m} z^k \qquad (k \geq 2). \tag{3.1}$$

Then $f(z) \in \tilde{K}(\gamma, \mu, m, \beta)$*, if and only if it can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) \tag{3.2}$$

where $\lambda_k \geq 0$ *and* $\sum_{k=1}^{\infty} \lambda_k = 1$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} \lambda_k f_k(z) \\ &= z + \sum_{k=2}^{\infty} \lambda_k \frac{|\gamma|\beta}{(1+(k-1)\mu)(1+(k-1)\delta)^m} z^k. \end{aligned}$$

Then

$$\begin{aligned} \sum_{k=2}^{\infty} (1+(k-1)\mu)(1+(k-1)\delta)^m & \frac{|\gamma|\beta}{(1+(k-1)\mu)(1+(k-1)\delta)^m} \lambda_k \\ &= |\gamma|\beta \sum_{k=2}^{\infty} \lambda_k \\ &\leq |\gamma|\beta\,(1-\lambda_1) \\ &\leq |\gamma|\beta. \end{aligned}$$

Thus, in view of Theorem 2.1, $f(z) \in \tilde{K}(\gamma, \mu, m, \beta)$.

Conversely, suppose that $f(z) \in \tilde{K}(\gamma, \mu, m, \beta)$. Setting

$$\lambda_k = \frac{(1+(k-1)\mu)(1+(k-1)\delta)^m}{|\gamma|\beta} a_k \quad \text{and} \quad \lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k,$$

we obtain

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z).$$

□

Corollary 3.2. *The extreme points of* $\tilde{K}(\gamma,\mu,m,\beta)$ *are the functions* $f_1(z)=z$ *and*

$$f_k(z)=z+\frac{|\gamma|\beta}{(1+(k-1)\mu)(1+(k-1)\delta)^m}\,z^k \qquad (k=2,3,\dots).$$

4. INTEGRAL MEAN INEQUALITIES FOR A DIFFERENTIAL OPERATOR

Theorem 4.1. *Let* $f(z)\in\tilde{K}(\gamma,\mu,m,\beta)$ *and suppose that*

$$\sum_{k=2}^{\infty}[(1-\lambda)k^n+\lambda\,{}^nC_{n+k-1}]|a_k|\le\frac{|\gamma|\beta[(1-\lambda)j^n+\lambda\,{}^nC_{n+j-1}]}{(1+\mu(j-1))(1+\delta(j-1))^m}. \tag{4.1}$$

Also, let the function

$$f_j(z)=z+\frac{|\gamma|\beta}{(1+\mu(j-1))(1+\delta(j-1))^m}z^j \qquad (j\ge 2). \tag{4.2}$$

If there exists an analytic function $w(z)$ *given by*

$$w(z)^{j-1}=\frac{(1+\mu(j-1))(1+\delta(j-1))^m}{|\gamma|\beta[(1-\lambda)j^n+\lambda\,{}^nC_{n+j-1}]}\sum_{k=2}^{\infty}[(1-\lambda)k^n+\lambda\,{}^nC_{n+k-1}]a_kz^{k-1},$$

then for $z=re^{i\theta}$ *with* $0<r<1$,

$$\int_0^{2\pi}|D_\lambda^nf(z)|^pd\theta\le\int_0^{2\pi}|D_\lambda^nf_j(z)|^pd\theta \qquad (0\le\lambda\le 1, p>0)$$

for the differential operator defined in (1.17).

Proof. By Definition 1.6 and by virtue of relation (1.17), we have

$$D_\lambda^nf(z)=z+\sum_{k=2}^{\infty}[(1-\lambda)k^n+\lambda\,{}^nC_{n+k-1}]a_kz^k. \tag{4.3}$$

Likewise,

$$D_\lambda^nf_j(z)=z+\frac{|\gamma|\beta[(1-\lambda)j^n+\lambda\,{}^nC_{n+j-1}]}{(1+\mu(j-1))(1+\delta(j-1))^m}\,z^j. \tag{4.4}$$

For $z=re^{i\theta}, 0<r<1$, we need to show that

$$\int_0^{2\pi}\left|1+\sum_{k=2}^{\infty}[(1-\lambda)k^n+\lambda\,{}^nC_{n+k-1}]a_kz^{k-1}\right|^pd\theta \le\int_0^{2\pi}\left|1+\frac{|\gamma|\beta[(1-\lambda)j^n+\lambda\,{}^nC_{n+j-1}]}{(1+\mu(j-1))(1+\delta(j-1))^m}z^{j-1}\right|d\theta \quad (p>0). \tag{4.5}$$

By applying Littlewood's subordination theorem, it would be sufficient to show that

$$1+\sum_{k=2}^{\infty}[(1-\lambda)k^n+\lambda\,{}^nC_{n+k-1}]a_kz^{k-1}\prec 1+\frac{|\gamma|\beta[(1-\lambda)j^n+\lambda\,{}^nC_{n+j-1}]}{(1+\mu(j-1))(1+\delta(j-1))^m}z^{j-1}. \tag{4.6}$$

Set

$$1+\sum_{k=2}^{\infty}[(1-\lambda)k^n+\lambda\,{}^nC_{n+k-1}]a_kz^{k-1}=1+\frac{|\gamma|\beta[(1-\lambda)j^n+\lambda\,{}^nC_{n+j-1}]}{(1+\mu(j-1))(1+\delta(j-1))^m}w(z)^{j-1}.$$

We note that

$$(w(z))^{j-1} = \frac{(1+\mu(j-1))(1+\delta(j-1))^m}{|\gamma|\beta[(1-\lambda)j^n + \lambda\, {}^nC_{n+j-1}]} \sum_{k=2}^{\infty} [(1-\lambda)k^n + \lambda\, {}^nC_{n+k-1}] a_k z^{k-1}, \tag{4.7}$$

and $w(0) = 0$. Moreover, we prove that the analytic function $w(z)$ satisfies $|w(z)| < 1$, $z \in U$

$$\begin{aligned} |w(z)|^{j-1} &\leq \left| \frac{(1+\mu(j-1))(1+\delta(j-1))^m}{|\gamma|\beta[(1-\lambda)j^n + \lambda\, {}^nC_{n+j-1}]} \sum_{k=2}^{\infty} [(1-\lambda)k^n + \lambda\, {}^nC_{n+k-1}] a_k z^{k-1} \right| \\ &\leq \frac{(1+\mu(j-1))(1+\delta(j-1))^m}{|\gamma|\beta[(1-\lambda)j^n + \lambda\, {}^nC_{n+j-1}]} \sum_{k=2}^{\infty} [(1-\lambda)k^n + \lambda\, {}^nC_{n+k-1}] |a_k| |z|^{k-1} \\ &\leq |z| \frac{(1+\mu(j-1))(1+\delta(j-1))^m}{|\gamma|\beta[(1-\lambda)j^n + \lambda\, {}^nC_{n+j-1}]} \sum_{k=2}^{\infty} [(1-\lambda)k^n + \lambda\, {}^nC_{n+k-1}] |a_k| \\ &\leq |z| < 1 \quad \text{by hypothesis (4.1).} \end{aligned}$$

This completes the proof of Theorem 4.1. □

As a particular case of Theorem 4.1, we can derive the following result when $n = 0$. That is, for $D_\lambda^0 f(z) = f(z)$.

Corollary 4.2. *Let $f(z) \in \tilde{K}(\gamma, \mu, m, \beta)$ be given by (1.1), then for $z = re^{i\theta}$ $(0 < r < 1)$*

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |f_j(re^{i\theta})|^p d\theta \qquad (p > 0),$$

where

$$f_j(z) = z + \frac{|\gamma|\beta}{(1+\mu(j-1))(1+\delta(j-1))^m} z^j \qquad (j \geq 2).$$

We conclude this section by observing that by specializing the parameters in Theorem 4.1, several integral mean inequalities can be deduced for $S^n f(z)$, $R^n f(z)$, the class of close-to-convex functions and the class of close-to-starlike functions as mentioned in Remark 1.

5. SUBORDINATION RESULTS FOR THE CLASS $T(\eta, f)$

In proving the main subordination results we need the following lemma due to [8, p. 132].

Lemma 5.1. *Let q be univalent in U and θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$, when $w \in q(U)$. Set*

$$Q(z) = zq'(z) \cdot \phi[q(z)], \qquad h(z) = \theta[q(z)] + Q(z)$$

and suppose that either:

(i) *Q is starlike or*
(ii) *h is convex.*
In addition, assume that
(iii) $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{\theta'(q(z))}{\phi(q(z))} + z\frac{Q'(z)}{Q(z)}\right) > 0.$

If P is analytic in U, with $P(0) = q(0)$, $P(U) \subset D$ and

$$\theta[P(z)] = zP'(z) \cdot \phi[P(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)] = h(z)$$

then $P \prec q$, and q is the best dominant.

Lemma 5.2. *Let $q \in H = \{f \in A : f(z) = 1 + b_1 z + b_2 z^2 + \cdots\}$ be univalent and satisfy the following conditions: $q(z)$ is convex and*

$$\mathrm{Re}\left\{\left(\frac{1}{\eta}+1\right)+\frac{zq''(z)}{q'(z)}\right\} > 0 \tag{5.1}$$

for $\eta \neq 0$ and all $z \in U$. For $P \in H$ in U if

$$P(z) + \eta z P'(z) \prec q(z) + \eta z q'(z), \tag{5.2}$$

then $P \prec q$ and q is the best dominant.

Proof. For $\eta \neq 0$ a real number, we define θ and ϕ by

$$\theta(w) := w, \quad \phi(w) := \eta, \quad D = \{w : w \neq 0\} \tag{5.3}$$

in Lemma 5.1. Then the functions

$$Q(z) = zq'(z)\phi(q(z)) = \eta z q'(z)$$
$$h(z) = \theta(q(z)) + Q(z) = q(z) + \eta z q'(z).$$

Using (5.1), we notice that $Q(z)$ is starlike in U and $\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for all $z \in U$ and $\eta \neq 0$.

Thus the hypotheses of Lemma 5.1 are satisfied. Therefore, from (5.2) it follows that $P \prec q$ and q is the best dominant. □

Theorem 5.3. *Let $q \in H$ be univalent and satisfy the condition (5.1) in Lemma 5.2. For $D^m f$ if*

$$T(\eta, D^m f) \prec q(z) + \eta z q'(z), \tag{5.4}$$

then $\frac{D^m f(z)}{z} \prec q(z)$ and $q(z)$ is the best dominant.

Proof. Substituting $P(z) = \frac{D^m f(z)}{z}$, where $P(0) = 1$, we have

$$P(z) + \eta z P'(z) = T(\eta, D^m f).$$

Thus using (5.4) and Lemma 5.2, we get the required result. □

Corollary 5.4. *Let $q \in H$ be univalent and satisfy the conditions (5.1) in Lemma 5.2. For $f \in A$, if $T(\eta, f) \prec q(z) + \eta z q'(z)$, then $\frac{f(z)}{z} \prec q(z)$ and q is the best dominant.*

Proof. By substituting $m = 0$ in Theorem 5.3 we obtain Corollary 5.4. □

Corollary 5.5. *Let $q \in H$ be univalent and convex for all $z \in U$. For $P \in H$ in U if*

$$P(z) + zP'(z) \prec q(z) + zq'(z), \tag{5.5}$$

then $P \prec q$, and q is the best dominant.

Proof. Take $\eta = 1$ in Lemma 5.2. □

Corollary 5.6. *Let $q \in S$ be convex. For $f \in A$ if*

$$f'(z) \prec q(z) + zq'(z),$$

then $\frac{f(z)}{z} \prec q(z)$ and q is the best dominant.

Proof. Take $\eta = 1$ in Corollary 5.4. □

Corollary 5.7. *Let $q \in S$ satisfy*

$$T(\eta, f) \prec \frac{1 + 2(\eta - \alpha - \eta\alpha)z - (1 - 2\alpha)z^2}{(1 - z)^2}$$

where $f \in A$. Then $\frac{f(z)}{z} \in CS^*(\alpha)$ and q is the best dominant.

Proof. Take $q(z) = \frac{1+(1-2\alpha)z}{1-z}$ in Corollary 5.4. Then it follows that

$$\frac{f(z)}{z} \prec \frac{1+(1-2\alpha)z}{1-z},$$

which is equivalent to $\mathrm{Re}\left\{\frac{f(z)}{z}\right\} > \alpha$. Therefore

$$\frac{f(z)}{z} \in CS^*(\alpha).$$

□

Corollary 5.8. *Let $q \in S$ satisfy*

$$f'(z) \prec \frac{1+2(1-2\alpha)z-(1-2\alpha)z^2}{(1-z)^2},$$

where $f \in A$. Then $\frac{f(z)}{z} \in CS^*(\alpha)$ and q is the best dominant.

Proof. Substituting $\eta = 1$ in Corollary 5.7, we get the desired result. □

Corollary 5.9. *Let $q \in S$ satisfy*

$$f'(z) \prec \frac{1+2z-z^2}{(1-z)^2},$$

where $f \in A$. Then $f(z) \in CS^*$ and q is the best dominant.

Proof. Take $\alpha = 0$ in Corollary 5.8. □

6. PARTIAL SUMS

In line with the earlier works of Silverman [12] and Silvia [13] on the partial sums of analytic functions, we investigate in this section the partial sums of functions in the class $K(\gamma,\mu,m,\beta)$. We obtain sharp lower bounds for the ratios of the real part of $f(z)$ to $f_N(z)$ and $f'(z)$ to $f'_N(z)$.

Theorem 6.1. *Let $f(z)$ of the form (1.1) belong to $K(\gamma,\mu,m,\beta)$ and $h(N+1,\gamma,\mu,m,\beta) \geq 1$. Then*

(6.1) $$\mathrm{Re}\left(\frac{f(z)}{f_N(z)}\right) \geq 1 - \frac{1}{h(N+1,\gamma,\mu,m,\beta)}$$

and

(6.2) $$\mathrm{Re}\left(\frac{f_N(z)}{f(z)}\right) \geq \frac{h(N+1,\gamma,\mu,m,\beta)}{h(N+1,\gamma,\mu,m,\beta)+1},$$

where

(6.3) $$h(k,\gamma,\mu,m,\beta) = \frac{(1+(k-1)\mu)(1+(k-1)\delta)^m}{|\gamma|\beta}.$$

The result is sharp for every N, with extremal functions given by

(6.4) $$f(z) = z + \frac{1}{h(N+1,\gamma,\mu,m,\beta)} \quad z^{N+1} \qquad (N \in \mathbb{N}\setminus\{1\}).$$

Proof. To prove (6.1), it is sufficient to show that

$$h(N+1,\gamma,\mu,m,\beta)\left[\frac{f(z)}{f_N(z)}-\left(1-\frac{1}{h(N+1,\gamma,\mu,m,\beta)}\right)\right]\prec\frac{1+z}{1-z}\qquad(z\in U).$$

By the subordination property (1.11), we can write

$$h(N+1,\gamma,\mu,m,\beta)\left[\frac{1+\sum_{k=2}^{\infty}a_kz^{k-1}}{1+\sum_{k=2}^{N}a_kz^{k-1}}-\left(1-\frac{1}{h(N+1,\gamma,\mu,m,\beta)}\right)\right]=\frac{1+w(z)}{1-w(z)}.$$

Notice that $w(0)=0$ and

$$|w(z)|\le\frac{h(N+1,\gamma,\mu,m,\beta)\sum_{k=N+1}^{\infty}|a_k|}{2-2\sum_{k=2}^{N}|a_k|-h(N+1,\gamma,\mu,m,\beta)\sum_{k=N+1}^{\infty}|a_k|}$$

$|w(z)|<1$ if and only if

$$\sum_{k=2}^{N}|a_k|+h(N+1,\gamma,\mu,m,\beta)\sum_{k=N+1}^{\infty}|a_k|\le 1.$$

In view of (2.1), we can equivalently show that

$$\sum_{k=2}^{N}(h(k,\gamma,\mu,m,\beta)-1)|a_k|+\sum_{k=N+1}^{\infty}((h(k,\gamma,\mu,m,\beta)-h(N+1,\gamma,\mu,m,\beta))|a_k|\ge 0.$$

The above inequality holds because $h(k,\gamma,\mu,m,\beta)$ is a non-decreasing sequence. This completes the proof of (6.1). Finally, it is observed that equality in (6.1) is attained for the function given by (6.4) when $z=re^{2\pi i/N}$ as $r\to 1^-$. The proof of (6.2) is similar to that of (6.1), and is hence omitted. □

Using a similar method, we can prove the following theorem.

Theorem 6.2. *Let $f(z)$ of the form (1.1) belong to $K(\gamma,\mu,m,\beta)$, and $h(N+1,\gamma,\mu,m,\beta)\ge N+1$. Then*

$$\operatorname{Re}\left(\frac{f'(z)}{f'_N(z)}\right)\ge 1-\frac{N+1}{h(N+1,\gamma,\mu,m,\beta)}$$

and

$$\operatorname{Re}\left(\frac{f'_N(z)}{f'(z)}\right)\ge\frac{h(N+1,\gamma,\mu,m,\beta)}{N+1+h(N+1,\gamma,\mu,m,\beta)},$$

where $h(N+1,\gamma,\mu,m,\beta)$ is given by (6.3). The result is sharp for every N, with extremal functions given by (6.4).

REFERENCES

[1] F.M. AL-OBOUDI, On univalent functions defined by a generalized Sălăgean operator, *Internat. J. Math. and Mathematical Sciences*, **27** (2004), 1429–1436.

[2] P.L. DUREN, Univalent functions, *Grundlehren der Mathematischen Wissenchaften*, **259**, Springer-Verlag, New York, USA, 1983.

[3] S.S. EKER AND H.Ö. GÜNEY, A new subclass of analytic functions involving Al-Oboudi differential operator, *Journal of Inequalities and Applications*, (2008), Art. ID. 452057.

[4] S.M. KHAIRNAR AND M. MORE, A class of analytic functions defined by Hurwitz-Lerch Zeta function, *Internat. J. Math. and Computation*, **1**(8) (2008), 106–123.

[5] S.M. KHAIRNAR AND M. MORE, A new class of analytic and multivalent functions defined by subordination property with applications to generalized hypergeometric functions, *Gen. Math.*, (to appear in 2009).

[6] Ö.Ö. KIHC, Sufficient conditions for subordination of multivalent functions, *Journal of Inequalities and Applications*, (2008), Art. ID. 374756.

[7] J.E. LITTLEWOOD, On inequalities in the theory of functions, *Proc. London Math. Soc.*, **23** (1925), 481–519.

[8] S.S. MILLER AND P.T. MOCANU, *Differential Subordinations, Theory and Applications*, Vol. **225**, of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, 2000.

[9] G. MURUGUSUNDARAMOORTHY, T. ROSY AND K. MUTHUNAGAI, Carlson-Shaffer operator and their applications to certain subclass of uniformly convex functions, *Gen. Math.*, **15**(4) (2007), 131–143.

[10] R.K. RAINA AND D. BANSAL, Some properties of a new class of analytic functions defined in terms of a Hadamard product, *J. Inequal. Pure and Appl. Math.*, **9**(1) (2008), Art. 22. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=957`]

[11] G.S. SĂLĂGEAN, Subclass of univalent functions in Complex Analysis, 5th Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), Vol. 1013 of Lecture Notes in Mathematics, Springer, Berlin, Germany, (1983), 362–372.

[12] H. SILVERMAN, Partial sums of starlike and convex functions, *J. Math. Anal. and Appl.*, **209** (1997), 221–227.

[13] E.M. SILVIA, Partial sums of convex functions of order α, *Houston J. Math., Math. Soc.*, **11**(3) (1985), 397–404.