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AN INEQUALITY FOR DIVIDED DIFFERENCES IN HIGH DIMENSIONS

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ABSTRACT. This paper is devoted to an inequality for divided differences in the multivariate case which is similar to the inequality obtained by [J. Pečarić, and M. Rodić Lipanović, On an inequality for divided differences, *Asian-European Journal of Mathematics*, Vol. 1, No. 1 (2008), 113-120].

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1. INTRODUCTION

Recently, Pečarić and Lipanović [3] have proved the following inequality for divided differences.

Theorem 1.1. *Let f, g be two $n-1$ times continuously differentiable functions on the interval $I \subseteq \mathbb{R}$ and n times differentiable on the interior I° of I, with the properties that $g^{(n)}(x) > 0$ on I°, and that the function $\frac{f^{(n)}(x)}{g^{(n)}(x)}$ is bounded on I°. Then for $x_i, y_i \in I$ $(i = 1, 2, \dots, n)$ such that $x_i \geq y_i$ for all $i = 1, 2, \dots, n$ and $\sum_{i=1}^n (x_i - y_i) \neq 0$, the following estimation holds true:*

$$\inf_{x \in I^\circ} \frac{f^{(n)}(x)}{g^{(n)}(x)} \leq \frac{[x_1, \dots, x_n]f - [y_1, \dots, y_n]f}{[x_1, \dots, x_n]g - [y_1, \dots, y_n]g} \leq \sup_{x \in I^\circ} \frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

This theorem generalized the following result obtained by [2].

Corollary 1.2. *Let f, g be two continuously differentiable functions on $[a, b]$ and twice differentiable on (a, b),with the properties that $g'' > 0$ on (a, b), and that the function $\frac{f''}{g''}$ is bounded on (a, b). Then for $a < c \leq d < b$, the following estimation holds:*

$$\inf_{x \in (a,b)} \frac{f''(x)}{g''(x)} \leq \frac{\frac{f(b)-f(d)}{b-d} - \frac{f(c)-f(a)}{c-a}}{\frac{g(b)-g(d)}{b-d} - \frac{g(c)-g(a)}{c-a}} \leq \sup_{x \in (a,b)} \frac{f''(x)}{g''(x)}.$$

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It is worth noting that the technique of the proof for Theorem 1.1 in [3] is very natural and useful. In this paper, using the technique and following the definition of mixed partial divided difference proposed by [1], we present a similar inequality for divided differences in the multivariate case.

2. NOTATIONS AND DEFINITIONS

The following notations will be used in this paper.

We denote by $\mathbb{R}^m$ the $m-$dimensional Euclidean space. Let $x \in \mathbb{R}^m$ be a vector denoted by $(x_1, x_2, \ldots, x_m)$. Let $\mathbb{N}_0$ be the set of nonnegative integers. Then it is obvious that $\mathbb{N}_0^m \subseteq \mathbb{R}^m$. Denote by $e^i \in \mathbb{N}_0^m$ a unit vector whose jth component is δ_{ij}, where

$$\delta_{ij} = \begin{cases} 0, & j \neq i; \\ 1, & j = i. \end{cases}$$

Let $0^0 = 1$. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}_0^m$, we define $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$, and then we have $x_i = x^{e^i}$. Define $|\alpha| = \sum_{i=1}^m \alpha_i$, $\alpha! = \prod_{i=1}^m \alpha_i!$. For $x, y \in \mathbb{R}^m$, we denote $x \geq y$, if $x_i \geq y_i$, $i = 1, 2, \ldots, m$.

Further, let

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m}$$

be a mixed partial differential operator of order $|\alpha|$.

For $x^0, x^1, \ldots, x^n \in \mathbb{R}^m$, we denote by

$$\left\langle x^0, x^1, \ldots, x^n \right\rangle = \left\{ \left(1 - \sum_{j=1}^n t_j\right) x^0 + t_1 x^1 + \cdots + t_n x^n \middle|\, t_j \geq 0,\ \sum_{j=1}^n t_j \leq 1 \right\}$$

the convex hull of $x^0, x^1, \ldots, x^n \in \mathbb{R}^m$. Then according to the Hermite-Genocchi formula for univariate divided difference, the multivariate divided difference (or mixed partial divided difference) of order n can be defined by the following formula.

Definition 2.1 ([1], see also [4, 5]). Let $\alpha \in \mathbb{N}_0^m$ with $|\alpha| = n$, and $x^0, x^1, \ldots, x^n \in \mathbb{R}^m$. Then the mixed partial divided difference of order n of f is defined by

$$[x^0, x^1, \ldots, x^n]_\alpha f = \int_{S^n} D^\alpha f\left(\left(1 - \sum_{j=1}^n t_j\right) x^0 + t_1 x^1 + \cdots + t_n x^n\right) dt_1 dt_2 \ldots dt_n,$$

where

$$S^n = \left\{ (t_1, t_2, \ldots, t_n) \middle|\, t_j \geq 0,\ j = 1, 2, \ldots, n;\ \sum_{j=1}^n t_j \leq 1 \right\}.$$

It is easy to see that if we let $m = 1$, then $[x^0, x^1, \ldots, x^n]_\alpha f$ is the ordinary divided difference in the univariate case. By the definition of the mixed partial divided difference, we also conclude that

$$[x^{\sigma_0}, x^{\sigma_1}, \ldots, x^{\sigma_n}]_\alpha f = [x^0, x^1, \ldots, x^n]_\alpha f$$

if $(\sigma_0, \sigma_1, \ldots, \sigma_n)$ is a permutation of $(0, 1, \ldots, n)$. Finally, we give another definition to end this section.

Definition 2.2 ([4, 5]). Let $\alpha \in \mathbb{N}_0^m$ with $|\alpha| = n$, and $x^0, x^1, \ldots, x^n \in \mathbb{R}^m$. Then the Newton fundamental functions are defined by

$$\omega_\alpha(x, \{x^j\}_{j=0}^{n-1}) = \begin{cases} 1, & n = 0, \\ \sum\limits_{e^{i_1} + \cdots + e^{i_n} = \alpha} \prod_{j=1}^n (x - x^{j-1})^{e^{i_j}}, & n > 0. \end{cases}$$

3. MAIN RESULT

We start this section with two lemmas. Using the definition of the mixed partial divided difference of f we have the following lemma.

Lemma 3.1 (cf. [4, 5]). *Let $\alpha \in \mathbb{N}_0^m$ with $|\alpha| = n$. If $f \in C^n(\langle x^0, x^1, \dots, x^n \rangle)$, then there exists a point $\xi \in \langle x^0, x^1, \dots, x^n \rangle$ such that*

$$[x^0, x^1, \dots, x^n]_\alpha f = \frac{1}{n!} D^\alpha f(\xi).$$

Also using the definition, we have the recurrence relations of the divided differences.

Lemma 3.2. *For $\beta \in \mathbb{N}_0^m$ with $|\beta| = n-1$, we have*

$$[x^1, x^2, \dots, x^n]_\beta f - [x^0, x^1, \dots, x^{n-1}]_\beta f = \sum_{i=1}^m (x^n - x^0)^{e^i} [x^0, x^1, \dots, x^n]_{\beta+e^i} f.$$

Proof. By the chain rule for the derivative of the composite function, we have

$$\begin{aligned} &\frac{\partial}{\partial t_n} D^\beta f \left(\left(1 - \sum_{j=1}^n t_j \right) x^0 + \dots + t_n x^n \right) \\ &\qquad = \sum_{i=1}^m D^{\beta+e^i} f \left(\left(1 - \sum_{j=1}^n t_j \right) x^0 + \dots + t_n x^n \right) (x^n - x^0)^{e^i}. \end{aligned}$$

Hence,

$$\begin{aligned} &\int_0^{1-\sum_{j=1}^{n-1} t_j} \sum_{i=1}^m D^{\beta+e^i} f \left(\left(1 - \sum_{j=1}^n t_j \right) x^0 + \dots + t_n x^n \right) (x^n - x^0)^{e^i} dt_n \\ &\qquad = D^\beta f \left(t_1 x^1 + \dots + \left(1 - \sum_{j=1}^{n-1} t_j \right) x^n \right) \\ &\qquad\qquad - D^\beta f \left(\left(1 - \sum_{j=1}^{n-1} t_j \right) x^0 + \dots + t_{n-1} x^{n-1} \right). \end{aligned}$$

Thus,

$$\begin{aligned} &\int_{S^n} \sum_{i=1}^m D^{\beta+e^i} f \left(\left(1 - \sum_{j=1}^n t_j \right) x^0 + \dots + t_n x^n \right) (x^n - x^0)^{e^i} dt_n \dots dt_1 \\ &\qquad = \int_{S^{n-1}} D^\beta f \left(t_1 x^1 + \dots + \left(1 - \sum_{j=1}^{n-1} t_j \right) x^n \right) dt_1 \dots dt_{n-1} \\ &\qquad\qquad - \int_{S^{n-1}} D^\beta f \left(\left(1 - \sum_{j=1}^{n-1} t_j \right) x^0 + \dots + t_{n-1} x^{n-1} \right) dt_1 \dots dt_{n-1}, \end{aligned}$$

which implies

$$\sum_{i=1}^m (x^n - x^0)^{e^i} [x^0, x^1, \dots, x^n]_{\beta+e^i} f = [x^1, x^2, \dots, x^n]_\beta f - [x^0, x^1, \dots, x^{n-1}]_\beta f.$$

This completes the proof. □

Let $\langle x^0, \ldots, x^n, y^0, \ldots, y^n\rangle$ be the convex hull of $x^0, x^1, \ldots, x^n, y^0, y^1, \ldots, y^n$. Now, we state our main theorem as follows.

Theorem 3.3. *Let $f, g \in C^{n+1}(\langle x^0, \ldots, x^n, y^0, \ldots, y^n\rangle)$, $\alpha \in \mathbb{N}_0^m$, and $|\alpha| = n$. If for all $z \in \langle x^0, \ldots, x^n, y^0, \ldots, y^n\rangle$ we have $D^{\alpha+e^i}g(z) > 0$ $(i = 1, 2, \ldots, m)$ and for all x^j, y^j $(j = 0, 1, \ldots, n)$ we have $x^j \geq y^j$ and $\sum_{j=0}^{n}\sum_{i=1}^{m}(x^j - y^j)^{e^i} \neq 0$, then*

$$L \leq \frac{[x^0, x^1, \ldots, x^n]_\alpha f - [y^0, y^1, \ldots, y^n]_\alpha f}{[x^0, x^1, \ldots, x^n]_\alpha g - [y^0, y^1, \ldots, y^n]_\alpha g} \leq U,$$

where

$$L = \min_{1\leq i\leq m} \inf_{z\in\langle x^0,\ldots,x^n,y^0,\ldots,y^n\rangle} \frac{D^{\alpha+e^i}f(z)}{D^{\alpha+e^i}g(z)},$$
$$U = \max_{1\leq i\leq m} \sup_{z\in\langle x^0,\ldots,x^n,y^0,\ldots,y^n\rangle} \frac{D^{\alpha+e^i}f(z)}{D^{\alpha+e^i}g(z)}.$$

Proof. It is evident that

$$\begin{aligned} L &\leq \inf_{z\in\langle x^0,\ldots,x^n,y^0,\ldots,y^n\rangle} \frac{D^{\alpha+e^i}f(z)}{D^{\alpha+e^i}g(z)} \\ &\leq \frac{D^{\alpha+e^i}f(z)}{D^{\alpha+e^i}g(z)} \leq \sup_{z\in\langle x^0,\ldots,x^n,y^0,\ldots,y^n\rangle} \frac{D^{\alpha+e^i}f(z)}{D^{\alpha+e^i}g(z)} \leq U. \end{aligned}$$

Since $D^{\alpha+e^i}g(z) > 0,\ 1 \leq i \leq m$, then

$$LD^{\alpha+e^i}g(z) \leq D^{\alpha+e^i}f(z) \leq UD^{\alpha+e^i}g(z). \tag{3.1}$$

Let

$$\bar{x} = \left(1 - \sum_{j=1}^{n} t_j\right)x^0 + t_1x^1 + \cdots + t_nx^n,$$
$$\bar{y} = \left(1 - \sum_{j=1}^{n} t_j\right)y^0 + t_1y^1 + \cdots + t_ny^n.$$

Since $f, g \in C^{n+1}(\langle x^0, \ldots, x^n, y^0, \ldots, y^n\rangle)$, $D^{\alpha+e^i}g(z)$ and $D^{\alpha+e^i}f(z)$ are continuous on each of the contours between the points $\bar{y}$ and $\bar{x}$. Then we can find three line integrals satisfying

$$L\int_{\bar{y}}^{\bar{x}}\sum_{i=1}^{m} D^{\alpha+e^i}g(z)dz_i \leq \int_{\bar{y}}^{\bar{x}}\sum_{i=1}^{m} D^{\alpha+e^i}f(z)dz_i \leq U\int_{\bar{y}}^{\bar{x}}\sum_{i=1}^{m} D^{\alpha+e^i}g(z)dz_i.$$

This implies that

$$L[D^\alpha g(\bar{x}) - D^\alpha g(\bar{y})] \leq D^\alpha f(\bar{x}) - D^\alpha f(\bar{y}) \leq U[D^\alpha g(\bar{x}) - D^\alpha g(\bar{y})]. \tag{3.2}$$

Integrating (3.2) with respect to $t_1, t_2, \ldots, t_n$ over the n-dimensional simplex S_n as defined in the previous section, we arrive at

$$\begin{aligned} L([x^0, x^1, \ldots,&x^n]_\alpha g - [y^0, y^1, \ldots, y^n]_\alpha g) \\ &\leq [x^0, x^1, \ldots, x^n]_\alpha f - [y^0, y^1, \ldots, y^n]_\alpha f \\ &\leq U([x^0, x^1, \ldots, x^n]_\alpha g - [y^0, y^1, \ldots, y^n]_\alpha g). \end{aligned}$$

Using Lemmas 3.1 and 3.2, we have

$$\begin{aligned}[x^0, x^1, \dots, x^n]_\alpha g - [y^0, y^1, \dots, y^n]_\alpha g &= \sum_{j=0}^{n} \sum_{i=1}^{m} (x^j - y^j)^{e^i} [y^0, \dots, y^j, x^j, \dots, x^n]_{\alpha+e^i} g \\ &= \frac{1}{(n+1)!} \sum_{j=0}^{n} \sum_{i=1}^{m} (x^j - y^j)^{e^i} D^{\alpha+e^i} g(\xi_{i,j}),\end{aligned}$$

where $\xi_{i,j} \in \langle x^0, \dots, x^n, y^0, \dots, y^n \rangle$. Since $x^j \geq y^j$, $\sum_{j=0}^{n} \sum_{i=1}^{m} (x^j - y^j)^{e^i} \neq 0$ and $D^{\alpha+e^i} g(z) > 0$, we have

$$[x^0, x^1, \dots, x^n]_\alpha g - [y^0, y^1, \dots, y^n]_\alpha g > 0.$$

Thus,

$$L \leq \frac{[x^0, x^1, \dots, x^n]_\alpha f - [y^0, y^1, \dots, y^n]_\alpha f}{[x^0, x^1, \dots, x^n]_\alpha g - [y^0, y^1, \dots, y^n]_\alpha g} \leq U.$$

This completes the proof. □

Considering $p_i(z) = \sum_{e^{i_0}+e^{i_1}+\dots+e^{i_n}=\alpha+e^i} z^{\alpha+e^i}$, $i = 1, 2, \dots, m$, we can obtain that

$$p_i(z) = \omega_{\alpha+e^i}(z, \{0\}_{j=0}^{n}).$$

Further, let

$$p(z) = \frac{1}{(n+1)!} \sum_{i=1}^{m} p_i(z).$$

By calculating, we have

$$p(z) = \sum_{i=1}^{m} \frac{1}{(\alpha + e^i)!} z^{\alpha+e^i}.$$

This implies that, for $1 \leq i \leq m$,

$$D^{\alpha+e^i} p(z) = 1 > 0,$$

and

$$D^\alpha p(z) = \sum_{i=1}^{m} z^{e^i}.$$

Then

$$\begin{aligned}[x^0, x^1, \dots, x^n]_\alpha p &= \int_{S^n} D^\alpha p \left(\left(1 - \sum_{j=1}^{n} t_j\right) x^0 + t_1 x^1 + \dots + t_n x^n \right) dt_1 dt_2 \cdots dt_n \\ &= \sum_{i=1}^{m} \int_{S^n} \left(\left(1 - \sum_{j=1}^{n} t_j\right) x^0 + t_1 x^1 + \dots + t_n x^n \right)^{e^i} dt_1 dt_2 \cdots dt_n \\ &= \sum_{i=1}^{m} \int_{S^n} \left(\left(1 - \sum_{j=1}^{n} t_j\right) (x^0)^{e^i} + t_1 (x^1)^{e^i} + \dots + t_n (x^n)^{e^i} \right) dt_1 dt_2 \cdots dt_n \\ &= \frac{1}{(n+1)!} \sum_{j=0}^{n} \sum_{i=1}^{m} (x^j)^{e^i}.\end{aligned}$$

Therefore, if we take $g(z) = p(z)$ in Theorem 3.3, we have the following corollary.

Corollary 3.4. *Let* $f \in C^{n+1}\left(\langle x^0, \ldots, x^n, y^0, \ldots, y^n \rangle\right)$, $\alpha \in \mathbb{N}_0^m$, *and* $|\alpha| = n$. *If for all* x^j, y^j $(j = 0, 1, \ldots, n)$ *we have* $x^j \geq y^j$ *and* $\sum_{j=0}^{n} \sum_{i=1}^{m} (x^j - y^j)^{e^i} \neq 0$, *then*

$$L' \leq [x^0, x^1, \ldots, x^n]_\alpha f - [y^0, y^1, \ldots, y^n]_\alpha f \leq U',$$

where

$$L' = \frac{1}{(n+1)!} \min_{1 \leq i \leq m} \inf_{z \in \langle x^0, \ldots, x^n, y^0, \ldots, y^n \rangle} D^{\alpha + e^i} f(z) \sum_{j=0}^{n} \sum_{i=1}^{m} (x^j - y^j)^{e^i},$$

$$U' = \frac{1}{(n+1)!} \max_{1 \leq i \leq m} \sup_{z \in \langle x^0, \ldots, x^n, y^0, \ldots, y^n \rangle} D^{\alpha + e^i} f(z) \sum_{j=0}^{n} \sum_{i=1}^{m} (x^j - y^j)^{e^i}.$$

In fact, from the procedure of the proof of Theorem 3.3, it is not difficult to find that the conditions of the corollary can be weakened. If we replace $x^j \geq y^j$ and $\sum_{j=0}^{n} \sum_{i=1}^{m} (x^j - y^j)^{e^i} \neq 0$ by $\sum_{j=0}^{n} \sum_{i=1}^{m} (x^j - y^j)^{e^i} > 0$, the corollary holds true as well.

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