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NOTE ON AN OPEN PROBLEM

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ABSTRACT. The aim of this short note is to establish an integral inequality and its reverse which give an affirmative answer to an open problem posed by QUÔC ANH NGÔ, DU DUC THANG, TRANT TAT DAT, and DANG ANH TUAN, in the paper [*Notes on an integral inequality,* J. Ineq. Pure and Appl. Math., 7(4)(2006), Art. 120.]

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1. INTRODUCTION

Very recently, in the paper [1] the authors studied some integral inequalities and proposed the following open problem:

Problem 1.1. Let f be a continuous function on $[0, 1]$ satisfying

$$\int_x^1 f(t)dt \geq \int_x^1 tdt, \qquad \forall x \in [0, 1]. \tag{1.1}$$

Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx \tag{1.2}$$

hold for α and β?

This type of integral inequality is a complement, variant and continuation of Qi's inequality [2]. Before giving an affirmative answer to Problem 1.1 and its reverse, we establish the following essential lemma:

Lemma 1.1. *Let $f(x)$ be nonnegative function, continuous on $[a, b]$ and differentiable on (a, b). If $\int_x^b f(t)dt \leq \int_x^b (t-a)dt,\ \forall x \in [a, b]$, then*

$$f(x) \leq x - a. \tag{1.3}$$

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If $\int_x^b f(t)dt \geq \int_x^b (t-a)dt,\ \forall x \in [a,b]$, *then*

$$f(x) \geq x - a. \tag{1.4}$$

Proof. In order to prove (1.3), set

$$G(x) = \int_x^b [f(t) - (t-a)]dt \leq 0, \quad \forall x \in [a,b],$$

we have

$$G^{'}(x) = x - a - f(x), \quad \forall x \in [a,b].$$

We shall give an indirect proof, we suppose $f(x) \geq x - a$, then $G^{'}(x) \leq 0$, $G(x)$ decreases, and $G(x) \geq 0$, because of $G(b) = 0$. This contradiction establishes (1.3).

The proof of (1.4) is the same as the proof of (1.3). □

Now, our results can be stated as follows:

2. MAIN RESULTS

Theorem 2.1. *Let $f(x)$ be a nonnegative function, continuous on $[a,b]$ and differentiable on (a,b), and let α and β be positive numbers.*
If $\int_x^b f(t)dt \leq \int_x^b (t-a)dt,\ \forall x \in [a,b]$, *then*

$$\int_a^b f^{\alpha+\beta}(x)dx \leq \int_a^b (x-a)^{\alpha} f^{\beta}(x)dx. \tag{2.1}$$

If $\int_x^b f(t)dt \geq \int_x^b (t-a)dt,\ \forall x \in [a,b]$, *then*

$$\int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b (x-a)^{\alpha} f^{\beta}(x)dx. \tag{2.2}$$

Proof. Set

$$F(x) = \int_a^x \left[f^{\alpha+\beta}(t) - (t-a)^{\alpha} f^{\beta}(t)dt\right], \quad \forall x \in [a,b].$$

We can see that

$$F^{'}(x) = f^{\alpha+\beta}(x) - (x-a)^{\alpha} f^{\beta}(x),$$

so that

$$F^{'}(x) = \left[f^{\alpha}(x) - (x-a)^{\alpha}\right] f^{\beta}(x).$$

If

$$\int_x^b f(t)dt \leq \int_x^b (t-a)dt, \quad \forall x \in [a,b],$$

and from (1.3) of Lemma 1.1, we have, $f(x) \leq (x-a)$, so that $f^{\alpha}(x) \leq (x-a)^{\alpha}$. Thus $F^{'}(x) \leq 0$, and $F(x)$ is decreasing on $[a,b]$. Since $F(a) = 0$, we have $F(x) \leq 0,\ \forall x \in [a,b]$, which gives the inequality (2.1).

When

$$\int_x^b f(t)dt \geq \int_x^b (t-a)dt, \quad \forall x \in [a,b],$$

we have from (1.4) $f^{\alpha}(x) \geq (x-a)^{\alpha}$, and the rest of the proof is the same as that of (2.1). □

Remark 2.2. If $f(x) = 0$ or $f(x) = x - a$, the equality in (2.1) holds.

Now we establish new integral inequalities similar to (2.1) and (2.2) involving n functions: $f_i(x),\ i = 1, \ldots, n$.

Theorem 2.3. *Let* $f_i(x),\ i = 1, 2, \dots, n$ *be nonnegative functions, continuous on* $[a, b]$ *and differentiable on* (a, b), *and let* $\alpha_i,\ \beta_i,\ i = 1, \dots, n$ *be positive numbers.*
If

$$\int_x^b f_i(t)dt \le \int_x^b (t-a)dt, \quad \forall x \in [a, b],$$

then

$$\int_a^b \prod_{i=1}^n f_i^{\alpha_i+\beta_i}(x)dx \le \int_a^b (x-a)^{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n f_i^{\beta_i}(x)dx. \tag{2.3}$$

If

$$\int_x^b f_i(t)dt \ge \int_x^b (t-a)dt, \quad \forall x \in [a, b],$$

then

$$\int_a^b \prod_{i=1}^n f_i^{\alpha_i+\beta_i}(x)dx \ge \int_a^b (x-a)^{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n f_i^{\beta_i}(x)dx. \tag{2.4}$$

Proof. As in proof of (2.1) and (2.2), we let

$$F(x) = \int_a^x \left[\prod_{i=1}^n f_i^{\alpha_i+\beta_i}(t) - (t-a)^{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n f_i^{\beta_i}(t)\right] dt,$$

thus

$$F^{'}(x) = \left[\prod_{i=1}^n f_i^{\alpha_i}(x) - (x-a)^{\sum_{i=1}^n \alpha_i}\right] \prod_{i=1}^n f_i^{\beta_i}(x),$$

when

$$\int_x^b f_i(t)dt \le \int_x^b (t-a)dt \quad \left(\text{resp. } \int_x^b f_i(t)dt \ge \int_x^b (t-a)dt\right), \quad \forall x \in [a, b],$$

using Lemma 1.1, we have

$$f_i(x) \le x - a \quad (\text{resp. } f_i(x) \ge x - a), \quad i = 1, 2, \dots, n,$$

and

$$f_i^{\alpha_i}(x) \le (x-a)^{\alpha_i} \quad (\text{resp. } f_i^{\alpha_i}(x) \ge (x-a)^{\alpha_i}), \quad i = 1, 2, \dots, n,$$

thus

$$\prod_{i=1}^n f_i^{\alpha_i}(x) \le (x-a)^{\sum_{i=1}^n \alpha_i} \quad \left(\text{resp. } \prod_{i=1}^n f_i^{\alpha_i}(x) \ge (x-a)^{\sum_{i=1}^n \alpha_i}\right).$$

The rest of the proof is the same as in Theorem 2.1, and we omit (2.3) and (2.4). □

In order to illustrate a possible practical use of this result, we shall give in the following a simple example in which we can apply Theorem 2.1.

Example 2.1. For $\alpha = \beta = 1$:

i) Let $f(t) = \cos t + t$ on $[0, \frac{\pi}{2}]$, we see that all the conditions of Theorem 2.1 are fulfilled. Indeed,

$$\begin{aligned}\int_x^{\frac{\pi}{2}} (\cos t + t)\, dt &= 1 - \sin x + \frac{1}{2}\left(\frac{\pi^2}{4} - x^2\right) \\ &\ge \int_x^{\frac{\pi}{2}} tdt = \frac{1}{2}\left(\frac{\pi^2}{4} - x^2\right), \quad \forall x \in \left[0, \frac{\pi}{2}\right],\end{aligned}$$

and straightforward computation yields

$$\int_0^{\frac{\pi}{2}} (\cos x + x)^2 \, dx = \frac{5\pi}{4} + \frac{\pi^3}{24} - 2$$
$$> \int_0^{\frac{\pi}{2}} x (\cos x + x) \, dx$$
$$= \frac{\pi}{2} + \frac{\pi^3}{24} - 1.$$

That is,

$$\int_0^{\frac{\pi}{2}} f^2(x) dx > \int_0^{\frac{\pi}{2}} x f(x) dx.$$

ii) Let $f(t) = t - \frac{\pi}{2} + \cos t$ on $[\frac{\pi}{2}, \pi]$, all the conditions of Theorem 2.1 be satisfied. We see that

$$\int_x^{\pi} \left(t - \frac{\pi}{2} + \cos t \right) dt \leq \int_x^{\pi} \left(t - \frac{\pi}{2} \right) dt, \quad \forall x \in \left[\frac{\pi}{2}, \pi \right],$$

which is equivalent to

$$\int_x^{\pi} \cos t dt = - \sin x \leq 0, \quad x \in \left[\frac{\pi}{2}, \pi \right],$$

and direct computation yields

$$\int_{\frac{\pi}{2}}^{\pi} \left(x - \frac{\pi}{2} + \cos x \right)^2 dx = \frac{\pi}{4} + \frac{\pi^3}{24} - 2$$
$$< \int_{\frac{\pi}{2}}^{\pi} \left(x - \frac{\pi}{2} \right) f(x) dx$$
$$= \frac{\pi^3}{24} - 1.$$

That is

$$\int_{\frac{\pi}{2}}^{\pi} f^2(x) dx < \int_{\frac{\pi}{2}}^{\pi} \left(x - \frac{\pi}{2} \right) f(x) dx.$$

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