A MATRIX INEQUALITY FOR MÖBIUS FUNCTIONS

OLIVIER BORDELLÈS
2 allée de la combe
43000 AIGUILHE (France)
EMail: borde43@wanadoo.fr

BENOIT CLOITRE
19 rue Louise Michel
92300 LEVALLOIS-PERRET (France)
EMail: benoit7848c@orange.fr

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Abstract: The aim of this note is the study of an integer matrix whose determinant is related to the Möbius function. We derive a number-theoretic inequality involving sums of a certain class of Möbius functions and obtain a sufficient condition for the Riemann hypothesis depending on an integer triangular matrix. We also provide an alternative proof of Redheffer's theorem based upon a LU decomposition of the Redheffer's matrix.

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1. Introduction

In what follows, $[t]$ is the integer part of t and, for integers $i, j \geqslant 1$, we set $\bmod(j,i) := j - i[j/i]$.

1.1. Arithmetic motivation

In 1977, Redheffer [5] introduced the matrix $R_n = (r_{ij}) \in \mathcal{M}_n(\{0,1\})$ defined by

$$r_{ij} = \begin{cases} 1, & \text{if } i \mid j \text{ or } j = 1; \\ 0, & \text{otherwise} \end{cases}$$

and has shown that (see appendix)

$$\det R_n = M(n) := \sum_{k=1}^{n} \mu(k),$$

where μ is the Möbius function and M is the Mertens function. This determinant is clearly related to two of the most famous problems in number theory, the Prime Number Theorem (PNT) and the Riemann Hypothesis (RH). Indeed, it is well-known that

$$\text{PNT} \iff M(n) = o(n) \quad \text{and} \quad \text{RH} \iff M(n) = O_\varepsilon\left(n^{1/2+\varepsilon}\right)$$

(for any $\varepsilon > 0$). These estimations for $|\det R_n|$ remain unproven, but Vaughan [6] showed that 1 is an eigenvalue of R_n with (algebraic) multiplicity $n - \left[\frac{\log n}{\log 2}\right] - 1$, that R_n has two "dominant" eigenvalues $\lambda_\pm$ such that $|\lambda_\pm| \asymp n^{1/2}$, and that the others eigenvalues satisfy $\lambda \ll (\log n)^{2/5}$.

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It should be mentioned that Hadamard's inequality, which states that

$$|\det R_n|^2 \leqslant \prod_{i=1}^{n} \|L_i\|_2^2,$$

where L_i is the ith row of R_n and $\|\cdot\|_2$ is the euclidean norm on $\mathbb{C}^n$, gives

$$(M(n))^2 \leqslant n \prod_{i=2}^{n} \left(1 + \left[\frac{n}{i}\right]\right) = 2^{n-[n/2]} n \prod_{i=2}^{[n/2]} \left(1 + \left[\frac{n}{i}\right]\right) \leqslant 2^{n-[n/2]} \binom{n + [n/2]}{n},$$

which is very far from the trivial bound $|M(n)| \leqslant n$ so that it seems likely that general matrix analysis tools cannot be used to provide an elementary proof of the PNT.

In this work we study an integer matrix whose determinant is also related to the Möbius function. This will provide a new criteria for the PNT and the RH (see Corollary 2.3 below). In an attempt to go further, we will prove an inequality for a class of Möbius functions and deduce a sufficient condition for the PNT and the RH in terms of the smallest singular value of a triangular matrix.

1.2. Convolution identities for the Möbius function

The function μ, which plays an important role in number theory, satisfies the following well-known convolution identity.

Lemma 1.1. *For every real number $x \geqslant 1$ we have*

$$\sum_{k \leqslant x} \mu(k) \left[\frac{x}{k}\right] = \sum_{d \leqslant x} M\left(\frac{x}{d}\right) = 1.$$

One can find a proof for example in [1]. The following corollary will be useful.

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Corollary 1.2. *For every integer $j \geqslant 1$ we have*

(i)

$$\sum_{k=1}^{j} \frac{\mu(k) \bmod (j,k)}{k} = j \sum_{k=1}^{j} \frac{\mu(k)}{k} - 1.$$

(ii)

$$\sum_{k=1}^{j} \left(\sum_{h=1}^{k} \frac{\mu(h)}{h} \right) (\operatorname{mod}(j,k+1) - \operatorname{mod}(j,k)) = 1.$$

Proof.

(i) We have

$$\sum_{k=1}^{j} \frac{\mu(k) \bmod (j,k)}{k} = \sum_{k=1}^{j} \frac{\mu(k)}{k} \left(j - k \left[\frac{j}{k} \right] \right) = j \sum_{k=1}^{j} \frac{\mu(k)}{k} - \sum_{k=1}^{j} \mu(k) \left[\frac{j}{k} \right]$$

and we conclude with Lemma 1.1.

(ii) Using Abel summation we get

$$\begin{aligned} &\sum_{k=1}^{j} \left(\sum_{h=1}^{k} \frac{\mu(h)}{h} \right) (\operatorname{mod}(j,k+1) - \operatorname{mod}(j,k)) \\ &\quad = \left(\sum_{h=1}^{j} \frac{\mu(h)}{h} \right) \sum_{k=1}^{j} (\operatorname{mod}(j,k+1) - \operatorname{mod}(j,k)) \\ &\qquad - \sum_{k=1}^{j-1} \left(\sum_{h=1}^{k+1} \frac{\mu(h)}{h} - \sum_{h=1}^{k} \frac{\mu(h)}{h} \right) \sum_{m=1}^{k} (\operatorname{mod}(j,m+1) - \operatorname{mod}(j,m)) \end{aligned}$$

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$$= j \sum_{k=1}^{j} \frac{\mu(k)}{k} - \sum_{k=1}^{j-1} \frac{\mu(k+1) \bmod (j, k+1)}{k+1}$$

$$= j \sum_{k=1}^{j} \frac{\mu(k)}{k} - \sum_{k=1}^{j} \frac{\mu(k) \bmod (j, k)}{k}$$

and we conclude using (i).

□

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2. An Integer Matrix Related to the Möbius Function

We now consider the matrix $\Gamma_n = (\gamma_{ij})$ defined by

$$\gamma_{ij} = \begin{cases} \operatorname{mod}(j,2) - 1, & \text{if } i = 1 \quad \text{and} \quad 2 \leqslant j \leqslant n; \\ \operatorname{mod}(j, i+1) - \operatorname{mod}(j,i), & \text{if } 2 \leqslant i \leqslant n-1 \quad \text{and} \quad 1 \leqslant j \leqslant n; \\ 1, & \text{if } (i,j) \in \{(1,1),(n,1)\}; \\ 0, & \text{otherwise.} \end{cases}$$

The matrix Γ_n is almost upper triangular except the entry $\gamma_{n1} = 1$ which is nonzero. Note that it is easy to check that $|\gamma_{ij}| \leqslant i$ for every $1 \leqslant i, j \leqslant n$ and that $\gamma_{ij} = -1$ if $[j/2] < i < j$.

Example 2.1.

$$\Gamma_8 = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 2 & -1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 3 & -1 & -1 & 2 & 2 & -2 \\ 0 & 0 & 0 & 4 & -1 & -1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 5 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 6 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2.1. The determinant of Γ_n

Theorem 2.1. *Let $n \geqslant 2$ be an integer and Γ_n defined as above. Then we have*

$$\det \Gamma_n = n! \sum_{k=1}^{n} \frac{\mu(k)}{k}.$$

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A possible proof of Theorem 2.1 uses a LU decomposition of the matrix Γ_n. Let $L_n = (l_{ij})$ and $U_n = (u_{ij})$ be the matrices defined by

$$u_{ij} = \begin{cases} 0, & \text{if } (i,j) = (n,1); \\ 1, & \text{if } (i,j) = (n,n); \\ \gamma_{ij}, & \text{otherwise} \end{cases}$$

and

$$l_{ij} = \begin{cases} 1, & \text{if } 1 \leqslant i = j \leqslant n-1; \\ \sum_{k=1}^{j} \frac{\mu(k)}{k}, & \text{if } i = n \quad \text{and} \quad 1 \leqslant j \leqslant n-1; \\ n \sum_{k=1}^{n} \frac{\mu(k)}{k}, & \text{if } (i,j) = (n,n); \\ 0, & \text{otherwise.} \end{cases}$$

The proof of Theorem 2.1 follows from the lemma below.

Lemma 2.2. *We have* $\Gamma_n = L_n U_n$.

Proof. Set $L_n U_n = (x_{ij})$. When $i = 1$ we immediately obtain $x_{1j} = u_{1j} = \gamma_{1j}$. We also have

$$x_{n1} = \sum_{k=1}^{n} l_{nk} u_{k1} = l_{n1} u_{11} = 1 = \gamma_{n1}.$$

Moreover, using Corollary 1.2 (ii) we get for $i = n$ and $2 \leqslant j \leqslant n-1$

$$x_{nj} = \sum_{k=1}^{n} l_{nk} u_{kj} = l_{n1} u_{1j} + \sum_{k=2}^{n} l_{nk} u_{kj}$$

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$$= \operatorname{mod}(j,2) - 1 + \sum_{k=2}^{j} \left(\sum_{h=1}^{k} \frac{\mu(h)}{h} \right) (\operatorname{mod}(j,k+1) - \operatorname{mod}(j,k))$$

$$= \sum_{k=1}^{j} \left(\sum_{h=1}^{k} \frac{\mu(h)}{h} \right) (\operatorname{mod}(j,k+1) - \operatorname{mod}(j,k)) - 1 = 0 = \gamma_{nj}$$

and, for $(i,j) = (n,n)$, we have similarly

$$x_{nn} = \sum_{k=1}^{n} l_{nk} u_{kn} = l_{n1} u_{1n} + \sum_{k=2}^{n-1} l_{nk} u_{kn} + l_{nn} u_{nn}$$

$$= \operatorname{mod}(n,2) - 1 + \sum_{k=2}^{n-1} \left(\sum_{h=1}^{k} \frac{\mu(h)}{h} \right) (\operatorname{mod}(n,k+1) - \operatorname{mod}(n,k))$$

$$+ n \sum_{k=1}^{n} \frac{\mu(k)}{k}$$

$$= \sum_{k=1}^{n} \left(\sum_{h=1}^{k} \frac{\mu(h)}{h} \right) (\operatorname{mod}(n,k+1) - \operatorname{mod}(n,k)) - 1 = 0 = \gamma_{nn}.$$

Finally, for $2 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant n$, we get

$$x_{ij} = \sum_{k=1}^{n} l_{ik} u_{kj} = l_{ii} u_{ij} = u_{ij} = \gamma_{ij}.$$

□

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Example 2.2.

$$\Gamma_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{30} & \frac{2}{15} & -\frac{1}{105} & -\frac{8}{105} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 2 & -1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 3 & -1 & -1 & 2 & 2 & -2 \\ 0 & 0 & 0 & 4 & -1 & -1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 5 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 6 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 2.1 now immediately follows from

$$\det \Gamma_n = \det L_n \det U_n = (n-1)! \det L_n = n! \sum_{k=1}^{n} \frac{\mu(k)}{k}.$$

We easily deduce the following criteria for the PNT and the RH.

Corollary 2.3. *For any real number $\varepsilon > 0$ we have*

$$\mathrm{PNT} \iff \det \Gamma_n = o(n!) \quad \textit{and} \quad \mathrm{RH} \iff \det \Gamma_n = O_\varepsilon(n^{-1/2+\varepsilon} n!).$$

2.2. A sufficient condition for the PNT and the RH

2.2.1. Computation of U_n^{-1}

The inverse of U_n uses a Möbius-type function denoted by μ_i which we define below.

Definition 2.4. *Set $\mu_1 = \mu$ the well-known Möbius function and, for any integer*

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$i \geqslant 2$, we define the Möbius function μ_i by $\mu_i(1) = 1$ and, for any integer $m \geqslant 2$, by

$$\mu_i(m) := \begin{cases} \mu\left(\frac{m}{i}\right), & \text{if } i \mid m \text{ and } (i+1) \nmid m; \\ -\mu\left(\frac{m}{i+1}\right), & \text{if } (i+1) \mid m \text{ and } i \nmid m; \\ \mu\left(\frac{m}{i}\right) - \mu\left(\frac{m}{i+1}\right), & \text{if } i(i+1) \mid m; \\ 0, & \text{otherwise.} \end{cases}$$

The following result completes and generalizes Lemma 1.1 and Corollary 1.2.

Lemma 2.5. *For all integers $i, j \geqslant 2$ we have*

$$\sum_{k=i}^{j} \mu_i(k) \left[\frac{j}{k}\right] = \delta_{ij},$$

$$\sum_{k=i}^{j} \frac{\mu_i(k) \bmod (j,k)}{k} = j \sum_{k=i}^{j} \frac{\mu_i(k)}{k} - \delta_{ij},$$

$$\sum_{k=i}^{j} \left(\sum_{h=i}^{k} \frac{\mu_i(h)}{h} \right) (\operatorname{mod}(j, k+1) - \operatorname{mod}(j,k)) = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol.

Proof. We only prove the first identity, the proof of the two others being strictly identical to the identities of Corollary 1.2. Without loss of generality, one can suppose that $2 \leqslant i \leqslant j$. If $i = j$ then we have

$$\sum_{k=i}^{j} \mu_i(k) \left[\frac{j}{k}\right] = \mu_j(j) \left[\frac{j}{j}\right] = \mu\left(\frac{j}{j}\right) = 1.$$

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Now suppose that $2 \leqslant i < j$. By Lemma 1.1, we have

$$\begin{aligned}
\sum_{k=i}^{j} \mu_i(k) \left[\frac{j}{k}\right] &= \sum_{\substack{k=i \\ i|k,\,(i+1)\nmid k}}^{j} \mu\left(\frac{k}{i}\right)\left[\frac{j}{k}\right] - \sum_{\substack{k=i \\ (i+1)|k,\,i\nmid k}}^{j} \mu\left(\frac{k}{i+1}\right)\left[\frac{j}{k}\right] \\
&\qquad + \sum_{\substack{k=i \\ i(i+1)|k}}^{j} \left(\mu\left(\frac{k}{i}\right) - \mu\left(\frac{k}{i+1}\right)\right)\left[\frac{j}{k}\right] \\
&= \sum_{\substack{k=i \\ i|k}}^{j} \mu\left(\frac{k}{i}\right)\left[\frac{j}{k}\right] - \sum_{\substack{k=i \\ (i+1)|k}}^{j} \mu\left(\frac{k}{i+1}\right)\left[\frac{j}{k}\right] \\
&= \sum_{h=1}^{[j/i]} \mu(h)\left[\frac{[j/i]}{h}\right] - \sum_{h=1}^{[j/(i+1)]} \mu(h)\left[\frac{[j/(i+1)]}{h}\right] \\
&= 1 - 1 = 0,
\end{aligned}$$

which concludes the proof. □

This result gives the inverse of U_n.

Corollary 2.6. *Set $U_n^{-1} = (\theta_{ij})$. Then we have*

$$\theta_{ij} = \sum_{k=i}^{j} \frac{\mu_i(k)}{k} \qquad (1 \leqslant i \leqslant j \leqslant n-1)$$

$$\theta_{in} = n \sum_{k=i}^{n} \frac{\mu_i(k)}{k} \qquad (1 \leqslant i \leqslant n).$$

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Proof. Since U_n^{-1} is upper triangular, it suffices to show that, for all integers $1 \leqslant i \leqslant j \leqslant n$, we have

$$\sum_{k=i}^{j} \theta_{ik} u_{kj} = \delta_{ij}.$$

In what follows, we set S_{ij} as the sum on the left-hand side

We easily check that $S_{jj} = 1$ for every integer $1 \leqslant j \leqslant n$. Now suppose that $1 \leqslant i < j \leqslant n-1$. By Corollary 1.2, we first have

$$\begin{aligned} S_{1j} &= \sum_{k=1}^{j} \theta_{1k} u_{kj} = \theta_{11} u_{1j} + \sum_{k=2}^{j} \theta_{1k} u_{kj} \\ &= \operatorname{mod}(j,2) - 1 + \sum_{k=2}^{j} \left(\sum_{h=1}^{k} \frac{\mu(h)}{h} \right) (\operatorname{mod}(j,k+1) - \operatorname{mod}(j,k)) \\ &= \sum_{k=1}^{j} \left(\sum_{h=1}^{k} \frac{\mu(h)}{h} \right) (\operatorname{mod}(j,k+1) - \operatorname{mod}(j,k)) - 1 = 0 \end{aligned}$$

and, if $2 \leqslant i < j \leqslant n-1$, then by Lemma 2.5, we have

$$S_{ij} = \sum_{k=i}^{j} \left(\sum_{h=1}^{k} \frac{\mu_i(h)}{h} \right) (\operatorname{mod}(j,k+1) - \operatorname{mod}(j,k)) = \delta_{ij} = 0.$$

Now suppose that $j = n$. By Corollary 1.2, we first have

$$\begin{aligned} S_{1n} &= \sum_{k=1}^{n} \theta_{1k} u_{kn} = \theta_{11} u_{1n} + \sum_{k=2}^{n-1} \left(\sum_{h=1}^{k} \frac{\mu(h)}{h} \right) (\operatorname{mod}(n,k+1) - \operatorname{mod}(n,k)) + \theta_{1n} u_{nn} \\ &= \operatorname{mod}(n,2) - 1 + \sum_{k=2}^{n-1} \left(\sum_{h=1}^{k} \frac{\mu(h)}{h} \right) (\operatorname{mod}(n,k+1) - \operatorname{mod}(n,k)) + n \sum_{k=1}^{n} \frac{\mu(k)}{k} \end{aligned}$$

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$$= \sum_{k=1}^{n} \left(\sum_{h=1}^{k} \frac{\mu(h)}{h} \right) (\operatorname{mod}(n, k+1) - \operatorname{mod}(n, k)) - 1 = 0$$

and, if $2 \leqslant i \leqslant n-1$, we have

$$\begin{aligned} S_{in} &= \sum_{k=i}^{n-1} \left(\sum_{h=i}^{k} \frac{\mu_i(h)}{h} \right) (\operatorname{mod}(n, k+1) - \operatorname{mod}(n, k)) + \theta_{in} u_{nn} \\ &= \sum_{k=i}^{n-1} \left(\sum_{h=i}^{k} \frac{\mu_i(h)}{h} \right) (\operatorname{mod}(n, k+1) - \operatorname{mod}(n, k)) + n \sum_{k=i}^{n} \frac{\mu_i(k)}{k} \\ &= \sum_{k=i}^{n} \left(\sum_{h=i}^{k} \frac{\mu_i(h)}{h} \right) (\operatorname{mod}(n, k+1) - \operatorname{mod}(n, k)) = \delta_{in} = 0 \end{aligned}$$

which completes the proof. □

For example, we get

$$U_8^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{30} & \frac{2}{15} & -\frac{1}{105} & -\frac{8}{105} \\ 0 & \frac{1}{2} & \frac{1}{6} & -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & -\frac{2}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} & -\frac{3}{5} \\ 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{30} & \frac{1}{30} & \frac{4}{15} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{42} & \frac{4}{21} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

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2.2.2. A sufficient condition for the PNT and the RH

Corollary 2.7. *For all integers $i \geqslant 1$ and $n \geqslant 2$ we have*

$$\left| \sum_{k=i}^{n} \frac{\mu_i(k)}{k} \right| \leqslant \frac{1}{n\sigma_n},$$

where σ_n is the smallest singular value of U_n. Thus any estimate of the form

$$\sigma_n \gg_\varepsilon n^{-1+\varepsilon},$$

where $\varepsilon > 0$ is any real number, is sufficient to prove the PNT. Similarly, any estimate of the form

$$\sigma_n \gg_\varepsilon n^{-1/2-\varepsilon}$$

where $\varepsilon > 0$ is any real number, is sufficient to prove the RH.

Proof. The result follows at once from the well-known inequalities

$$\|U_n^{-1}\|_2 \geqslant \max_{1 \leqslant i,j \leqslant n} |\theta_{ij}| \geqslant |\theta_{in}|$$

(see [3]), where $\|\cdot\|_2$ is the spectral norm, and the fact that $\sigma_n = \|U_n^{-1}\|_2^{-1}$. □

Smallest singular values of triangular matrices have been studied by many authors. For example (see [2, 4]), it is known that, if $A_n = (a_{ij})$ is an invertible upper triangular matrix such that $|a_{ii}| \geqslant |a_{ij}|$ for $i < j$, then we have

$$\sigma_n \geqslant \frac{\min |a_{ii}|}{2^{n-1}}$$

but, applied here, such a bound is still very far from the PNT.

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Appendix : A Proof of Redheffer's Theorem

Let $S_n = (s_{ij})$ and $T_n = (t_{ij})$ be the matrices defined by

$$s_{ij} = \begin{cases} 1, & \text{if } i \mid j; \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad t_{ij} = \begin{cases} M(n/i), & \text{if } j = 1; \\ 1, & \text{if } i = j \geqslant 2; \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.8. *We have $R_n = S_n T_n$. In particular,* $\det R_n = M(n)$.

Proof. Set $S_n T_n = (x_{ij})$. If $j = 1$, by Lemma 1.1, we have

$$x_{i1} = \sum_{k=1}^{n} s_{ik} t_{k1} = \sum_{\substack{k \leqslant n \\ i \mid k}} M\left(\frac{n}{k}\right) = \sum_{d \leqslant n/i} M\left(\frac{n/i}{d}\right) = 1 = r_{i1}.$$

If $j \geqslant 2$, then $t_{1j} = 0$ and thus

$$x_{ij} = \sum_{k=2}^{n} s_{ik} t_{kj} = s_{ij} = \begin{cases} 1, & \text{if } i \mid j \\ 0, & \text{otherwise} \end{cases} = r_{ij}$$

which is the desired result. The second assertion follows at once from

$$\det R_n = \det S_n \det T_n = \det T_n = M(n).$$

The proof is complete. □

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