Journal of Inequalities in Pure and Applied Mathematics

A NOTE ON MULTIPLICATIVELY e-PERFECT NUMBERS

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ISSN (electronic): 1443-5756
313-05

volume 7, issue 3, article 99, 2006.

Received 30 March, 2005; accepted 03 February, 2006.

Communicated by: J. Sándor

Abstract

Let $T_e(n)$ denote the product of all exponential divisors of n. An integer n is called multiplicatively e-perfect if $T_e(n) = n^2$ and multiplicatively e-superperfect if $T_e(T_e(n)) = n^2$. In this note, we give an alternative proof for characterization of multiplicatively e-perfect and multiplicatively e-superperfect numbers.

2000 Mathematics Subject Classification: 11A25, 11A99.
Key words: Perfect number, Exponential divisor, Multiplicatively perfect, Sum of divisors, Number of divisors.

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1. Introduction

Let $\sigma(n)$ be the sum of all divisors of n. An integer n is called perfect if $\sigma(n) = 2n$ and superperfect if $\sigma(\sigma(n)) = 2n$. If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the prime factorization of $n > 1$, a divisor $d \mid n$, called an exponential divisor (e-divisor) of n is $d = p_1^{\beta_1} \cdots p_k^{\beta_k}$ with $\beta_i \mid \alpha_i$ for all $1 \leq i \leq k$. Let $T_e(n)$ denote the product of all exponential divisors of n. The concepts of multiplicatively e-perfect and multiplicatively e-superperfect numbers were first introduced by Sándor in [1].

Definition 1.1. *An integer n is called multiplicatively e-perfect if $T_e(n) = n^2$ and multiplicatively e-superperfect if $T_e(T_e(n)) = n^2$.*

In [1], Sándor completely characterizes multiplicatively e-perfect and multiplicatively e-superperfect numbers.

Theorem 1.1 ([1]).

a) *An integer n is multiplicatively e-perfect if and only if $n = p^\alpha$, where p is prime and α is a perfect number.*

b) *An interger n is multiplicatively e-superperfect if and only if $n = p^\alpha$, where p is a prime, and α is a superperfect number.*

Sándor's proof is based on an explicit expression of $T_e(n)$. In this note, we offer an alternative proof of Theorem 1.1.

2. Proof of Theorem 1.1

a) Suppose that n is multiplicatively e-perfect; that is $T_e(n) = n^2$. If n has more than one prime factor then $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ for some $k \geq 2$, $\alpha_i \geq 1$ and $p_1, \dots, p_k$ are k distinct primes. We have three separate cases.

1. Suppose that $\alpha_1 = \cdots = \alpha_k = 1$. Then d is an exponential divisor of n if and only if $d = p_1 \cdots p_k = n$. Hence $T_e(n) = n$, which is a contradiction.

2. Suppose that two of $\alpha_1, \dots, \alpha_k$ are greater 1. Without loss of generality, we may assume that $\alpha_1, \alpha_2 > 1$. Then $d_1 = p_1 p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $d_2 = p_1^{\alpha_1} p_2 p_3^{\alpha_3} \cdots p_k^{\alpha_k}$, $d_3 = n$ are three distinct exponential divisors of n. Hence $d_1 d_2 d_3 \mid T_e(n)$. However, $p_1^{2\alpha_1+1} \mid d_1 d_2 d_3$ so $T_e(n) \neq n^2$, which is a contradiction.

3. Suppose that there is exactly one of $\alpha_1, \dots, \alpha_k$ which is greater than 1. Without loss of generality, we may assume that $\alpha_1 > 1$ and $\alpha_2 = \cdots = \alpha_k = 1$. We have that if d is an exponential divisor of n then $d = p_1^{\beta_1} p_2 \cdots p_k$ for some $\beta_1 \mid \alpha_1$. Hence if n has more than two distinct exponential divisors then $p_2^3 \mid T_e(n) = p_1^{2\alpha_1} p_2^2 \cdots p_k^2$, which is a contradiction. However, $d_1 = p_1 p_2 \cdots p_k$, $d_2 = p_1^{\alpha_1} p_2 p_3 \cdots p_k$ are two distinct exponential divisors of n so d_1, d_2 are all exponential divisors of n. Hence $T_e(n) = p_1^{\alpha_1+1} p_2^2 \cdots p_k^2 = p_1^{2\alpha_1} p_2^2 \cdots p_k^2$. This implies that $\alpha_1 = 1$, which is a contradiction.

Thus n has only one prime factor; that is, $n = p^\alpha$ for some prime p. In this case then $T_e(n) = p^{\sigma(\alpha)}$. Hence $T_e(n) = n^2 = p^{2\alpha}$ if and only if $\sigma(\alpha) = 2\alpha$. This concludes the proof.

b) Suppose that n is multiplicatively e-superperfect; that is $T_e(T_e(n)) = n^2$. If n has more than one prime factor then $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ for some $k \geq 2$, $\alpha_i \geq 1$ and $p_1, \ldots, p_k$ are k distinct primes. We have two separate cases.

1. Suppose that $\alpha_1 = \cdots = \alpha_k = 1$. Then d is an exponential divisor of n if and only if $d = p_1 \cdots p_k = n$. Hence $T_e(n) = n$ and $T_e(T_e(n)) = T_e(n) = n$ which is a contradiction.

2. Suppose that there is at least one of $\alpha_1, \ldots, \alpha_k$ which is greater 1. Without loss of generality, we may assume that $\alpha_1 > 1$. Then $d_1 = p_1 p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $d_2 = n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$, are two distinct exponential divisors of n. Hence $d_1 d_2 \mid T_e(n)$. However, $d_1 d_2 = p_1^{\alpha_1+1} p_2^{2\alpha_2} \cdots p_k^{2\alpha_k}$ so $T_e(n) = p_1^{\gamma_1} \cdots p_k^{\gamma_k}$ where $\gamma_1 \geq \alpha_1 + 1$, $\gamma_i \geq 2\alpha_i \geq 2$ for $i = 2, \ldots, k$. Thus, $t_1 = p_1^{\gamma_1} p_2 p_3^{\gamma_3} \cdots p_k^{\gamma_k}$ and $t_2 = T_e(n) = p_1^{\gamma_1} p_2^{\gamma_2} p_3^{\gamma_3} \cdots p_k^{\gamma_k}$ are two distinct exponential divisors of $T_e(n)$. Hence $t_1 t_2 \mid T_e(T_e(n))$. However, $p_1^{2\gamma_1} \mid t_1 t_2$ and $\gamma_1 > \alpha_1$, which is a contradiction.

Thus n has only one prime factor; that is $n = p^\alpha$ for some prime p. In this case then $T_e(n) = p^{\sigma(\alpha)}$ and $T_e(T_e(n)) = p^{\sigma(\sigma(n))}$. Hence $T_e(T_e(n)) = n^2 = p^{2\alpha}$ if and only if $\sigma(\sigma(\alpha)) = 2\alpha$. This concludes the proof.

Remark 1. *In an e-mail message, Professor Sándor has provided the authors some more recent references related to the arithmetic function $T_e(n)$, as well as connected notions on e-perfect numbers and generalizations. These are [2], [3], and [4].*

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