A LATER NOTE ON THE RELATIONSHIPS OF NUMERICAL SEQUENCES

L. LEINDLER
University of Szeged, Bolyai Institute
Aradi vértanúk tere 1
H-6720 Szeged, Hungary
EMail: leindler@math.u-szeged.hu

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Contents

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1. Introduction

T.W. Chaundy and A.E. Jolliffe [1] proved the following classical theorem:

Suppose that $b_n \geqq b_{n+1}$ *and* $b_n \to 0$. *Then a necessary and sufficient condition for the uniform convergence of the series*

$$\sum_{n=1}^{\infty} b_n \sin nx \tag{1.1}$$

is $n\, b_n \to 0$.

Near fifty years later S.M. Shah [11] showed that *any classical quasimonotone sequence* $(CQMS)$ could replace the monotone one in (1.1).

For notions and notations, please see the second section.

In [3, 4], we defined the class of *sequences of rest bounded variation* $(RBVS)$ and verified that Chaundy-Jolliffe's theorem also remains valid by these sequences.

In connection with these two results, S.A. Telyakovskiĭ raised the following problem (personal communication): Are the classes $CQMS$ and $RBVS$ comparable? This problem implicitly includes the question: which result is better, that of Shah or ours?

In [5] we gave a negative answer, that is, these classes are not comparable. Thus these two results are disconnected.

Recently a group of authors (see e.g. R.J. Le and S.P. Zhou [2], D.S. Yu and S.P. Zhou [13], S. Tikhonov [12], L. Leindler [6, 7]) have generalized further the notion of monotonicity by keeping some good properties of decreasing sequences.

Among others, D.S. Yu and S.P. Zhou [13] proved that their *newly defined sequences* $(NBVS)$ could replace the monotone ones in (1.1).

In [8] we proved a similar result for *sequences of mean group bounded variation* $(MGBVS)$.

vol. 8, iss. 4, art. 103, 2007

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The latter two results have again offered to investigate the relation of the classes $NBVS$ and $MGBVS$.

Now, first we shall prove that these classes are not comparable. Furthermore we also show the class of *sequences of mean rest bounded variation* $(MRBVS)$, defined in [9] and used in [7], is not comparable to either $NBVS$ or $MGBVS$.

We mention that in [10] we already analyzed the relationships of seven similar numerical sequences. In the papers [2], [12] and [13] we can also read analogous investigations.

vol. 8, iss. 4, art. 103, 2007

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2. Notions and Notations

We recall some definitions and notations.

We shall only consider sequences with nonnegative terms. For a sequence $\mathbf{c} := \{c_n\}$, denote $\Delta c_n := c_n - c_{n+1}$. The capital letters K, K_1 and $K(\cdot)$ denote positive constants, or constants depending upon the given parameters. We shall also use the following notation: we write $L \ll R$ if there exists a constant K such that $L \leqq KR$, but not necessarily the same K at each occurrence.

The well-known *classical quasimonotone sequences* $(CQMS)$ will be defined here by $0 < \alpha \leqq 1$ and

$$c_{n+1} \leqq c_n \left(1 + \frac{\alpha}{n}\right), \quad n = 1, 2, \dots .$$

Let $\gamma := \{\gamma_n\}$ be a positive sequence. A null-sequence $\mathbf{c}$ $(c_n \to 0)$ satisfying the inequalities

$$\sum_{n=m}^{\infty} |\Delta c_n| \leqq K(\mathbf{c})\gamma_m, \quad m = 1, 2, \dots \tag{2.1}$$

is said to be a *sequence of* γ *rest bounded variation*, in symbolic form: $\mathbf{c} \in \gamma RBVS$ (see e.g. [6]).

If $\gamma \equiv \mathbf{c}$ and every $c_n > 0$, then we get the *class of sequences of rest bounded variation* $(RBVS)$.

If γ is given by

$$\gamma_m := \frac{1}{m} \sum_{n=m}^{2m-1} c_n, \tag{2.2}$$

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and

$$\sum_{n=2m}^{\infty} |\Delta c_n| \leqq K(\mathbf{c})\gamma_m \tag{2.3}$$

holds, then we say that $\mathbf{c}$ belongs to the *class of mean rest bounded variation sequences* $(MRBVS)$.

We remark that if γ is given by (2.2) then $\gamma RBVS$ does not necessarily include the monotone sequences, but $MRBVS$ does (see e.g. $c_n = 2^{-n}$).

If we claim

$$\sum_{n=m}^{2m} |\Delta c_n| \leqq K(\mathbf{c})\gamma_m, \quad m = 1, 2, \ldots \tag{2.4}$$

instead of (2.1) then we get the class $\gamma GBVS$ (see [6]).

If in (2.4) γ is given by $\gamma_m := c_m + c_{2m}$, then we obtain the *new class of sequences defined by Yu and Zhou* [13], which will be denoted by $NBVS$.

Finally if in (2.4) γ is given by (2.2) then we get the class of *sequences of mean group bounded variation* $(MGBVS)$.

If two classes of sequences A and B are not comparable we shall denote this by $A \nsim B$.

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3. A Theorem

Now we formulate our assertions in a terse form.

Theorem 3.1. *The following relations hold:*

$$NBVS \not\sim MGBVS, \tag{3.1}$$

$$NBVS \not\sim MRBVS, \tag{3.2}$$

$$MGBVS \not\sim MRBVS. \tag{3.3}$$

Relationships of Numerical Sequences
L. Leindler
vol. 8, iss. 4, art. 103, 2007

journal of inequalities in pure and applied mathematics
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4. Proof of Theorem 3.1

Proof of (3.1). Let

$$c_n := 2^{-n}, \quad n = 1, 2, \dots . \tag{4.1}$$

This sequence clearly belongs to $NBVS$, but it does not belong to the class $MGBVS$, namely

$$K\, 2^{-m} \leqq \sum_{n=m}^{2m} |\Delta\, c_n| \leqq K_1\, 2^{-m}$$

and

$$\frac{1}{m} \sum_{n=m}^{2m-1} c_n \leqq \frac{2}{m} 2^{-m}.$$

Next we define a sequence $\mathbf{d} := \{d_n\}$ such that $\mathbf{d} \notin NBVS$, but $\mathbf{d} \in MGBVS$. Let $d_1 = 1$ and

$$d_n := \begin{cases} 0, & \text{if } n = 2^\nu \\ 2^{-\nu}, & \text{if } 2^\nu < n < 2^{\nu+1}, \quad \nu = 1, 2, \dots . \end{cases} \tag{4.2}$$

Then

$$K\, m^{-1} \leqq \sum_{n=m}^{2m} |\Delta\, d_n| \leqq K_1\, m^{-1}, \quad m \geqq 2, \tag{4.3}$$

holds, and if $m = 2^\nu$, then

$$d_m + d_{2m} = 0,$$

thus $\mathbf{d}$ does not belong to $NBVS$, namely (2.4) does not hold if $c_n = d_n$, $m = 2^\nu$ $(\nu \geqq 1)$ and $\gamma_m = d_m + d_{2m}$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

On the other hand, the inequality (2.4) plainly holds if $c_n = d_n$ and

$$\gamma_m := m^{-1} \sum_{n=m}^{2m-1} d_n \ (\geqq K\, m^{-1}), \tag{4.4}$$

that is, $\mathbf{d} \in MGBVS$.

Herewith (3.1) is proved. □

Proof of (3.2). As we have seen above, the sequence $\mathbf{d}$ defined in (4.2) does not belong to $NBVS$, but by (4.3) and (4.4), it is easy to see that if $c_n = d_n$, then (2.3) is satisfied, whence $\mathbf{d} \in MRBVS$ holds.

Next we consider the following sequence δ defined as follows:

$$\delta_n := \begin{cases} 0, & \text{if } n = 2^\nu + 1, \\ \nu^{-1}, & \text{if } 2^\nu + 1 < n < 2^{\nu+1} + 1. \end{cases}$$

Elementary consideration gives that if $m \geqq 2$, then

$$\delta_m + \delta_{2m} \gg \sum_{n=m}^{2m} |\Delta\, \delta_n| \gg (\log m)^{-1} \tag{4.5}$$

and

$$m^{-1} \sum_{n=m}^{2m-1} \delta_n \ll (\log m)^{-1}. \tag{4.6}$$

The first inequality of (4.5) clearly shows that $\delta \in NBVS$, but the second inequality of (4.5) and (4.6) convey that $\delta \notin MRBVS$, namely

$$\sum_{k=1}^{\infty} (\log 2^k m)^{-1} = \infty. \tag{4.7}$$

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The facts proved above verify (3.2). □

Proof of (3.3). In the proof of (3.1) we have verified that the sequence $\mathbf{c}$ defined in (4.1) does not belong to $MGBVS$, but it clearly belongs to $MRBVS$, because $2^{-2m} < m^{-1}\,2^{-m}$.

Next we show that the following sequence α defined by $\alpha_1 = 1$ and for $n \geqq 2$

$$\alpha_n := \begin{cases} 0, & \text{if } n = 2^\nu, \\ \nu^{-1}, & \text{if } 2^\nu < n < 2^{\nu+1}, \quad \nu = 1, 2, \dots \end{cases}$$

has a contrary property.

It is clear that

$$(\log m)^{-1} \ll m^{-1} \sum_{n=m}^{2m-1} \alpha_n \ll (\log m)^{-1}, \quad m \geqq 2,$$

and

$$(\log m)^{-1} \ll \sum_{n=m}^{2m} |\Delta\, \alpha_n| \ll (\log m)^{-1}.$$

The latter two estimates prove that $\alpha \in MGBVS$, and since (4.7) holds, thus $\alpha \notin MRBVS$ also holds.

Herewith (3.3) is proved, and our theorem is also proved. □

journal of inequalities in pure and applied mathematics
issn: 1443-5756

References

[1] T.W. CHAUNDY AND A.E. JOLLIFFE, The uniform convergence of a certain class of trigonometric series, *Proc. London Math. Soc.*, **15** (1916), 214–216.

[2] R.J. LE AND S.P. ZHOU, A new condition for uniform convergence of certain trigonometric series, *Acta Math. Hungar.*, **10**(1-2) (2005), 161–169.

[3] L. LEINDLER, Embedding results pertaining to strong approximation of Fourier series. II, *Analysis Math.*, **23** (1997), 223–240.

[4] L. LEINDLER, On the uniform convergence and boundedness of a certain class of sine series, *Analysis Math.*, **27** (2001), 279–285.

[5] L. LEINDLER, A new class of numerical sequences and its applications to sine and cosine series, *Analysis Math.*, **28** (2002), 279–286.

[6] L. LEINDLER, A new extension of monotone sequences and its applications, *J. Inequal. Pure and Appl. Math.*, **7**(1) (2006), Art. 39. [ONLINE: http://jipam.vu.edu.au/article.php?sid=656].

[7] L. LEINDLER, Integrability conditions pertaining to Orlicz space, *J. Inequal. Pure and Appl. Math.*, **8**(2) (2007), Art. 38. [ONLINE: http://jipam.vu.edu.au/article.php?sid=860].

[8] L. LEINDLER, Necessary and sufficient conditions for uniform convergence and boundedness of a general class of sine series, *Australian J. of Math. Anal. and Appl.*, **4**(1) (2007), Art. 10. [ONLINE: http://ajmaa.vu.edu.au/.]

[9] L. LEINDLER, Embedding results pertaining to strong approximation of Fourier series. VI, *Analysis Math.*, (in print).

journal of inequalities in pure and applied mathematics

issn: 1443-5756

[10] L. LEINDLER, On the relationships of seven numerical sequences, *Acta Math. Hungar.*, **114**(3) (2007), 227–234.

[11] S.M. SHAH, Trigonometric series with quasi-monotone coefficients, *Proc. Amer. Math. Soc.*, **13** (1962), 266–273.

[12] S. TIKHONOV, Trigonometric series of Nikol'skii classes, *Acta Math. Hungar.*, **114**(1-2) (2007), 67–78.

[13] D.S. YU AND S.P. ZHOU, A generalization of the monotonicity condition and applications, *Acta Math. Hungar.*, **115**(3) (2007), 247–267.

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journal of inequalities in pure and applied mathematics

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