

A LATER NOTE ON THE RELATIONSHIPS OF NUMERICAL SEQUENCES

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Abstract: We analyze the relationships of three recently defined classes of numerical sequences.



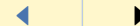
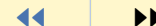
**Relationships of Numerical
Sequences**

L. Leindler

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1. Introduction

T.W. Chaundy and A.E. Jolliffe [1] proved the following classical theorem:

Suppose that $b_n \geq b_{n+1}$ and $b_n \rightarrow 0$. Then a necessary and sufficient condition for the uniform convergence of the series

$$(1.1) \quad \sum_{n=1}^{\infty} b_n \sin nx$$

is $nb_n \rightarrow 0$.

Near fifty years later S.M. Shah [11] showed that *any classical quasimonotone sequence (CQMS)* could replace the monotone one in (1.1).

For notions and notations, please see the second section.

In [3, 4], we defined the class of *sequences of rest bounded variation (RBVS)* and verified that Chaundy-Jolliffe's theorem also remains valid by these sequences.

In connection with these two results, S.A. Telyakovskii raised the following problem (personal communication): Are the classes *CQMS* and *RBVS* comparable? This problem implicitly includes the question: which result is better, that of Shah or ours?

In [5] we gave a negative answer, that is, these classes are not comparable. Thus these two results are disconnected.

Recently a group of authors (see e.g. R.J. Le and S.P. Zhou [2], D.S. Yu and S.P. Zhou [13], S. Tikhonov [12], L. Leindler [6, 7]) have generalized further the notion of monotonicity by keeping some good properties of decreasing sequences.

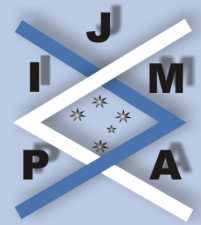
Among others, D.S. Yu and S.P. Zhou [13] proved that their *newly defined sequences (NBVS)* could replace the monotone ones in (1.1).

In [8] we proved a similar result for *sequences of mean group bounded variation (MGBVS)*.

The latter two results have again offered to investigate the relation of the classes $NBVS$ and $MGBVS$.

Now, first we shall prove that these classes are not comparable. Furthermore we also show the class of *sequences of mean rest bounded variation* ($MRBVS$), defined in [9] and used in [7], is not comparable to either $NBVS$ or $MGBVS$.

We mention that in [10] we already analyzed the relationships of seven similar numerical sequences. In the papers [2], [12] and [13] we can also read analogous investigations.



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2. Notions and Notations

We recall some definitions and notations.

We shall only consider sequences with nonnegative terms. For a sequence $\mathbf{c} := \{c_n\}$, denote $\Delta c_n := c_n - c_{n+1}$. The capital letters K, K_1 and $K(\cdot)$ denote positive constants, or constants depending upon the given parameters. We shall also use the following notation: we write $L \ll R$ if there exists a constant K such that $L \leq KR$, but not necessarily the same K at each occurrence.

The well-known *classical quasimonotone sequences (CQMS)* will be defined here by $0 < \alpha \leq 1$ and

$$c_{n+1} \leq c_n \left(1 + \frac{\alpha}{n}\right), \quad n = 1, 2, \dots$$

Let $\gamma := \{\gamma_n\}$ be a positive sequence. A null-sequence \mathbf{c} ($c_n \rightarrow 0$) satisfying the inequalities

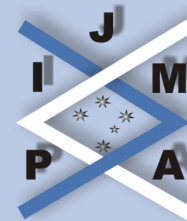
$$(2.1) \quad \sum_{n=m}^{\infty} |\Delta c_n| \leq K(\mathbf{c})\gamma_m, \quad m = 1, 2, \dots$$

is said to be a *sequence of γ rest bounded variation*, in symbolic form: $\mathbf{c} \in \gamma RBVS$ (see e.g. [6]).

If $\gamma \equiv \mathbf{c}$ and every $c_n > 0$, then we get the *class of sequences of rest bounded variation (RBVS)*.

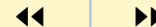
If γ is given by

$$(2.2) \quad \gamma_m := \frac{1}{m} \sum_{n=m}^{2m-1} c_n,$$



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and

$$(2.3) \quad \sum_{n=2m}^{\infty} |\Delta c_n| \leq K(\mathbf{c})\gamma_m$$

holds, then we say that \mathbf{c} belongs to the *class of mean rest bounded variation sequences (MRBVS)*.

We remark that if γ is given by (2.2) then $\gamma RBVS$ does not necessarily include the monotone sequences, but *MRBVS* does (see e.g. $c_n = 2^{-n}$).

If we claim

$$(2.4) \quad \sum_{n=m}^{2m} |\Delta c_n| \leq K(\mathbf{c})\gamma_m, \quad m = 1, 2, \dots$$

instead of (2.1) then we get the class $\gamma GBVS$ (see [6]).

If in (2.4) γ is given by $\gamma_m := c_m + c_{2m}$, then we obtain the *new class of sequences defined by Yu and Zhou [13]*, which will be denoted by *NBVS*.

Finally if in (2.4) γ is given by (2.2) then we get the class of *sequences of mean group bounded variation (MGBVS)*.

If two classes of sequences A and B are not comparable we shall denote this by $A \approx B$.



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3. A Theorem

Now we formulate our assertions in a terse form.

Theorem 3.1. *The following relations hold:*

$$(3.1) \quad NBVS \approx MGBVS,$$

$$(3.2) \quad NBVS \approx MRBVS,$$

$$(3.3) \quad MGBVS \approx MRBVS.$$



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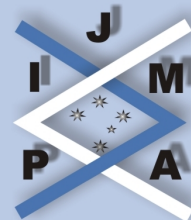


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4. Proof of Theorem 3.1

Proof of (3.1). Let

$$(4.1) \quad c_n := 2^{-n}, \quad n = 1, 2, \dots$$

This sequence clearly belongs to $NBVS$, but it does not belong to the class $MGBVS$, namely

$$K 2^{-m} \leq \sum_{n=m}^{2m} |\Delta c_n| \leq K_1 2^{-m}$$

and

$$\frac{1}{m} \sum_{n=m}^{2m-1} c_n \leq \frac{2}{m} 2^{-m}.$$

Next we define a sequence $\mathbf{d} := \{d_n\}$ such that $\mathbf{d} \notin NBVS$, but $\mathbf{d} \in MGBVS$. Let $d_1 = 1$ and

$$(4.2) \quad d_n := \begin{cases} 0, & \text{if } n = 2^\nu \\ 2^{-\nu}, & \text{if } 2^\nu < n < 2^{\nu+1}, \quad \nu = 1, 2, \dots \end{cases}$$

Then

$$(4.3) \quad K m^{-1} \leq \sum_{n=m}^{2m} |\Delta d_n| \leq K_1 m^{-1}, \quad m \geq 2,$$

holds, and if $m = 2^\nu$, then

$$d_m + d_{2m} = 0,$$

thus \mathbf{d} does not belong to $NBVS$, namely (2.4) does not hold if $c_n = d_n$, $m = 2^\nu$ ($\nu \geq 1$) and $\gamma_m = d_m + d_{2m}$.



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On the other hand, the inequality (2.4) plainly holds if $c_n = d_n$ and

$$(4.4) \quad \gamma_m := m^{-1} \sum_{n=m}^{2m-1} d_n (\geq K m^{-1}),$$

that is, $\mathbf{d} \in MGBVS$.

Herewith (3.1) is proved. □

Proof of (3.2). As we have seen above, the sequence \mathbf{d} defined in (4.2) does not belong to $NBVS$, but by (4.3) and (4.4), it is easy to see that if $c_n = d_n$, then (2.3) is satisfied, whence $\mathbf{d} \in MRBVS$ holds.

Next we consider the following sequence δ defined as follows:

$$\delta_n := \begin{cases} 0, & \text{if } n = 2^\nu + 1, \\ \nu^{-1}, & \text{if } 2^\nu + 1 < n < 2^{\nu+1} + 1. \end{cases}$$

Elementary consideration gives that if $m \geq 2$, then

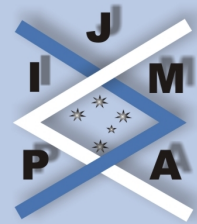
$$(4.5) \quad \delta_m + \delta_{2m} \gg \sum_{n=m}^{2m} |\Delta \delta_n| \gg (\log m)^{-1}$$

and

$$(4.6) \quad m^{-1} \sum_{n=m}^{2m-1} \delta_n \ll (\log m)^{-1}.$$

The first inequality of (4.5) clearly shows that $\delta \in NBVS$, but the second inequality of (4.5) and (4.6) convey that $\delta \notin MRBVS$, namely

$$(4.7) \quad \sum_{k=1}^{\infty} (\log 2^k m)^{-1} = \infty.$$



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The facts proved above verify (3.2). □

Proof of (3.3). In the proof of (3.1) we have verified that the sequence c defined in (4.1) does not belong to $MGBVS$, but it clearly belongs to $MRBVS$, because $2^{-2m} < m^{-1} 2^{-m}$.

Next we show that the following sequence α defined by $\alpha_1 = 1$ and for $n \geq 2$

$$\alpha_n := \begin{cases} 0, & \text{if } n = 2^\nu, \\ \nu^{-1}, & \text{if } 2^\nu < n < 2^{\nu+1}, \quad \nu = 1, 2, \dots \end{cases}$$

has a contrary property.

It is clear that

$$(\log m)^{-1} \ll m^{-1} \sum_{n=m}^{2m-1} \alpha_n \ll (\log m)^{-1}, \quad m \geq 2,$$

and

$$(\log m)^{-1} \ll \sum_{n=m}^{2m} |\Delta \alpha_n| \ll (\log m)^{-1}.$$

The latter two estimates prove that $\alpha \in MGBVS$, and since (4.7) holds, thus $\alpha \notin MRBVS$ also holds.

Herewith (3.3) is proved, and our theorem is also proved. □

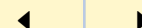
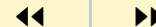
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