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APPLICATIONS OF THE EXTENDED HERMITE-HADAMARD INEQUALITY

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ABSTRACT. In this paper we give some combinatorial applications according to a new extension of the classical Hermite-Hadamard inequality proved in [1].

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1. INTRODUCTION

In the paper [1], we have proved the following generalization of the classical Hermite-Hadamard inequality for convex functions, extended it to n nodes: Suppose that $-\infty \leq a < b \leq \infty$, $f : (a,b) \to \mathbb{R}$ is a strict convex function, $x_i \in (a,b)$, $i = 1,\dots,n$ such that $x_i \neq x_j$ if $1 \leq i < j \leq n$. Then the following inequality holds:

$$\sum_{i=1}^{n} \frac{F^{(n-1)}(x_i)}{\Pi_i(x_1,\dots,x_n)} < \frac{1}{n!} \sum_{i=1}^{n} f(x_i).$$

In the concave case $<$ sign is changed to $>$. Here $\Pi_i(x_1,\dots,x_n) := \prod_{\substack{k=1\\k\neq i}}^{n}(x_i - x_k)$ and $F^{(0)}(s)$, $F^{(1)}(s),\dots, F^{(n-1)}(s),\dots$ is a sequence of functions defined recursively by $F^{(0)}(s) = f(s)$ and $\frac{d}{ds}F^{(n)}(s) = F^{(n-1)}(s)$, $n = 1,2,\dots$. These sequences of functions of f are known as the iterated integrals of f. As an application of this main result we have proved the following identities:

$$\sum_{k=1}^{n} \frac{x_k^n}{\Pi_k(x_1,\dots,x_n)} = \sum_{k=1}^{n} x_k, \tag{1.1}$$

$$\sum_{k=1}^{n} \frac{x_k^{n-1}}{\Pi_k(x_1,\dots,x_n)} = 1, \tag{1.2}$$

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$$\sum_{k=1}^{n}\frac{1}{x_k}=(-1)^{n-1}\prod_{k=1}^{n}x_k\sum_{k=1}^{n}\frac{1}{x_k^2\Pi_k(x_1,\dots,x_n)}\quad(x_k\neq 0)\tag{1.3}$$

and

$$\prod_{k=1}^{n}\frac{1}{x_k}=(-1)^{n-1}\sum_{k=1}^{n}\frac{1}{x_k\Pi_k(x_1,\dots,x_n)}\quad(x_k\neq 0).\tag{1.4}$$

In this paper we apply the formulae above to obtain closed combinatorial formulae and to investigate the asymptotic behaviours of these sums. For the sake of the convenience of the reader we collect the transformational regulations for the quantity $\Pi_k(x_1,\dots,x_n)$ which plays a significant role in all of the formulae. Note that all of them are simple consequences of the definition of $\Pi_k(x_1,\dots,x_n)$.

a) If $\alpha\in\mathbb{R}$, then

$$\Pi_k(\alpha+x_1,\dots,\alpha+x_n)=\Pi_k(x_1,\dots,x_n),$$

that is, Π_k is shift invariant.

b)

$$\Pi_k(\alpha x_1,\dots,\alpha x_n)=\alpha^{n-1}\Pi_k(x_1,\dots,x_n).$$

c)

$$\Pi_k(\alpha-x_1,\dots,\alpha-x_n)=(-1)^{n-1}\Pi_k(x_1,\dots,x_n).$$

d) If $x_k\neq 0$, $k=1,\dots,n$, then

$$\Pi_k\left(\frac{1}{x_1},\dots,\frac{1}{x_n}\right)=(-1)^{n-1}\frac{\Pi_k(x_1,\dots,x_n)}{x_k^{n-2}\Pi_{j=1}^{n}x_j}.$$

2. Asymptotical Formulae

a) Let $1<x_1<\cdots<x_n$ be variables and $s_1=x_1$, $s_2=x_1+x_2$, …, $s_n=x_1+\cdots+x_n$. Then $s_i\neq s_j$ if $1\le i<j\le n$, hence formula (1.1) can be applied to $s_1,\dots,s_n$, consequently we have

$$\sum_{j=1}^{n}s_j=\sum_{j=1}^{n}\frac{s_j^n}{\Pi_j(s_1,\dots,s_n)}.\tag{2.1}$$

For the left-hand side of (2.1)

$$\sum_{j=1}^{n}s_j=x_1+(x_1+x_2)+\cdots+(x_1+\cdots+x_n)=\sum_{j=1}^{n}j\cdot x_{n-j+1}.$$

The definition of Π_j implies

$$\begin{aligned}&\Pi_j(s_1,\dots,s_n)\\&=(s_j-s_1)\cdots(s_j-s_{j-1})(s_j-s_{j+1})\cdots(s_j-s_n)\\&=(x_2+\cdots+x_j)\cdots(x_{j-1}+x_j)\cdot x_j(-1)^{n-j}x_{j+1}(x_{j+1}+x_{j+2})\cdots(x_{j+1}+\cdots+x_n).\end{aligned}$$

Using the above expressions we have the following identities for all $1<x_1<\cdots<x_n$:

$$\sum_{j=1}^{n}j\cdot x_{n-j+1}=\sum_{j=1}^{n}\frac{(-1)^{n-j}(x_1+\cdots+x_j)^n}{(x_2+\cdots+x_j)\cdots(x_{j-1}+x_j)x_jx_{j+1}\cdots(x_{j+1}+\cdots+x_n)}.\tag{2.2}$$

Now, if $x_k \to 1$, $k = 1, \dots, n$ then (2.2) gives

$$\binom{n+1}{2} = \sum_{j=1}^{n} j = \sum_{j=1}^{n} \frac{(-1)^{n-j} j^n}{(j-1)!(n-j)!}.$$

Multiplying both sides by $(n-1)!$ leads us to

$$\sum_{j=1}^{n} (-1)^{n-j} j^n \binom{n-1}{j-1} = \frac{(n+1)!}{2}.$$

b) Apply now formula (1.3) to $x_k = 1 + \frac{k}{n}$, $k = 1, \dots, n$. Hence

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} &= (-1)^{n-1} \prod_{k=1}^{n} \left(1 + \frac{k}{n}\right) \sum_{k=1}^{n} \frac{1}{(1+\frac{k}{n})^2 \Pi_k(1+\frac{1}{n}, \dots, 1+\frac{n}{n})} \\
&= (-1)^{n-1} \frac{1}{n^n} \prod_{k=1}^{n} (n+k) \sum_{k=1}^{n} \frac{n^2}{(n+k)^2 \Pi_k(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n})} \\
&= (-1)^{n-1} \frac{1}{n^n} \prod_{k=1}^{n} (n+k) \sum_{k=1}^{n} \frac{n^2}{(n+k)^2 \frac{1}{n^{n-1}} \Pi_k(1, \dots, n)} \\
&= (-1)^{n-1} \prod_{k=1}^{n} (n+k) \sum_{k=1}^{n} \frac{n}{(n+k)^2 (k-1)! (-1)^{n-k} (n-k)!} \\
&= n \cdot \binom{2n}{n} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(n+k)^2} \binom{n-1}{k-1}.
\end{aligned}$$

From this equality chain we conclude that

$$\binom{2n}{n} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(n+k)^2} \binom{n-1}{k-1} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} = \sum_{k=1}^{n} \frac{1}{n+k}. \tag{2.3}$$

Since

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} = \int_0^1 \frac{1}{1+u} du = \log 2,$$

hence the left-hand side of (2.3) is convergent and

$$\lim_{n\to\infty} \binom{2n}{n} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(n+k)^2} \binom{n-1}{k-1} = \log 2.$$

Pursuant to the Stirling formula $\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$ $(n \to \infty)$, consequently we have the following asymptotic expression for the sum on the left:

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{(n+k)^2} \binom{n-1}{k-1} \sim \sqrt{\pi} \log 2 \frac{\sqrt{n}}{4^n} \quad (n \to \infty).$$

c) This example follows along the lines above but choosing $x_k = 1 + \left(\frac{k}{n}\right)^2$ $(k = 1, \dots, n)$. Without the detailed computation we have

$$\frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+(\frac{k}{n})^2} = 2n \frac{\prod_{k=1}^{n}(n^2+k^2)}{(n!)^2} \sum_{k=1}^{n} \frac{(-1)^{k-1} k^2}{(n^2+k^2)^2} \cdot \frac{\binom{n}{k}}{\binom{n+k}{n}}. \tag{2.4}$$

From this equality the limit of the right-hand side exists and

$$\lim_{n\to\infty} 2n \cdot \frac{\prod_{k=1}^{n}(n^2+k^2)}{(n!)^2} \sum_{k=1}^{n} \frac{(-1)^{k-1}k^2}{(n^2+k^2)^2} \cdot \frac{\binom{n}{k}}{\binom{n+k}{n}} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+(\frac{k}{n})^2}$$
$$= \int_0^1 \frac{1}{1+u^2} du$$
$$= \arctan 1 = \frac{\pi}{4}.$$

Combining these observations we have

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}k^2}{(n^2+k^2)^2} \frac{\binom{n}{k}}{\binom{n+k}{n}} \sim \frac{\pi}{8n} \cdot \frac{(n!)^2}{\Pi_{k=1}^{n}(n^2+k^2)} \quad (n\to\infty).$$

Moreover $(n!)^2 \sim 2\pi n(\frac{n}{e})^{2n}$ and

$$\prod_{k=1}^{n}(n^2+k^2) = n^{2n} \prod_{k=1}^{n} \left[1+\left(\frac{k}{n}\right)^2\right].$$

On the other hand

$$\frac{1}{n} \log \prod_{k=1}^{n} \left[1+\left(\frac{k}{n}\right)^2\right] = \frac{1}{n} \sum_{k=1}^{n} \log \left[1+\left(\frac{k}{n}\right)^2\right] \to \int_0^1 \log(1+x^2)dx \quad (n\to\infty),$$

hence

$$\prod_{k=1}^{n} \left[1+\left(\frac{k}{n}\right)^2\right] \sim e^{n\int_0^1 \log(1+x^2)dx}.$$

Using integration by parts to evaluate the integral in the exponent gives

$$\int_0^1 \log(1+x^2)dx = \log 2 + \frac{\pi}{2} - 2 = \alpha > 0.$$

Putting together these results gives the following asymptotical expression:

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}k^2}{(n^2+k^2)^2} \frac{\binom{n}{k}}{\binom{n+k}{n}} \sim \frac{\pi^2}{4} e^{-n(\log 2 + \frac{\pi}{2})} \quad (n\to\infty).$$

3. FORMULAE FOR POWER SUMS

Let $u_1,\dots,u_n \in \mathbb{R}$ such that $u_i \neq 0$, $|u_i| < 1$, $u_i \neq u_j$ if $1 \le i < j \le n$ and $x_i = 1-u_i$ $(i = 1,\dots,n)$. Then applying (1.3) gives

$$\sum_{i=1}^{n} \frac{1}{1-u_i} = (-1)^{n-1} \prod_{k=1}^{n}(1-u_k) \sum_{i=1}^{n} \frac{1}{(1-u_i)^2 \Pi_i(1-u_1,\dots,1-u_n)}$$
$$= \prod_{k=1}^{n}(1-u_k) \sum_{i=1}^{n} \frac{1}{(1-u_i)^2 \Pi_i(u_1,\dots,u_n)}.$$

Under the above conditions $\frac{1}{1-u_i} = \sum_{\ell=0}^{\infty} u_i^{\ell}$, hence

$$\sum_{i=1}^{n} \frac{1}{1-u_i} = \sum_{i=1}^{n}\sum_{\ell=0}^{\infty} u_i^{\ell} = \sum_{\ell=0}^{\infty}\sum_{i=1}^{n} u_i^{\ell} = \sum_{\ell=0}^{\infty} P(\ell), \tag{3.1}$$

where we use the denotation $P(\ell) := u_1^\ell + \cdots + u_n^\ell$ for $(\ell = 0, 1, 2, \ldots)$, and thus we have the following identities for $u_1, \ldots, u_n$:

$$\sum_{\ell=0}^{\infty} P(\ell) = \prod_{k=1}^{n}(1-u_k) \sum_{i=1}^{n} \frac{1}{(1-u_i)^2 \Pi_i(u_1, \ldots, u_n)}. \tag{3.2}$$

Apply now (3.2) to $u_k = \frac{k}{n+1}$ $(k = 1, \ldots, n)$. For the left-hand side:

$$\sum_{\ell=0}^{\infty} P(\ell) = \sum_{\ell=0}^{\infty}\sum_{k=1}^{n}\left(\frac{k}{n+1}\right)^{\ell} = \sum_{\ell=0}^{\infty}\frac{1}{(n+1)^\ell}\sum_{k=1}^{n}k^\ell = \sum_{\ell=0}^{\infty}\frac{\tilde{P}(\ell)}{(n+1)^\ell},$$

where $\tilde{P}(\ell) = 1^\ell + 2^\ell + \cdots + n^\ell$ $(\ell = 0, 1, 2, \ldots)$. On the other hand

$$\prod_{k=1}^{n}\left(1 - \frac{k}{n+1}\right) = \frac{n!}{(n+1)^n} \quad \text{and}$$

$$\Pi_i\left(\frac{1}{n+1}, \ldots, \frac{n}{n+1}\right) = \frac{(-1)^{n-i}(i-1)!(n-i)!}{(n+1)^{n-1}}.$$

Putting these expressions into the right-hand side of (3.2) we obtain

$$\sum_{\ell=0}^{\infty}\frac{\tilde{P}(\ell)}{(n+1)^\ell} = n(n+1)\sum_{j=1}^{n}\frac{(-1)^{j-1}}{j^2}\binom{n-1}{j-1}. \tag{3.3}$$

A simple rearrangemant of (3.3) and using $\frac{n}{j}\binom{n-1}{j-1} = \binom{n}{j}$ gives

$$\sum_{\ell=0}^{\infty}\frac{\tilde{P}(\ell)}{(n+1)^{\ell+1}} = \sum_{j=1}^{n}\frac{(-1)^{j-1}}{j}\binom{n}{j},$$

and since it is known that

$$\sum_{j=1}^{n}\frac{(-1)^{j-1}}{j}\binom{n}{j} = \int_0^1 \frac{1-(1-s)^n}{s}ds = \sum_{k=1}^{n}\frac{1}{k},$$

consequently we have the following identity for all fixed n:

$$\sum_{\ell=0}^{\infty}\frac{\tilde{P}(\ell)}{(n+1)^{\ell+1}} = \sum_{k=1}^{n}\frac{1}{k}.$$

Let us start now with (3.1) substituting $u_k = \frac{1}{x_k}$, $x_k > 1$ $(k = 1, \ldots, n)$ and $x_i \neq x_j$ if $1 \leq i < j \leq n$. Using the same technique applied above and the transformational rule d) we get the following identity for $x_1, \ldots, x_n$:

$$\sum_{\ell=0}^{\infty}\sum_{k=1}^{n}\frac{1}{x_k^\ell} = (-1)^{n-1}\prod_{k=1}^{n}(x_k - 1)\sum_{i=1}^{n}\frac{x_i^n}{(x_i-1)^2\Pi_i(x_1, \ldots, x_n)}. \tag{3.4}$$

Putting $x_k = k+1$ $(k = 1, \ldots, n)$ in (3.4) and simplifying the right-hand side we have the following equality:

$$\sum_{\ell=0}^{\infty}\left[\frac{1}{2^\ell} + \cdots + \frac{1}{(n+1)^\ell}\right] = \sum_{i=1}^{n}\frac{(-1)^{i-1}}{i}(i+1)^n\binom{n}{i}. \tag{3.5}$$

Observe that $\frac{1}{2^\ell} + \cdots + \frac{1}{(n+1)^\ell} = \zeta^{[n+1]}(\ell) - 1$, where $\zeta^{[n+1]}(\ell)$ denotes the $(n+1)$th partial sum of the series $\zeta(\ell) = \sum_{k=1}^{\infty}\frac{1}{k^\ell}$, if $\ell \geq 2$. Moreover let H_n denote the nth partial sum of the harmonic series, that is $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Using this quantity and separating the summands corresponding to $\ell = 0$ and $\ell = 1$ gives

$$n + H_{n+1} - 1 + \sum_{\ell=2}^{\infty}[\zeta^{[n+1]}(\ell) - 1] = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i}(i+1)^n \binom{n}{i}. \tag{3.6}$$

Rearranging (3.6) and taking the limit as $n \to \infty$ yields that:

$$\sum_{\ell=2}^{\infty}[\zeta(\ell) - 1] = 1 + \lim_{n\to\infty}\left[\sum_{1=1}^{n} \frac{(-1)^{i-1}}{i}(i+1)^n \binom{n}{i} - n - H_{n+1}\right]. \tag{3.7}$$

This relation prompted us to investigate the quantities arising in both sides of the equality. On one hand the sum of the series is equal to 1 as an easy computation shows below:

$$\begin{aligned}
\sum_{\ell=2}^{\infty}[\zeta(\ell) - 1] &= \sum_{\ell=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{k^\ell} = \sum_{k=2}^{\infty}\frac{1}{k^2}\sum_{j=0}^{\infty}\frac{1}{k^j} \\
&= \sum_{k=2}^{\infty}\frac{1}{k^2}\frac{1}{1-\frac{1}{k}} \\
&= \sum_{k=2}^{\infty}\frac{1}{k(k-1)} = 1.
\end{aligned}$$

Hence

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{(-1)^{i-1}}{i}(i+1)^n \binom{n}{i} - n - H_{n+1}\right] = 0.$$

In fact we prove more in the following lemma.

Lemma 3.1. *For all $n \geq 1$ the identity below holds*

$$\sum_{i=1}^{n} \frac{(-1)^{i-1}}{i}(i+1)^n \binom{n}{i} = n + H_n.$$

Proof. In the paper [1] we proved that if $x_1, \dots, x_n \in \mathbb{R}$ such that $x_i \neq x_j$ if $1 \leq i < j \leq n$ then

$$\sum_{i=1}^{n} \frac{x_i^j}{\Pi_i(x_i, \dots, x_n)} = \begin{cases} \sum_{k=1}^{n} x_k & \text{if} \quad j = n \\ 1 & \text{if} \quad j = n-1 \\ 0 & \text{if} \quad 0 \leq j \leq n-2. \end{cases}$$

Applying this result to $x_i = \frac{1}{i}$ $(i = 1, \dots, n)$ gives that

$$\sum_{i=1}^{n} \frac{(-1)^{i-1}}{i} i^{n-j} \binom{n}{i} = \begin{cases} H_n & \text{if} \quad j = n \\ 1 & \text{if} \quad j = n-1 \\ 0 & \text{if} \quad 0 \leq j \leq n-2. \end{cases}$$

Moreover

$$\begin{aligned}\sum_{i=1}^{n}\frac{(-1)^{i-1}}{i}(i+1)^n\binom{n}{i} &= \sum_{i=1}^{n}\frac{(-1)^{i-1}}{i}\left[\sum_{k=0}^{n} i^{n-k}\binom{n}{k}\right]\binom{n}{i} \\ &= \sum_{k=0}^{n}\left[\sum_{i=1}^{n}\frac{(-1)^{i-1}}{i} i^{n-k}\binom{n}{i}\right]\binom{n}{k} \\ &= \binom{n}{n-1} + H_n\binom{n}{n} = n + H_n\end{aligned}$$

as we stated. □

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