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GENERALIZED λ-NEWTON INEQUALITIES REVISITED

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ABSTRACT. We present in this work a new and shorter proof of the generalized λ-Newton inequalities for elementary symmetric functions defined on a self-conjugate set which lies essentially in the open right half-plane. We also point out some interesting consequences of the generalized λ-Newton inequalities. In particular, we establish an improved complex version of the arithmetic mean-geometric mean inequality along with the corresponding determinant-trace inequality for positive stable matrices.

Key words and phrases: Elementary symmetric functions, λ-Newton inequalities, generalized λ-Newton inequalities, arithmetic mean-geometric mean inequality, positive stable matrices, determinant-trace inequality.

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1. INTRODUCTION

The elementary symmetric functions on a set $S = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{C}$ are defined to be $E_0(x_1, x_2, \ldots, x_n) = 1$ and

$$E_k(x_1, x_2, \ldots, x_n) = \frac{\sum_{1\le j_1<j_2<\cdots<j_k\le n} x_{j_1}x_{j_2}\cdots x_{j_k}}{\binom{n}{k}}, \quad k = 1, 2, \ldots, n.$$

Throughout this paper, we simply write such functions as E_k, $\widetilde{E}_k$, or $\widehat{E}_k$ if the set S is specified or is clear from the context. In addition, we denote by $\#S$ the cardinality of S. We comment that if S represents the spectrum of some matrix A, then the elementary symmetric functions can be formulated in terms of the principal minors of A. The elementary symmetric functions can also be interpreted as the normalized coefficients in the monic polynomial whose zeros are given by S, counting multiplicities.

The celebrated Newton's inequalities concern a quadratic type relationship among the elementary symmetric functions, provided that S consists of real numbers. Specifically, this relationship can be expressed as follows: On any $S \subset \mathbb{R}$ with $\#S = n$,

$$E_k^2 \ge E_{k-1}E_{k+1}, \quad 1 \le k \le n-1.$$

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For background material with respect to Newton's inequalities, we refer the reader to [3, 8, 9]. In [8, 9], such inequalities are also extended to include higher order terms involving the elementary symmetric functions.

In light of the circumstances as mentioned earlier, in which S stands for the spectrum of a matrix or the zeros of a polynomial, it is natural to raise the question of whether Newton's inequalities continue to hold in the complex domain, i.e. on $S \subset \mathbb{C}$. In this scenario, the set S is always assumed to be self-conjugate, meaning that the non-real elements of S appear in conjugate pairs. Such a condition on S ensures that the elementary symmetric functions remain real-valued.

Continuing with the question regarding Newton's inequalities on a self-conjugate set, the answer turns out to be, in general, negative. Nevertheless, it is shown in [6] that for any self-conjugate S in the open right half-plane, possibly including zero elements, with $\#S = n$, there exists some $0 < \lambda \leq 1$ such that

$$E_k^2 \geq \lambda E_{k-1} E_{k+1}, \quad 1 \leq k \leq n-1.$$

These inequalities are developed independently in [7] over a self-conjugate set representing the spectrum of the Drazin inverse of a singular M-matrix. In addition, they are termed in [7] the Newton-like inequalities. In order to avoid potential ambiguity, from now on, we shall refer to such inequalities as the λ-Newton inequalities.[1] Obviously, the λ-Newton inequalities reflect a generalized quadratic type relationship among the elementary symmetric functions when it comes to the complex domain.

The results of [6, 7] are further broadened in [11]. It is illustrated there that the following generalized λ-Newton inequalities are fulfilled under the same assumptions as in [6]:

$$E_k E_l \geq \lambda E_{k-1} E_{l+1}, \quad 1 \leq k \leq l \leq n-1.$$

As pointed out in [11], the above formulation includes the λ-Newton, with $l = k$, as well as Newton's, with $l = k$ and $\lambda = 1$, inequalities; moreover, it constitutes a stronger result in that it does not follow from the λ-Newton inequalities.

We mention that the notion of generalized λ-Newton inequalities is also motivated by the literature regarding log-concave, or second order Pólya-frequency, sequences [1, 10]. In fact, a sequence $\{E_k\}$ consisting of nonnegative numbers E_k is said to be log-concave if $E_k^2 \geq E_{k-1}E_{k+1}$ for all k. It is well-known that $\{E_k\}$ is log-concave iff $E_k E_l \geq E_{k-1}E_{l+1}$ for all $k \leq l$, assuming that $\{E_k\}$ has no internal zeros. This shows the close connection, in the special case when $\lambda = 1$, between the λ-Newton and the generalized λ-Newton inequalities, prompting us to look into the latter for the overall situation with $0 < \lambda \leq 1$.

The method in [11] is, in essence, in line with that of [3]. It reveals how the elementary symmetric functions change as the set S is augmented by a real number or a conjugate pair. Such an approach, therefore, may be useful for further investigating, for example, the λ-Newton inequalities involving higher order terms as studied in [6, 8] and other problems related to the λ-Newton inequalities as treated in [4, 7]. The proof in [11], however, is quite lengthy.

As a follow-up to [11], we demonstrate in this work that the generalized λ-Newton inequalities can be confirmed in a more elegant fashion without explicit knowledge of the variations in the elementary symmetric functions due to the changes in $\#S$. This new and briefer proof is largely inspired by [6, 8, 9]. In addition to the proof, we derive some interesting implications of the generalized λ-Newton inequalities. In particular, we strengthen a complex version of the arithmetic mean-geometric mean inequality which appears in [6]. The associated determinant-trace inequality for positive stable matrices is also established.

[1] The author would like to thank Professor Charles R. Johnson for discussion on this terminology.

2. MAIN RESULTS

We begin with some necessary preliminary results. The following conclusion can be found in [8, 9], whose proof is included here for completeness.

Lemma 2.1 ([8, 9])**.** *Let $p(x)$ be a monic polynomial of degree n whose zeros are $x_1, x_2, \dots, x_n \in \mathbb{C}$, counting multiplicities. Denote the zeros of $p'(x)$, the derivative of $p(x)$, by $y_1, y_2, \dots, y_{n-1}$, again counting multiplicities. Then for all $0 \le k \le n-1$,*

$$E_k(x_1, x_2, \dots, x_n) = E_k(y_1, y_2, \dots, y_{n-1}).$$

Proof. Denote that $E_k = E_k(x_1, x_2, \dots, x_n)$ and $\widetilde{E}_k = E_k(y_1, y_2, \dots, y_{n-1})$. It is a familiar fact that

$$p(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} E_j x^{n-j}. \tag{2.1}$$

From this, the monic polynomial associated with $p'(x)$ can be written as

$$q(x) = \frac{1}{n} p'(x) = \sum_{j=0}^{n-1} (-1)^j \frac{n-j}{n} \binom{n}{j} E_j x^{n-j-1}.$$

On the other hand, we notice that, similar to (2.1),

$$q(x) = \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \widetilde{E}_j x^{n-j-1}.$$

The conclusion now follows immediately from a comparison of the two expressions for $q(x)$ in terms of E_j and $\widetilde{E}_j$, respectively. ☐

The next result is a direct consequence of Lemma 2.1.

Theorem 2.2. *Suppose that $p(x)$ is a monic polynomial of degree n with zeros $x_1, x_2, \dots, x_n \in \mathbb{C}$, counting multiplicities. For any $1 \le m \le n$, denote the zeros of $p^{(n-m)}(x)$, the $(n-m)$-th derivative of $p(x)$, by $y_1, y_2, \dots, y_m$, also counting multiplicities. Then for all $0 \le k \le m$,*

$$E_k(x_1, x_2, \dots, x_n) = E_k(y_1, y_2, \dots, y_m).$$

We also need the conclusion below, which is stated in [8, 9] for the case of real numbers. The proof is straightforward and thus is omitted.

Lemma 2.3 ([8, 9])**.** *Suppose that $x_1, x_2, \dots, x_n \in \mathbb{C}$ are such that $x_j \neq 0$, $1 \le j \le n$. Set $z_j = x_j^{-1}$ for all j. Denote that $E_k = E_k(x_1, x_2, \dots, x_n)$ and $\widehat{E}_k = E_k(z_1, z_2, \dots, z_n)$. Then for any $0 \le k \le n$, $E_k = E_n \widehat{E}_{n-k}$.*

In the sequel, we assume that $0 < \lambda \le 1$.

We are now ready to prove by induction that the generalized λ-Newton inequalities hold on any self-conjugate set $S = \{x_1, x_2, \dots, x_n\}$ under the assumption that $S \subset \Omega$, where Ω is a wedge in the form [6, 11]

$$\Omega = \left\{ z : |\arg z| \le \cos^{-1} \sqrt{\lambda} \right\}. \tag{2.2}$$

An immediate outcome of this assumption, i.e. $S \subset \Omega$, is that $E_k \ge 0$ for any k. Before proceeding, we also remark that this condition is equivalent to the following: For any nonzero $x_j \in S$,

$$\frac{\operatorname{Re} x_j}{|x_j|} \ge \sqrt{\lambda}. \tag{2.3}$$

Similar to [11], we first verify the cases when $n = 2,\ 3$. Seeing the fact that Newton's inequalities are satisfied whenever $S \subset \mathbb{R}$, we only consider here the situation that at least one nonzero conjugate pair is present in S. In what follows, a, b, and c are all real numbers.

Lemma 2.4. *Suppose that $S = \{a \pm bi\} \subset \Omega$, where $a > 0$ and Ω is given as in (2.2). Then the generalized λ-Newton inequality holds on S, i.e.*

$$E_1^2 \geq \lambda E_0 E_2.$$

Proof. Let $p(x) = (x - x_1)(x - x_2)$ be the monic polynomial with zeros $x_{1,2} = a \pm bi$. Clearly, $p(x) = x^2 - 2ax + a^2 + b^2$. Next, by comparing with (2.1), we obtain that $E_1 = a$ and $E_2 = a^2 + b^2$. Hence

$$E_1^2 - \lambda E_0 E_2 = a^2 - \lambda(a^2 + b^2) \geq 0.$$

□

We comment that, although it seems simple, the foregoing proof indeed suggests several important issues. First, equalities are possible in the generalized λ-Newton inequalities. Second, such inequalities may fail on S when it contains nonzero purely imaginary conjugate pairs. And finally, generally speaking, such inequalities may not hold if λ is chosen to be greater than 1.

Lemma 2.5. *Suppose that $S = \{a \pm bi,\ c\} \subset \Omega$, where $a > 0$ and Ω is given as in (2.2). Then the generalized λ-Newton inequalities hold on S, i.e.*

$$E_1^2 \geq \lambda E_0 E_2, \quad E_2^2 \geq \lambda E_1 E_3, \quad \textit{and} \quad E_1 E_2 \geq \lambda E_0 E_3.$$

Proof. In a similar fashion as in the proof of Lemma 2.4, we find that

$$E_1 = \frac{2a + c}{3}, \quad E_2 = \frac{a^2 + b^2 + 2ac}{3}, \quad \text{and} \quad E_3 = c(a^2 + b^2).$$

Hence we arrive at:

$$\begin{aligned} E_1^2 - \lambda E_0 E_2 &\geq E_1^2 - \frac{a^2}{a^2 + b^2} E_0 E_2 \\ &= \frac{(a - c)^2}{9} + \frac{2ab^2 c}{3(a^2 + b^2)} \geq 0, \end{aligned}$$

$$\begin{aligned} E_2^2 - \lambda E_1 E_3 &\geq E_2^2 - \frac{a^2}{a^2 + b^2} E_1 E_3 \\ &= \frac{1}{9}\left[a^2(a - c)^2 + 2a^2 b^2 + 4ab^2 c + b^4\right] \geq 0, \end{aligned}$$

and

$$\begin{aligned} E_1 E_2 - \lambda E_0 E_3 &\geq E_1 E_2 - \frac{a^2}{a^2 + b^2} E_0 E_3 \\ &= \frac{1}{9}\left[2a(a - c)^2 + 2ab^2 + b^2 c\right] \geq 0. \end{aligned}$$

□

The proof of Lemma 2.5 indicates the possible failure of the generalized λ-Newton inequalities for the case when S, except for its zero elements if present, does not lie entirely in the open right half-plane. One such instance can be observed by considering the lower bound estimate of $E_1 E_2 - \lambda E_0 E_3$, assuming that a and c are both negative. Thus the restriction that $S \subset \Omega$ is necessary to ensure the satisfaction of the generalized λ-Newton inequalities.

Next, we turn to an inductive hypothesis: Suppose that the generalized λ-Newton inequalities are realized on all self-conjugate $S \subset \Omega$ such that $\#S \leq n - 1$. The following result illustrates how this hypothesis guarantees that those inequalities continue to hold on any self-conjugate $S \subset \Omega$ with $\#S = n$.

Lemma 2.6. *Suppose that the generalized λ-Newton inequalities hold on any self-conjugate set $\widetilde{S} \subset \Omega$ with $\#\widetilde{S} \leq n-1$, where Ω is given as in (2.2). Then such inequalities are also satisfied on any self-conjugate set $S \subset \Omega$ with $\#S = n$. In other words, for any $1 \leq k \leq l \leq n-1$,*

$$E_k E_l \geq \lambda E_{k-1} E_{l+1}$$

holds on S.

Proof. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a self-conjugate set in Ω. Denote that $E_k = E_k(x_1, x_2, \ldots, x_n)$. Set $p(x) = \prod_{j=1}^n (x - x_j)$, the monic polynomial of degree n with zeros as in S.

For arbitrary, but fixed, $1 \leq m \leq n-1$, we consider $q(x) = p^{(n-m)}(x)$, the $(n-m)$-th derivative of $p(x)$. The zeros of $q(x)$ form a self-conjugate set $\widetilde{S} = \{y_1, y_2, \ldots, y_m\}$ with $\#\widetilde{S} \leq n-1$. By the Gauss-Lucas theorem [2], we see that $\widetilde{S} \subset \Omega$. Hence the generalized λ-Newton inequalities hold on $\widetilde{S}$, i.e. on letting $\widetilde{E}_k = E_k(y_1, y_2, \ldots, y_m)$, we have that

$$\widetilde{E}_k \widetilde{E}_l \geq \lambda \widetilde{E}_{k-1} \widetilde{E}_{l+1}$$

for all $1 \leq k \leq l \leq m-1$. This, according to Theorem 2.2, verifies that for all $1 \leq k \leq l \leq m-1$,

$$E_k E_l \geq \lambda E_{k-1} E_{l+1}.$$

Since $1 \leq m \leq n-1$ is arbitrary, we conclude that the generalized λ-Newton inequalities are satisfied on S for all $1 \leq k \leq l \leq n-2$.

It remains to show that for each $1 \leq k \leq n-1$,

$$E_k E_{n-1} \geq \lambda E_{k-1} E_n.$$

Obviously, this statement is true when $E_n = 0$. We assume, therefore, that $E_n > 0$, which translates into $x_j \neq 0$ for $1 \leq j \leq n$. Set $z_j = x_j^{-1}$ for all j. Notice that $\widehat{S} = \{z_1, z_2, \ldots, z_n\}$ is self-conjugate and, additionally, that $\widehat{S} \subset \Omega$. Denote that $\widehat{E}_k = E_k(z_1, z_2, \ldots, z_n)$. Then, by Lemma 2.3,

$$E_k = E_n \widehat{E}_{n-k}$$

for all k. We now observe that for any $1 \leq k \leq n-1$, $E_k E_{n-1} \geq \lambda E_{k-1} E_n$ iff

$$\widehat{E}_1 \widehat{E}_k \geq \lambda \widehat{E}_{k+1} = \lambda \widehat{E}_0 \widehat{E}_{k+1}.$$

Again, based on Theorem 2.2 and the Gauss-Lucas theorem, the validity of this latter statement can be justified whenever $1 \leq k \leq n-2$. Thus it is enough to establish that

$$\widehat{E}_1 \widehat{E}_{n-1} \geq \lambda \widehat{E}_n.$$

It is more convenient to write the above as $\widehat{E}_1 \frac{\widehat{E}_{n-1}}{\widehat{E}_n} \geq \lambda$. Note that

$$\widehat{E}_1 = \frac{1}{n} \sum_{j=1}^n z_j = \frac{1}{n} \sum_{j=1}^n \frac{\operatorname{Re} x_j}{|x_j|^2} \quad \text{and} \quad \frac{\widehat{E}_{n-1}}{\widehat{E}_n} = \frac{1}{n} \sum_{j=1}^n x_j = \frac{1}{n} \sum_{j=1}^n \operatorname{Re} x_j.$$

By Cauchy's inequality, we obtain that

$$\widehat{E}_1 \frac{\widehat{E}_{n-1}}{\widehat{E}_n} \geq \frac{1}{n^2} \left(\sum_{j=1}^n \frac{\operatorname{Re} x_j}{|x_j|} \right)^2 \geq \lambda.$$

This completes the proof. $\square$

With either Lemma 2.4 or Lemma 2.5, along with the fact that the stronger Newton's inequalities hold on sets of real numbers, it is clear that Lemma 2.6 serves as the final step towards an inductive proof of the generalized λ-Newton inequalities. Our main conclusion can be stated as follows.

Theorem 2.7. *For any fixed $0 < \lambda \leq 1$, let Ω be given as in (2.2). Suppose that S is a self-conjugate set such that $S \subset \Omega$ and that $\#S = n$. Then, for all $1 \leq k \leq l \leq n-1$, the generalized λ-Newton inequalities*

$$E_k E_l \geq \lambda E_{k-1} E_{l+1} \tag{2.4}$$

hold on S.

We comment that a similar conclusion follows from Theorem 2.7 when the wedge-shaped region is reflected across the imaginary axis. Specifically, for $0 < \lambda \leq 1$, we consider

$$\Omega = \left\{ z : |\arg z - \pi| \leq \cos^{-1} \sqrt{\lambda} \right\}$$

and a self-conjugate set $S = \{x_1, x_2, \dots, x_n\} \subset \Omega$. Let E_k be the elementary symmetric functions on S and $\widetilde{E}_k$ be those on $\widetilde{S} = \{-x_1, -x_2, \dots, -x_n\}$. It is observed in [6] that $\widetilde{E}_k = (-1)^k E_k$. Hence, by applying Theorem 2.7 to $\widetilde{E}_k$, we obtain that

$$|E_k E_l| \geq \lambda |E_{k-1} E_{l+1}| \tag{2.5}$$

for all $1 \leq k \leq l \leq n-1$.

As a final remark in this section, we mention that our results can also be interpreted in the context that the set S is prescribed while λ is allowed to vary in $(0, 1]$. In this alternative setting, by (2.3), we see that the best λ can be written as

$$\lambda_{\max} = \min_{0 \neq x_j \in S} \frac{\mathrm{Re}^2 x_j}{|x_j|^2},$$

provided that the trivial case is excluded, i.e. that $\{x \in S : x \neq 0\} \neq \emptyset$.

3. IMPLICATIONS

In this section, we discuss some interesting consequences of the generalized λ-Newton inequalities.

First, we look at a complex counterpart of the arithmetic mean-geometric mean inequality. It is illustrated in [6] that under the same assumptions as in Theorem 2.7,

$$E_1 \geq \lambda^{\frac{n-1}{2}} E_n^{\frac{1}{n}}.$$

In view of Theorem 2.7, this inequality can be improved as follows.

Theorem 3.1. *For any fixed $0 < \lambda \leq 1$, let Ω be as in (2.2). Suppose that $S = \{x_1, x_2, \dots, x_n\}$ is a self-conjugate set such that $S \subset \Omega$. Then*

$$E_1 \geq \lambda^{\frac{n-1}{n}} E_n^{\frac{1}{n}}, \tag{3.1}$$

i.e.

$$\frac{1}{n} \sum_{j=1}^{n} x_j \geq \lambda^{\frac{n-1}{n}} \left(\prod_{j=1}^{n} x_j \right)^{\frac{1}{n}}.$$

Proof. For any fixed $1 \le k \le n-1$, we see from (2.4) that

$$(E_kE_k)(E_kE_{k+1})\cdots(E_kE_{n-1}) \ge \lambda^{n-k}(E_{k-1}E_{k+1})(E_{k-1}E_{k+2})\cdots(E_{k-1}E_n),$$

which yields that

$$E_k^{n-k+1} \ge \lambda^{n-k}E_{k-1}^{n-k}E_n.$$

In particular, on setting $k = 1$, the above inequality reduces to (3.1). □

It should be pointed out that from (2.4), we can also derive an expression involving two consecutive E_l's. Specifically, fixing any $1 \le l \le n-1$, we obtain that

$$(E_1E_l)(E_2E_l)\cdots(E_lE_l) \ge \lambda^l(E_0E_{l+1})(E_1E_{l+1})\cdots(E_{l-1}E_{l+1})$$

and, consequently, that

$$E_l^{\frac{1}{l}} \ge \lambda^{\frac{1}{l+1}}E_{l+1}^{\frac{1}{l+1}} \tag{3.2}$$

for any $1 \le l \le n-1$. It is interesting to note that formula (3.2) provides another way of showing (3.1) on condition that $x_j \ne 0$. On letting $l = n-1$ and considering $\widehat{E}_k$ as defined in the proof of Theorem 2.7, we have that

$$\widehat{E}_{n-1}^n \ge \lambda^{n-1}\widehat{E}_n^{n-1},$$

which yields that

$$\frac{\widehat{E}_{n-1}}{\widehat{E}_n} \ge \lambda^{\frac{n-1}{n}}\widehat{E}_n^{-\frac{1}{n}},$$

and thus (3.1) after replacing z_j^{-1} with x_j.

We remark, however, that by taking $l = 1, 2, \dots, n-1$ in (3.2), it follows that

$$E_1 \ge \lambda^{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}E_n^{\frac{1}{n}},$$

which turns out to be not as tight as (3.1) since $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \ge \frac{n-1}{n}$.

Finally, we apply Theorem 3.1 to positive stable matrices. For references, see, for example, [5]. Obviously, given any $n \times n$ matrix A, its spectrum is self-conjugate, $E_1 = \frac{1}{n}\mathrm{tr}A$, and $E_n = \det A$, where E_k are defined on the spectrum of A. Recall that a matrix is said to be positive stable when its spectrum is located in the open right half-plane. Therefore, Theorem 3.1 can be rephrased in the following manner:

Theorem 3.2. *Let A be an $n \times n$ positive stable matrix whose spectrum $\sigma(A) \subset \Omega$, where Ω is defined as in (2.2). Then*

$$\left(\frac{1}{n}\mathrm{tr}A\right)^n \ge \lambda^{n-1}\det A. \tag{3.3}$$

We comment that a special case of (3.3) with $\lambda = 1$ applies to M- and inverse M-matrices, on which Newton's inequalities are indeed fulfilled [4]. It should be pointed out, however, that M- and inverse M-matrices form the only class of positive stable matrices with non-real eigenvalues which is known in the literature to satisfy Newton's inequalities. Hence (3.3) serves as an overall result which applies to general positive stable matrices.

4. CONCLUSIONS

Inspired by [6, 8, 9], we propose here a more elegant inductive proof of the generalized λ-Newton inequalities which are verified in [11]. We show that it is possible to confirm these inequalities without explicit formulations of the elementary symmetric functions being involved, which is a noteworthy difference between the current work and [11].

As illustrated in [11], the generalized λ-Newton inequalities are indeed in a stronger form as compared with the λ-Newton inequalities in [6, 7]. We also explore here several useful results which follow directly from the generalized λ-Newton inequalities. In particular, we show that it is possible to strengthen the complex version of the important arithmetic mean-geometric mean inequality as in [6].

Regarding potential future work, we mention herein a few topics: First, the generalized λ-Newton inequalities may be further improved by considering a subset of the wedge Ω. Second, it is an intriguing question as to fully characterize the case of equalities. Third, it remains to be answered whether similar inequalities can be developed on a self-conjugate set which does not lie entirely in the open right or left half-plane. Fourth, the generalized λ-Newton inequalities may be applied to, for example, other problems related to eigenvalues and even problems in combinatorics. To sum up, we strongly believe that much work still needs to be done concerning the generalized λ-Newton and associated inequalities.

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