

Journal of Inequalities in Pure and Applied Mathematics

ON THE STAR PARTIAL ORDERING OF NORMAL MATRICES

JORMA K. MERIKOSKI AND XIAOJI LIU

Department of Mathematics, Statistics and Philosophy
FI-33014 University of Tampere, Finland
EMail: jorma.merikoski@uta.fi

College of Computer and Information Science
Guangxi University for Nationalities
Nanning 530006, China.
EMail: xiaojiliu72@yahoo.com.cn

©2000 Victoria University
ISSN (electronic): 1443-5756
278-05



volume 7, issue 1, article 17,
2006.

*Received 19 September, 2005;
accepted 09 January, 2006.*

Communicated by: S. Puntanen

[Abstract](#)

[Contents](#)

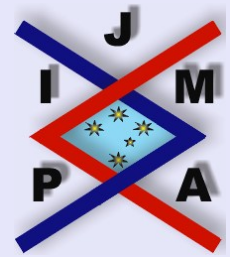


[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)



Abstract

We order the space of complex $n \times n$ matrices by the star partial ordering \leq^* . So $A \leq^* B$ means that $A^*A = A^*B$ and $AA^* = BA^*$. We find several characterizations for $A \leq^* B$ in the case of normal matrices. As an application, we study how $A \leq^* B$ relates to $A^2 \leq^* B^2$. The results can be extended to study how $A \leq^* B$ relates to $A^k \leq^* B^k$, where $k \geq 2$ is an integer.

2000 Mathematics Subject Classification: 15A45, 15A18.

Key words: Star partial ordering, Normal matrices, Eigenvalues.

We thank the referee for various suggestions that improved the presentation of this paper. The second author thanks Guangxi Science Foundation (0575032) for the support.

Contents

1	Introduction	3
2	Characterizations of $A \leq^* B$	4
3	Relationship between $A \leq^* B$ and $A^2 \leq^* B^2$	11
4	Remarks	15
	References	

On the Star Partial Ordering of Normal Matrices

Jorma K. Merikoski and Xiaoji Liu

Title Page

Contents



Go Back

Close

Quit

Page 2 of 16

1. Introduction

Throughout this paper, we consider the space of complex $n \times n$ matrices ($n \geq 2$). We order it by the star partial ordering \leq^* . So $A \leq^* B$ means that $A^*A = A^*B$ and $AA^* = BA^*$. Our motivation rises from the following

Theorem 1.1 (Baksalary and Pukelsheim [1, Theorem 3]). *Let A and B be Hermitian and nonnegative definite. Then $A^2 \leq^* B^2$ if and only if $A \leq^* B$.*

We cannot drop out the assumption on nonnegative definiteness.

Example 1.1. *Let*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $A^2 \leq^ B^2$, but not $A \leq^* B$.*

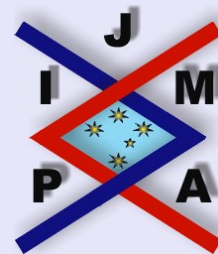
We will study how $A \leq^* B$ relates to $A^2 \leq^* B^2$ in the case of normal matrices. We will see (Theorem 3.1) that the “if” part of Theorem 1.1 remains valid. However, it is not valid for all matrices.

Example 1.2. *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Then $A \leq^ B$, but not $A^2 \leq^* B^2$.*

In Section 2, we will give several characterizations of $A \leq^* B$. Thereafter, in Section 3, we will apply some of them in discussing our problem. Finally, in Section 4, we will complete our paper with some remarks.



On the Star Partial Ordering of
Normal Matrices

Jorma K. Merikoski and Xiaoji Liu

Title Page

Contents



Go Back

Close

Quit

Page 3 of 16

2. Characterizations of $A \leq^* B$

Hartwig and Styan ([2, Theorem 2]) presented eleven characterizations of $A \leq^* B$ for general matrices. One of them uses singular value decompositions. In the case of normal matrices, spectral decompositions are more accessible.

Theorem 2.1. *Let A and B be normal matrices with $1 \leq \text{rank } A < \text{rank } B$. Then the following conditions are equivalent:*

- (a) $A \leq^* B$.
 (b) *There is a unitary matrix U such that*

$$U^*AU = \begin{pmatrix} D & O \\ O & O \end{pmatrix}, \quad U^*BU = \begin{pmatrix} D & O \\ O & E \end{pmatrix},$$

where D is a nonsingular diagonal matrix and $E \neq O$ is a diagonal matrix.

- (c) *There is a unitary matrix U such that*

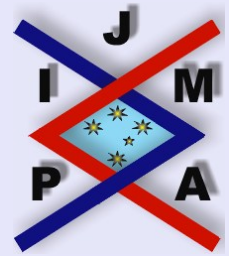
$$U^*AU = \begin{pmatrix} F & O \\ O & O \end{pmatrix}, \quad U^*BU = \begin{pmatrix} F & O \\ O & G \end{pmatrix},$$

where F is a nonsingular square matrix and $G \neq O$.

- (d) *If a unitary matrix U satisfies*

$$U^*AU = \begin{pmatrix} F & O \\ O & O \end{pmatrix}, \quad U^*BU = \begin{pmatrix} F' & O \\ O & G \end{pmatrix},$$

where F is a nonsingular square matrix, F' is a square matrix of the same dimension, and $G \neq O$, then $F = F'$.



On the Star Partial Ordering of Normal Matrices

Jorma K. Merikoski and Xiaoji Liu

Title Page

Contents



Go Back

Close

Quit

Page 4 of 16

(e) If a unitary matrix U satisfies

$$U^*AU = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \quad U^*BU = \begin{pmatrix} \mathbf{D}' & \mathbf{O} \\ \mathbf{O} & \mathbf{E} \end{pmatrix},$$

where \mathbf{D} is a nonsingular diagonal matrix, \mathbf{D}' is a diagonal matrix of the same dimension, and $\mathbf{E} \neq \mathbf{O}$ is a diagonal matrix, then $\mathbf{D} = \mathbf{D}'$.

(f) If a unitary matrix U satisfies

$$U^*AU = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix},$$

where \mathbf{D} is a nonsingular diagonal matrix, then

$$U^*BU = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{G} \end{pmatrix},$$

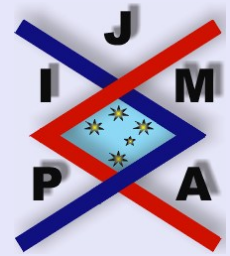
where $\mathbf{G} \neq \mathbf{O}$.

(g) All eigenvectors corresponding to nonzero eigenvalues of \mathbf{A} are eigenvectors of \mathbf{B} corresponding to the same eigenvalues.

The reason to assume $1 \leq \text{rank } \mathbf{A} < \text{rank } \mathbf{B}$ is to omit the trivial cases $\mathbf{A} = \mathbf{O}$ and $\mathbf{A} = \mathbf{B}$.

Proof. We prove this theorem in four parts.

Part 1. (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a).



On the Star Partial Ordering of Normal Matrices

Jorma K. Merikoski and Xiaoji Liu

Title Page

Contents



Go Back

Close

Quit

Page 5 of 16

(a) \Rightarrow (b). Assume (a). Then, by normality, A^* and B commute and are therefore simultaneously diagonalizable (see, e.g., [3, Theorem 1.3.19]). Since A and A^* have the same eigenvectors (see, e.g., [3, Problem 2.5.20]), also A and B are simultaneously diagonalizable. Hence (recall the assumption on the ranks) there exists a unitary matrix U such that

$$U^*AU = \begin{pmatrix} D & O \\ O & O \end{pmatrix}, \quad U^*BU = \begin{pmatrix} D' & O \\ O & E \end{pmatrix},$$

where D is a nonsingular diagonal matrix, D' is a diagonal matrix of the same dimension, and $E \neq O$ is a diagonal matrix. Now $A^*A = A^*B$ implies $D^*D = D^*D'$ and further $D = D'$. Hence (b) is valid.

(b) \Rightarrow (c). Trivial.

(c) \Rightarrow (a). Direct calculation.

Part 2. (a) \Rightarrow (d) \Rightarrow (e) \Rightarrow (a).

This is a trivial modification of Part 1.

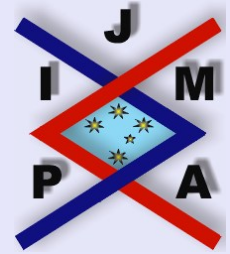
Part 3. (b) \Leftrightarrow (f).

(b) \Rightarrow (f). Assume (b). Let U be a unitary matrix satisfying

$$U^*AU = \begin{pmatrix} D & O \\ O & O \end{pmatrix}.$$

By (b), there exists a unitary matrix V such that

$$V^*AV = \begin{pmatrix} D' & O \\ O & O \end{pmatrix}, \quad V^*BV = \begin{pmatrix} D' & O \\ O & E \end{pmatrix},$$



On the Star Partial Ordering of Normal Matrices

Jorma K. Merikoski and Xiaoji Liu

Title Page

Contents



Go Back

Close

Quit

Page 6 of 16

where D' is a nonsingular diagonal matrix and $E \neq O$ is a diagonal matrix. Interchanging the columns of V if necessary, we can assume $D' = D$.

Let $U = (U_1 \ U_2)$ be such a partition that

$$U^*AU = \begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} A (U_1 \ U_2) = \begin{pmatrix} U_1^*AU_1 & U_1^*AU_2 \\ U_2^*AU_1 & U_2^*AU_2 \end{pmatrix} = \begin{pmatrix} D & O \\ O & O \end{pmatrix}.$$

Then, for the corresponding partition $V = (V_1 \ V_2)$, we have

$$V^*AV = \begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix} A (V_1 \ V_2) = \begin{pmatrix} V_1^*AV_1 & V_1^*AV_2 \\ V_2^*AV_1 & V_2^*AV_2 \end{pmatrix} = \begin{pmatrix} D & O \\ O & O \end{pmatrix}$$

and

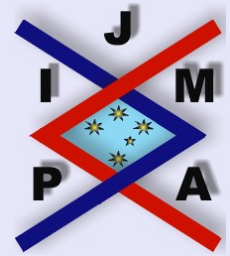
$$V^*BV = \begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix} B (V_1 \ V_2) = \begin{pmatrix} V_1^*BV_1 & V_1^*BV_2 \\ V_2^*BV_1 & V_2^*BV_2 \end{pmatrix} = \begin{pmatrix} D & O \\ O & E \end{pmatrix}.$$

Noting that

$$A = (V_1 \ V_2) \begin{pmatrix} D & O \\ O & O \end{pmatrix} \begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix} = (V_1 \ V_2) \begin{pmatrix} DV_1^* \\ O \end{pmatrix} = V_1DV_1^*,$$

we therefore have

$$\begin{aligned} U^*BU &= \begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} (V_1 \ V_2) \begin{pmatrix} D & O \\ O & E \end{pmatrix} \begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix} (U_1 \ U_2) \\ &= \begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} (V_1 \ V_2) \begin{pmatrix} DV_1^* \\ EV_2^* \end{pmatrix} (U_1 \ U_2) \end{aligned}$$



On the Star Partial Ordering of Normal Matrices

Jorma K. Merikoski and Xiaoji Liu

Title Page

Contents



Go Back

Close

Quit

Page 7 of 16

$$\begin{aligned}
&= \begin{pmatrix} U_1^*V_1 & U_1^*V_2 \\ U_2^*V_1 & U_2^*V_2 \end{pmatrix} \begin{pmatrix} DV_1^*U_1 & DV_1^*U_2 \\ EV_2^*U_1 & EV_2^*U_2 \end{pmatrix} \\
&= \begin{pmatrix} U_1^*V_1 & O \\ O & U_2^*V_2 \end{pmatrix} \begin{pmatrix} DV_1^*U_1 & O \\ O & EV_2^*U_2 \end{pmatrix} \\
&= \begin{pmatrix} U_1^*V_1DV_1^*U_1 & O \\ O & U_2^*V_2EV_2^*U_2 \end{pmatrix} \\
&= \begin{pmatrix} U_1^*AU_1 & O \\ O & U_2^*V_2EV_2^*U_2 \end{pmatrix} = \begin{pmatrix} D & O \\ O & U_2^*V_2EV_2^*U_2 \end{pmatrix},
\end{aligned}$$

and so (f) follows.

(f) \Rightarrow (b). Assume (f). Let U be a unitary matrix such that

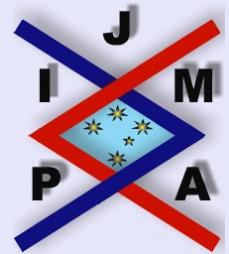
$$U^*AU = \begin{pmatrix} D & O \\ O & O \end{pmatrix},$$

where D is a nonsingular diagonal matrix. Then, by (f),

$$U^*BU = \begin{pmatrix} D & O \\ O & G \end{pmatrix},$$

where $G \neq O$. Since G is normal, there exists a unitary matrix W such that $E = W^*GW$ is a diagonal matrix. Let

$$V = U \begin{pmatrix} I & O \\ O & W \end{pmatrix}.$$



On the Star Partial Ordering of
Normal Matrices

Jorma K. Merikoski and Xiaoji Liu

Title Page

Contents



Go Back

Close

Quit

Page 8 of 16

Then

$$\begin{aligned} V^*AV &= \begin{pmatrix} I & O \\ O & W^* \end{pmatrix} U^*AU \begin{pmatrix} I & O \\ O & W \end{pmatrix} \\ &= \begin{pmatrix} I & O \\ O & W^* \end{pmatrix} \begin{pmatrix} D & O \\ O & O \end{pmatrix} \begin{pmatrix} I & O \\ O & W \end{pmatrix} = \begin{pmatrix} D & O \\ O & O \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} V^*BV &= \begin{pmatrix} I & O \\ O & W^* \end{pmatrix} U^*BU \begin{pmatrix} I & O \\ O & W \end{pmatrix} \\ &= \begin{pmatrix} I & O \\ O & W^* \end{pmatrix} \begin{pmatrix} D & O \\ O & G \end{pmatrix} \begin{pmatrix} I & O \\ O & W \end{pmatrix} = \begin{pmatrix} D & O \\ O & E \end{pmatrix}. \end{aligned}$$

Thus (b) follows.

Part 4. (b) \Leftrightarrow (g).

This is an elementary fact. □

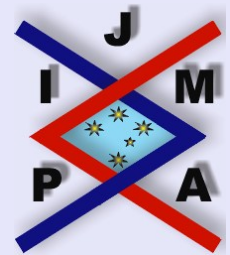
Corollary 2.2. *Let A and B be normal matrices. If $A \leq^* B$, then $AB = BA$.*

Proof. Apply (b). □

The converse does not hold (even assuming $\text{rank } A < \text{rank } B$), see Example 2.1. The normality assumption cannot be dropped out, see Example 2.2.

Example 2.1. *Let*

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$



On the Star Partial Ordering of Normal Matrices

Jorma K. Merikoski and Xiaoji Liu

Title Page

Contents



Go Back

Close

Quit

Page 9 of 16

Then $\mathbf{AB} = \mathbf{BA}$ and $\text{rank } \mathbf{A} < \text{rank } \mathbf{B}$, but $\mathbf{A} \leq^* \mathbf{B}$ does not hold. However, $\frac{1}{2}\mathbf{A} \leq^* \mathbf{B}$, which makes us look for a counterexample such that $c\mathbf{A} \leq^* \mathbf{B}$ does not hold for any $c \neq 0$. It is easy to see that we must have $n \geq 3$. The matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

obviously have the required properties.

Example 2.2. Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $\mathbf{A} \leq^* \mathbf{B}$, but $\mathbf{AB} \neq \mathbf{BA}$.



On the Star Partial Ordering of
Normal Matrices

Jorma K. Merikoski and Xiaoji Liu

Title Page

Contents



Go Back

Close

Quit

Page 10 of 16

3. Relationship between $A \leq^* B$ and $A^2 \leq^* B^2$

We will see that $A \leq^* B \Rightarrow A^2 \leq^* B^2$ for normal matrices, but the converse needs an extra condition, which we first present using eigenvalues.

Theorem 3.1. *Let A and B be normal matrices with $1 \leq \text{rank } A < \text{rank } B$. Then*

(a)
$$A \leq^* B$$

is equivalent to the following:

(b)
$$A^2 \leq^* B^2$$

and if A and B have nonzero eigenvalues α and respectively β such that α^2 and β^2 are eigenvalues of A^2 and respectively B^2 with a common eigenvector x , then $\alpha = \beta$ and x is a common eigenvector of A and B .

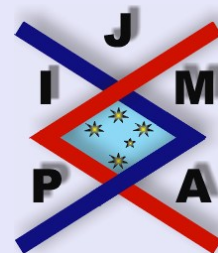
Proof. Assuming (a), we have

$$U^*AU = \begin{pmatrix} D & O \\ O & O \end{pmatrix}, \quad U^*BU = \begin{pmatrix} D & O \\ O & E \end{pmatrix}$$

as in (b) of Theorem 2.1, and so

$$U^*A^2U = \begin{pmatrix} D^2 & O \\ O & O \end{pmatrix}, \quad U^*B^2U = \begin{pmatrix} D^2 & O \\ O & E^2 \end{pmatrix}.$$

Hence, by Theorem 2.1, the first part of (b) follows. The second part is trivial.



On the Star Partial Ordering of Normal Matrices

Jorma K. Merikoski and Xiaoji Liu

Title Page

Contents



Go Back

Close

Quit

Page 11 of 16

Conversely, assume (b). Then

$$U^*A^2U = \begin{pmatrix} \Delta & O \\ O & O \end{pmatrix}, \quad U^*B^2U = \begin{pmatrix} \Delta & O \\ O & \Gamma \end{pmatrix},$$

where U , Δ , and Γ are matrices obtained by applying (b) of Theorem 2.1 to A^2 and B^2 . Let u_1, \dots, u_n be the column vectors of U and denote $r = \text{rank } A$.

For $i = 1, \dots, r$, we have $A^2u_i = B^2u_i = \delta_i u_i$, where $(\delta_i) = \text{diag } \Delta$. So, by the second part of (b), there exist complex numbers d_1, \dots, d_r such that, for all $i = 1, \dots, r$, we have $d_i^2 = \delta_i$ and $Au_i = Bu_i = \delta_i u_i$. Let D be the diagonal matrix with $(d_i) = \text{diag } D$.

For $i = r + 1, \dots, n$, we have $B^2u_i = \gamma_{i-r} u_i$, where $(\gamma_j) = \text{diag } \Gamma$. Take complex numbers e_1, \dots, e_{n-r} satisfying $e_i^2 = \gamma_i$ for $i = 1, \dots, n - r$. Let E be the diagonal matrix with $(e_i) = \text{diag } E$. Then

$$U^*AU = \begin{pmatrix} D & O \\ O & O \end{pmatrix}, \quad U^*BU = \begin{pmatrix} D & O \\ O & E \end{pmatrix},$$

and (a) follows from Theorem 2.1. □

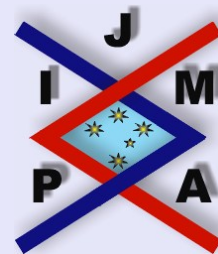
As an immediate corollary, we obtain a generalization of Theorem 1.1.

Corollary 3.2. *Let A and B be normal matrices whose all eigenvalues have nonnegative real parts. Then $A^2 \leq^* B^2$ if and only if $A \leq^* B$.*

Next, we present the extra condition using diagonalization.

Theorem 3.3. *Let A and B be normal matrices with $1 \leq \text{rank } A < \text{rank } B$. Then*

(a)
$$A \leq^* B$$



On the Star Partial Ordering of Normal Matrices

Jorma K. Merikoski and Xiaoji Liu

Title Page

Contents



Go Back

Close

Quit

Page 12 of 16

is equivalent to the following:

$$(b) \quad \mathbf{A}^2 \leq^* \mathbf{B}^2$$

and if

$$\mathbf{U}^* \mathbf{A} \mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \quad \mathbf{U}^* \mathbf{B} \mathbf{U} = \begin{pmatrix} \mathbf{D} \mathbf{H} & \mathbf{O} \\ \mathbf{O} & \mathbf{E} \end{pmatrix},$$

where \mathbf{U} is a unitary matrix, \mathbf{D} is a nonsingular diagonal matrix, \mathbf{H} is a unitary diagonal matrix, and $\mathbf{E} \neq \mathbf{O}$ is a diagonal matrix, then $\mathbf{H} = \mathbf{I}$.

(Note that the second part of (b) is weaker than (e) of Theorem 2.1. Otherwise Theorem 3.3 would be nonsense.)

Proof. For (a) \Rightarrow the first part of (b), see the proof of Theorem 3.1. For (a) \Rightarrow the second part of (b), see (e) of Theorem 2.1.

Conversely, assume (b). As in the proof of Theorem 3.1, we have

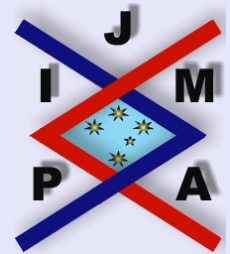
$$\mathbf{U}^* \mathbf{A}^2 \mathbf{U} = \begin{pmatrix} \Delta & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \quad \mathbf{U}^* \mathbf{B}^2 \mathbf{U} = \begin{pmatrix} \Delta & \mathbf{O} \\ \mathbf{O} & \Gamma \end{pmatrix}.$$

Hence

$$\mathbf{U}^* \mathbf{A} \mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \quad \mathbf{U}^* \mathbf{B} \mathbf{U} = \begin{pmatrix} \mathbf{D}' & \mathbf{O} \\ \mathbf{O} & \mathbf{E} \end{pmatrix},$$

where \mathbf{D} and \mathbf{D}' are diagonal matrices satisfying $\mathbf{D}^2 = (\mathbf{D}')^2 = \Delta$ and \mathbf{E} is a diagonal matrix satisfying $\mathbf{E}^2 = \Gamma$.

Denoting $(d_i) = \text{diag } \mathbf{D}$, $(d'_i) = \text{diag } \mathbf{D}'$, $r = \text{rank } \mathbf{A}$, we therefore have $d_i^2 = (d'_i)^2$ for all $i = 1, \dots, r$. Hence there are complex numbers h_1, \dots, h_r



On the Star Partial Ordering of Normal Matrices

Jorma K. Merikoski and Xiaoji Liu

Title Page

Contents



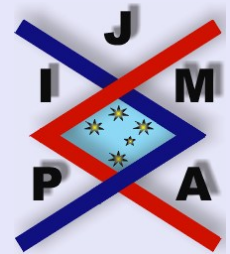
Go Back

Close

Quit

Page 13 of 16

such that $|h_1| = \dots = |h_r| = 1$ and $d'_i = d_i h_i$ for all $i = 1, \dots, r$. Let \mathbf{H} be the diagonal matrix with $(h_i) = \text{diag } \mathbf{H}$. Then $\mathbf{D}' = \mathbf{D}\mathbf{H}$, and so $\mathbf{D}' = \mathbf{D}$ by the second part of (b). Thus (b) of Theorem 2.1 is satisfied, and so (a) follows. \square



On the Star Partial Ordering of Normal Matrices

Jorma K. Merikoski and Xiaoji Liu

Title Page

Contents



Go Back

Close

Quit

Page 14 of 16

4. Remarks

We complete our paper with four remarks.

Remark 1. Let $k \geq 2$ be an integer. A natural further question is whether our discussion can be extended to describe how $\mathbf{A} \leq^* \mathbf{B}$ relates to $\mathbf{A}^k \leq^* \mathbf{B}^k$. As noted by Baksalary and Pukelsheim [1], Theorem 1.1 can be generalized in a similar way. In other words, for Hermitian nonnegative definite matrices, $\mathbf{A}^k \leq^* \mathbf{B}^k$ if and only if $\mathbf{A} \leq^* \mathbf{B}$. It can be seen also that Theorems 3.1 and 3.3 can be, with minor modifications, extended correspondingly.

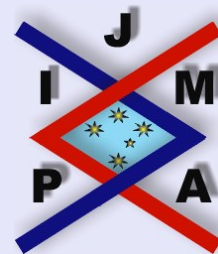
Remark 2. Let \mathbf{A} and \mathbf{B} be arbitrary $n \times n$ matrices with $\text{rank } \mathbf{A} < \text{rank } \mathbf{B}$. Hartwig and Styan ([2, Theorem 2]) proved that $\mathbf{A} \leq^* \mathbf{B}$ if and only if there are unitary matrices \mathbf{U} and \mathbf{V} such that

$$\mathbf{U}^* \mathbf{A} \mathbf{V} = \begin{pmatrix} \Sigma & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \quad \mathbf{U}^* \mathbf{B} \mathbf{V} = \begin{pmatrix} \Sigma & \mathbf{O} \\ \mathbf{O} & \Theta \end{pmatrix},$$

where Σ is a positive definite diagonal matrix and $\Theta \neq \mathbf{O}$ is a nonnegative definite diagonal matrix. This is analogous to (a) \Leftrightarrow (b) of Theorem 2.1. Actually it can be seen that all the characterizations of $\mathbf{A} \leq^* \mathbf{B}$ listed in Theorem 2.1 have singular value analogies in the general case.

Remark 3. The singular values of a normal matrix are absolute values of its eigenvalues (see e.g., [3, p. 417]). Hence it is relatively easy to see that if (and only if) \mathbf{A} and \mathbf{B} are normal, then \mathbf{U} and \mathbf{V} above can be chosen so that $\mathbf{U}^* \mathbf{V}$ is a diagonal matrix.

Remark 4. For normal matrices, it can be shown that Theorems 3.1 and 3.3 have singular value analogies. In the proof, it is crucial that $\mathbf{U}^* \mathbf{V}$ is a diagonal matrix. So these results do not remain valid without the normality assumption.



On the Star Partial Ordering of
Normal Matrices

Jorma K. Merikoski and Xiaoji Liu

Title Page

Contents



Go Back

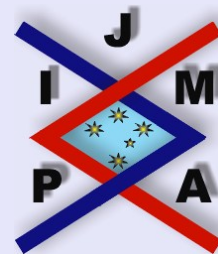
Close

Quit

Page 15 of 16

References

- [1] J.K. BAKSALARY AND F. PUKELSHEIM, On the Löwner, minus and star partial orderings of nonnegative definite matrices and their squares, *Linear Algebra Appl.*, **151** (1991), 135–141.
- [2] R.E. HARTWIG AND G.P.H. STYAN, On some characterizations on the “star” partial ordering for matrices and rank subtractivity, *Linear Algebra Appl.*, **82** (1986), 145–161.
- [3] R.A. HORN AND C.R. JOHNSON, *Matrix Analysis*, Cambridge University Press, 1985.



On the Star Partial Ordering of
Normal Matrices

Jorma K. Merikoski and Xiaoji Liu

Title Page

Contents



Go Back

Close

Quit

Page 16 of 16