

ON THE HÖLDER CONTINUITY OF MATRIX FUNCTIONS FOR NORMAL MATRICES

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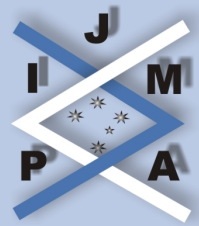
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Key words: Matrix functions, stability bounds, Hölder continuity.

Abstract: In this note, we shall investigate the Hölder continuity of matrix functions applied to normal matrices provided that the underlying scalar function is Hölder continuous. Furthermore, a few examples will be given.

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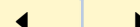
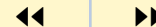
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1. Introduction

We consider a scalar function $f : D \rightarrow \mathbb{C}$ on a (possibly unbounded) subset D of the complex plane \mathbb{C} . In this note, we shall be particularly interested in the case where f is *Hölder continuous with exponent* α on D , that is, there exists a constant $\alpha \in (0, 1]$ such that the quantity

$$(1.1) \quad [f]_{\alpha, D} := \sup_{\substack{x, y \in D \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is bounded. We note that Hölder continuous functions are indeed continuous. Moreover, they are *Lipschitz continuous* if $\alpha = 1$; cf., e.g., [4].

Let us extend this concept to functions of matrices. To this end, consider

$$\mathbb{M}_{\text{normal}}^{n \times n}(\mathbb{C}) = \{ \mathbf{A} \in \mathbb{C}^{n \times n} : \mathbf{A}^H \mathbf{A} = \mathbf{A} \mathbf{A}^H \},$$

the set of all normal matrices with complex entries. Here, for a matrix $\mathbf{A} = [a_{ij}]_{i,j=1}^n$, we use the notation $\mathbf{A}^H = [\overline{a_{ji}}]_{i,j=1}^n$ to denote the conjugate transpose of \mathbf{A} . By the spectral theorem normal matrices are unitarily diagonalizable, i.e., for each $\mathbf{X} \in \mathbb{M}_{\text{normal}}^{n \times n}(\mathbb{C})$ there exists a unitary $n \times n$ -matrix \mathbf{U} , $\mathbf{U}^H \mathbf{U} = \mathbf{U} \mathbf{U}^H = \mathbf{1} = \text{diag}(1, 1, \dots, 1)$, such that

$$\mathbf{U}^H \mathbf{X} \mathbf{U} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n),$$

where the set $\sigma(\mathbf{X}) = \{\lambda_i\}_{i=1}^n$ is the spectrum of \mathbf{X} . For any function $f : D \rightarrow \mathbb{C}$, with $\sigma(\mathbf{X}) \subseteq D$, we can then define a corresponding matrix function “value” by

$$\mathbf{f}(\mathbf{X}) = \mathbf{U} \text{diag}(f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)) \mathbf{U}^H;$$

see, e.g., [5, 6]. Here, we use the bold face letter \mathbf{f} to denote the matrix function corresponding to the associated scalar function f .



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We can now easily widen the definition (1.1) of Hölder continuity for a scalar function $f : D \rightarrow \mathbb{C}$ to its associated matrix function \mathbf{f} applied to normal matrices: Given a subset $\mathbb{D} \subseteq \mathbb{M}_{\text{normal}}^{n \times n}(\mathbb{C})$, then we say that the matrix function $\mathbf{f} : \mathbb{D} \rightarrow \mathbb{C}^{n \times n}$ is Hölder continuous with exponent $\alpha \in (0, 1]$ on \mathbb{D} if

$$(1.2) \quad [\mathbf{f}]_{\alpha, \mathbb{D}} := \sup_{\substack{\mathbf{X}, \mathbf{Y} \in \mathbb{D} \\ \mathbf{X} \neq \mathbf{Y}}} \frac{\|\mathbf{f}(\mathbf{X}) - \mathbf{f}(\mathbf{Y})\|_F}{\|\mathbf{X} - \mathbf{Y}\|_F^\alpha}$$

is bounded. Here, for a matrix $\mathbf{X} = [x_{ij}]_{i,j=1}^n \in \mathbb{C}^{n \times n}$ we define $\|\mathbf{X}\|_F$ to be the Frobenius norm of \mathbf{X} given by

$$\|\mathbf{X}\|_F^2 = \text{trace}(\mathbf{X}^H \mathbf{X}) = \sum_{i,j=1}^n |x_{ij}|^2, \quad \mathbf{X} = (x_{ij})_{i,j=1}^n \in \mathbb{M}^{n \times n}(\mathbb{C}).$$

Evidently, for the definition (1.2) to make sense, it is necessary to assume that the scalar function f associated with the matrix function \mathbf{f} is well-defined on the spectra of all matrices $\mathbf{X} \in \mathbb{D}$, i.e.,

$$(1.3) \quad \bigcup_{\mathbf{X} \in \mathbb{D}} \sigma(\mathbf{X}) \subseteq D.$$

The goal of this note is to address the following question: Provided that a scalar function f is Hölder continuous, what can be said about the Hölder continuity of the corresponding matrix function \mathbf{f} ? The following theorem provides the answer:

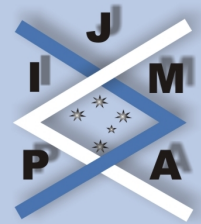
Theorem 1.1. *Let the scalar function $f : D \rightarrow \mathbb{C}$ be Hölder continuous with exponent $\alpha \in (0, 1]$, and $\mathbb{D} \subseteq \mathbb{M}_{\text{normal}}^{n \times n}(\mathbb{C})$ satisfy (1.3). Then, the associated matrix function $\mathbf{f} : \mathbb{D} \rightarrow \mathbb{C}^{n \times n}$ is Hölder continuous with exponent α and*

$$(1.4) \quad [\mathbf{f}]_{\alpha, \mathbb{D}} \leq n^{\frac{1-\alpha}{2}} [f]_{\alpha, D}$$

holds true. In particular, the bound

$$(1.5) \quad \|f(\mathbf{X}) - f(\mathbf{Y})\|_F \leq [f]_{\alpha, D} n^{\frac{1-\alpha}{2}} \|\mathbf{X} - \mathbf{Y}\|_F^\alpha,$$

holds for any $\mathbf{X}, \mathbf{Y} \in \mathbb{D}$.



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2. Proof of Theorem 1.1

We shall check the inequality (1.5). From this (1.4) follows immediately. Consider two matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{D}$. Since they are normal we can find two unitary matrices $\mathbf{V}, \mathbf{W} \in \mathbb{M}^{n \times n}(\mathbb{C})$ which diagonalize \mathbf{X} and \mathbf{Y} , respectively, i.e.,

$$\begin{aligned} \mathbf{V}^H \mathbf{X} \mathbf{V} &= \mathbf{D}_X = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n), \\ \mathbf{W}^H \mathbf{Y} \mathbf{W} &= \mathbf{D}_Y = \text{diag}(\mu_1, \mu_2, \dots, \mu_n), \end{aligned}$$

where $\{\lambda_i\}_{i=1}^n$ and $\{\mu_i\}_{i=1}^n$ are the eigenvalues of \mathbf{X} and \mathbf{Y} , respectively. Now we need to use the fact that the Frobenius norm is unitarily invariant. This means that for any matrix $\mathbf{X} \in \mathbb{C}^{n \times n}$ and any two unitary matrices $\mathbf{R}, \mathbf{U} \in \mathbb{C}^{n \times n}$ there holds

$$\|\mathbf{R} \mathbf{X} \mathbf{U}\|_F^2 = \|\mathbf{X}\|_F^2.$$

Therefore, it follows that

$$\begin{aligned} (2.1) \quad \|\mathbf{X} - \mathbf{Y}\|_F^2 &= \|\mathbf{V} \mathbf{D}_X \mathbf{V}^H - \mathbf{W} \mathbf{D}_Y \mathbf{W}^H\|_F^2 \\ &= \|\mathbf{W}^H \mathbf{V} \mathbf{D}_X \mathbf{V}^H \mathbf{V} - \mathbf{W}^H \mathbf{W} \mathbf{D}_Y \mathbf{W}^H \mathbf{V}\|_F^2 \\ &= \|\mathbf{W}^H \mathbf{V} \mathbf{D}_X - \mathbf{D}_Y \mathbf{W}^H \mathbf{V}\|_F^2 \\ &= \sum_{i,j=1}^n \left| (\mathbf{W}^H \mathbf{V} \mathbf{D}_X - \mathbf{D}_Y \mathbf{W}^H \mathbf{V})_{i,j} \right|^2 \\ &= \sum_{i,j=1}^n \left| \sum_{k=1}^n (\mathbf{W}^H \mathbf{V})_{i,k} (\mathbf{D}_X)_{k,j} - (\mathbf{D}_Y)_{i,k} (\mathbf{W}^H \mathbf{V})_{k,j} \right|^2 \\ &= \sum_{i,j=1}^n \left| (\mathbf{W}^H \mathbf{V})_{i,j} \right|^2 |\lambda_j - \mu_i|^2. \end{aligned}$$



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In the same way, noting that

$$\mathbf{f}(\mathbf{X}) = \mathbf{V}f(\mathbf{D}_X)\mathbf{V}^H, \quad \mathbf{f}(\mathbf{Y}) = \mathbf{W}f(\mathbf{D}_Y)\mathbf{W}^H,$$

we obtain

$$\|\mathbf{f}(\mathbf{X}) - \mathbf{f}(\mathbf{Y})\|_F^2 = \sum_{i,j=1}^n \left| (\mathbf{W}^H \mathbf{V})_{i,j} \right|^2 |f(\lambda_j) - f(\mu_i)|^2.$$

Employing the Hölder continuity of f , i.e.,

$$|f(x) - f(y)| \leq [f]_{\alpha,D} |x - y|^\alpha, \quad x, y \in D,$$

it follows that

$$(2.2) \quad \|\mathbf{f}(\mathbf{X}) - \mathbf{f}(\mathbf{Y})\|_F^2 \leq [f]_{\alpha,D}^2 \sum_{i,j=1}^n \left| (\mathbf{W}^H \mathbf{V})_{i,j} \right|^2 |\lambda_j - \mu_i|^{2\alpha}.$$

For $\alpha = 1$ the bound (1.5) results directly from (2.1) and (2.2). If $0 < \alpha < 1$, we apply Hölder's inequality. That is, for arbitrary numbers $s_i, t_i \in \mathbb{C}$, $i = 1, 2, \dots$, there holds

$$\sum_{i \geq 1} |s_i t_i| \leq \left(\sum_{i \geq 1} |s_i|^{\frac{1}{\alpha}} \right)^\alpha \left(\sum_{i \geq 1} |t_i|^{\frac{1}{1-\alpha}} \right)^{1-\alpha}.$$

In the present situation this yields

$$\begin{aligned} & \|\mathbf{f}(\mathbf{X}) - \mathbf{f}(\mathbf{Y})\|^2 \\ & \leq [f]_{\alpha,D}^2 \sum_{i,j=1}^n \left(|\lambda_j - \mu_i| \left| (\mathbf{W}^H \mathbf{V})_{i,j} \right| \right)^{2\alpha} \left| (\mathbf{W}^H \mathbf{V})_{i,j} \right|^{2-2\alpha} \\ & \leq [f]_{\alpha,D}^2 \left(\sum_{i,j=1}^n \left| (\mathbf{W}^H \mathbf{V})_{i,j} \right|^2 |\lambda_j - \mu_i|^2 \right)^\alpha \left(\sum_{i,j=1}^n \left| (\mathbf{W}^H \mathbf{V})_{i,j} \right|^2 \right)^{1-\alpha}. \end{aligned}$$

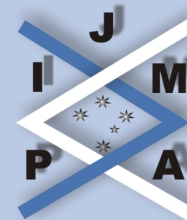
Therefore, using the identity (2.1), there holds

$$\|f(\mathbf{X}) - f(\mathbf{Y})\|_F \leq [f]_{\alpha,D} \|\mathbf{X} - \mathbf{Y}\|_F^\alpha \left(\sum_{i,j=1}^n |(\mathbf{W}^H \mathbf{V})_{i,j}|^2 \right)^{\frac{1-\alpha}{2}}.$$

Then, recalling again that $\|\cdot\|_F$ is unitarily invariant, yields

$$\left(\sum_{i,j=1}^n |(\mathbf{W}^H \mathbf{V})_{i,j}|^2 \right)^{\frac{1-\alpha}{2}} = \|\mathbf{W}^H \mathbf{V}\|_F^{1-\alpha} = \|\mathbf{1}\|_F^{1-\alpha} = n^{\frac{1-\alpha}{2}},$$

This implies the estimate (1.5).



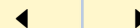
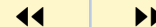
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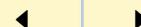
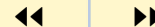
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3. Applications

We shall look at a few examples which fit in the framework of the previous analysis. Here, we consider the special case that all matrices are *real* and *symmetric*. In particular, they are normal and have only real eigenvalues.

Let us first study some functions $f : D \rightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}$ is an interval, which are continuously differentiable with bounded derivative on D . Then, by the mean value theorem, we have

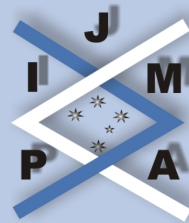
$$[f]_{1,D} = \sup_{\substack{x,y \in D \\ x \neq y}} \left| \frac{f(x) - f(y)}{x - y} \right| = \sup_{\xi \in D} |f'(\xi)| < \infty,$$

i.e., such functions are Lipschitz continuous.

Trigonometric Functions:

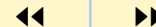
Let $m \in \mathbb{N}$. Then, the functions $t \mapsto \sin^m(t)$ and $t \mapsto \cos^m(t)$ are Lipschitz continuous on \mathbb{R} , with constant

$$\begin{aligned} L_m &:= [\sin^m]_{1,\mathbb{R}} = [\cos^m]_{1,\mathbb{R}} = \sup_{t \in \mathbb{R}} \left| \frac{d}{dt} \sin^m(t) \right| \\ &= \sup_{t \in \mathbb{R}} \left| \frac{d}{dt} \cos^m(t) \right| \\ &= \sqrt{m} \left(\frac{\sqrt{m-1}}{\sqrt{m}} \right)^{m-1}. \end{aligned}$$



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Thence, we immediately obtain the bounds

$$\|\sin^m(\mathbf{X}) - \sin^m(\mathbf{Y})\|_F \leq \sqrt{m} \left(\frac{\sqrt{m-1}}{\sqrt{m}} \right)^{m-1} \|\mathbf{X} - \mathbf{Y}\|_F$$

$$\|\cos^m(\mathbf{X}) - \cos^m(\mathbf{Y})\|_F \leq \sqrt{m} \left(\frac{\sqrt{m-1}}{\sqrt{m}} \right)^{m-1} \|\mathbf{X} - \mathbf{Y}\|_F$$

for any real symmetric $n \times n$ -matrices \mathbf{X}, \mathbf{Y} . We note that

$$\lim_{m \rightarrow \infty} \left(\frac{\sqrt{m-1}}{\sqrt{m}} \right)^{m-1} = e^{-\frac{1}{2}},$$

and hence $L_m \sim \sqrt{m}$ with $m \rightarrow \infty$.

Gaussian Function:

For fixed $m > 0$, the Gaussian function $f : t \mapsto \exp(-mt^2)$ is Lipschitz continuous on \mathbb{R} with constant $[f]_{1, \mathbb{R}} = \sqrt{2m} \exp(-\frac{1}{2})$. Consequently, we have for the matrix exponential that

$$\|\exp(-m\mathbf{X}^2) - \exp(-m\mathbf{Y}^2)\|_F \leq \sqrt{2m} e^{-\frac{1}{2}} \|\mathbf{X} - \mathbf{Y}\|_F,$$

for any real symmetric $n \times n$ -matrices \mathbf{X}, \mathbf{Y} .

We shall now consider some functions which are less smooth than in the previous examples. In particular, they are not differentiable at 0.

Absolute Value Function:

Due to the triangle inequality

$$||x| - |y|| \leq |x - y|, \quad x, y \in \mathbb{R},$$

the absolute value function $f : t \mapsto |t|$ is Lipschitz continuous with constant $[f]_{1, \mathbb{R}} = 1$, and hence

$$(3.1) \quad \left| \|X\| - \|Y\| \right|_F \leq \|X - Y\|_F,$$

for any real symmetric $n \times n$ -matrices X, Y . We note that, for general matrices, there is an additional factor of $\sqrt{2}$ on the right hand side of (3.1), whereas for symmetric matrices the factor 1 is optimal; see [1] and the references therein.

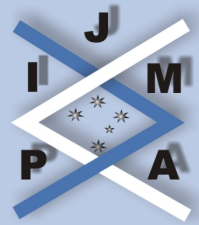
***p*-th Root of Positive Semi-Definite Matrices:**

Finally, let us consider the p -th root ($p > 1$) of a real symmetric positive semi-definite matrix. The spectrum of such matrices belongs to the non-negative real axes $D = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Here, we notice that the function $f : t \mapsto t^{\frac{1}{p}}$ is Hölder continuous on D with exponent $\alpha = \frac{1}{p}$ and $[f]_{\frac{1}{p}, D} = 1$. Hence, Theorem 1.1 applies. In particular, the inequality

$$(3.2) \quad \left\| X^{\frac{1}{p}} - Y^{\frac{1}{p}} \right\|_F^p \leq n^{\frac{p-1}{2}} \|X - Y\|_F$$

holds for any real symmetric positive-semidefinite $n \times n$ -matrices X, Y . We note that the estimate (3.2) is sharp. Indeed, there holds equality if X is chosen to be the identity matrix, and Y is the zero matrix.

We remark that an alternative proof of (3.2) has already been given in [2, Chapter X] in the context of operator monotone functions. Furthermore, closely related results on the Lipschitz continuity of matrix functions and the Hölder continuity of the p -th matrix root can be found in, e.g., [2, Chapter VII] and [3], respectively.



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