

THE GENERALIZED HYERS-ULAM-RASSIAS STABILITY OF A QUADRATIC FUNCTIONAL EQUATION

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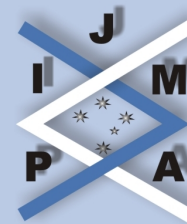
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Abstract: In this paper, we investigate the generalized Hyers - Ulam - Rassias stability of a new quadratic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 4f(x) - 2f(y).$$



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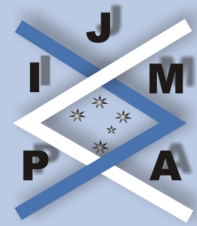
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1. Introduction

The problem of the stability of functional equations was originally stated by S.M.Ulam [20]. In 1941 D.H. Hyers [10] proved the stability of the linear functional equation for the case when the groups G_1 and G_2 are Banach spaces. In 1950, T. Aoki discussed the Hyers-Ulam stability theorem in [2]. His result was further generalized and rediscovered by Th.M. Rassias [17] in 1978. The stability problem for functional equations have been extensively investigated by a number of mathematicians [5], [8], [9], [12] – [16], [19].

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$(1.1) \quad f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

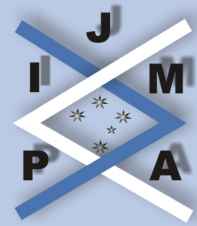
and therefore the equation (1.1) is called the quadratic functional equation.

The Hyers - Ulam stability theorem for the quadratic functional equation (1.1) was proved by F. Skof [19] for the functions $f : E_1 \rightarrow E_2$ where E_1 is a normed space and E_2 a Banach space. The result of Skof is still true if the relevant domain E_1 is replaced by an Abelian group and this was dealt with by P.W.Cholewa [6]. S.Czerwik [7] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation (1.1). This result was further generalized by Th.M. Rassias [18], C. Borelli and G.L. Forti [4].

In this paper, we discuss a new quadratic functional equation

$$(1.2) \quad f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 4f(x) - 2f(y).$$

The generalized Hyers-Ulam-Rassias stability of the equation (1.2) is dealt with here. As a result of the paper, we have a much better possible upper bound for (1.2) than S. Czerwik and Skof-Cholewa.



2. Hyers-Ulam-Rassias stability of (1.2)

In this section, let X be a real vector space and let Y be a Banach space. We will investigate the Hyers-Ulam-Rassias stability problem for the functional equation (1.2). Define

$$Df(x, y) = f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 4f(x) + 2f(y).$$

Now we state some theorems which will be useful in proving our results.

Theorem 2.1 ([7]). *If a function $f : G \rightarrow Y$, where G is an abelian group and Y a Banach space, satisfies the inequality*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \epsilon (\|x\|^p + \|y\|^q)$$

for $p \neq 2$ and for all $x, y \in G$, then there exists a unique quadratic function Q such that

$$\|f(x) - Q(x)\| \leq \frac{\epsilon \|x\|^p}{|4 - 2^p|} + \frac{\|f(0)\|}{3}$$

for all $x \in G$.

Theorem 2.2 ([6]). *If a function $f : G \rightarrow Y$, where G is an abelian group and Y is a Banach space, satisfies the inequality*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \epsilon$$

for all $x, y \in G$, then there exists a unique quadratic function Q such that

$$\|f(x) - Q(x)\| \leq \frac{\epsilon}{2}$$

for all $x \in G$, and for all $x \in G - 0$, and $\|f(0)\| = 0$.

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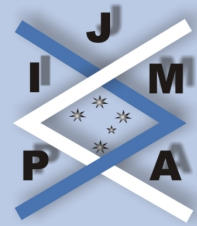
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Theorem 2.3. Let $\psi : X^2 \rightarrow \mathbb{R}^+$ be a function such that

$$(2.1) \quad \sum_{i=0}^{\infty} \frac{\psi(2^i x, 0)}{4^i} \text{ converges and } \lim_{n \rightarrow \infty} \frac{\psi(2^n x, 2^n y)}{4^n} = 0$$

for all $x, y \in X$. If a function $f : X \rightarrow Y$ satisfies

$$(2.2) \quad \|Df(x, y)\| \leq \psi(x, y)$$

for all $x, y \in X$, then there exists one and only one quadratic function $Q : X \rightarrow Y$ which satisfies equation (1.2) and the inequality

$$(2.3) \quad \|f(x) - Q(x)\| \leq \frac{1}{8} \sum_{i=0}^{\infty} \frac{\psi(2^i x, 0)}{4^i}$$

for all $x \in X$. The function Q is defined by

$$(2.4) \quad Q(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n}$$

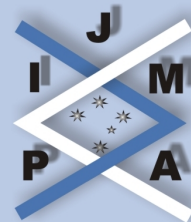
for all $x \in X$.

Proof. Letting $x = y = 0$ in (1.2), we get $f(0) = 0$. Putting $y = 0$ in (2.2) and dividing by 8, we have

$$(2.5) \quad \left\| f(x) - \frac{f(2x)}{4} \right\| \leq \frac{1}{8} \psi(x, 0)$$

for all $x \in X$. Replacing x by $2x$ in (2.5) and dividing by 4 and summing the resulting inequality with (2.5), we get

$$(2.6) \quad \left\| f(x) - \frac{f(2x)}{4} \right\| \leq \frac{1}{8} \left[\psi(x, 0) + \frac{\psi(2x, 0)}{4} \right]$$



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for all $x \in X$. Using induction on a positive integer n we obtain that

$$(2.7) \quad \left\| f(x) - \frac{f(2^n x)}{4^n} \right\| \leq \frac{1}{8} \sum_{i=0}^{n-1} \frac{\psi(2^i x, 0)}{4^i} \leq \frac{1}{8} \sum_{i=0}^{\infty} \frac{\psi(2^i x, 0)}{4^i}$$

for all $x \in X$.

Now, for $m, n > 0$

$$(2.8) \quad \begin{aligned} \left\| \frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n} \right\| &\leq \left\| \frac{f(2^{m+n-n} x)}{4^{m+n-n}} - \frac{f(2^n x)}{4^n} \right\| \\ &\leq \frac{1}{4^n} \left\| \frac{f(2^{m-n} 2^n x)}{4^{m-n}} - f(2^n x) \right\| \\ &\leq \frac{1}{8} \sum_{i=0}^{n-1} \frac{\psi(2^{i+n} x, 0)}{4^{i+n}} \\ &\leq \frac{1}{8} \sum_{i=0}^{\infty} \frac{\psi(2^{i+n} x, 0)}{4^{i+n}}. \end{aligned}$$

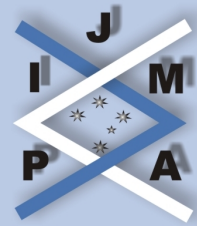
Since the right-hand side of the inequality (2.8) tends to 0 as n tends to infinity,

the sequence $\left\{ \frac{f(2^n x)}{4^n} \right\}$ is a Cauchy sequence. Therefore, we may define $Q(x) =$

$\lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n}$ for all $x \in X$. Letting $n \rightarrow \infty$ in (2.7), we arrive at (2.3).

Next, we have to show that Q satisfies (1.2). Replacing x, y by $2^n x, 2^n y$ in (2.2) and dividing by 4^n , it then follows that

$$\begin{aligned} &\frac{1}{4^n} \|f(2^n(2x+y)) + f(2^n(2x-y)) \\ &- 2f(2^n(x+y)) - 2f(2^n(x-y)) - 4f(2^n x) + 2f(2^n y)\| \leq \frac{1}{4^n} \psi(2^n x, 2^n y). \end{aligned}$$



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Taking the limit as $n \rightarrow \infty$, using (2.1) and (2.4), we see that

$$\|Q(2x + y) + Q(2x - y) - 2Q(x + y) - 2Q(x - y) - 4Q(x) + 2Q(y)\| \leq 0$$

which gives

$$Q(2x + y) + Q(2x - y) = 2Q(x + y) + 2Q(x - y) + 4Q(x) - 2Q(y).$$

Therefore, we have that Q satisfies (1.2) for all $x, y \in X$. To prove the uniqueness of the quadratic function Q , let us assume that there exists a quadratic function $Q' : X \rightarrow Y$ which satisfies (1.2) and the inequality (2.3). But we have $Q(2^n x) = 4^n Q(x)$ and $Q'(2^n x) = 4^n Q'(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Hence it follows from (2.3) that

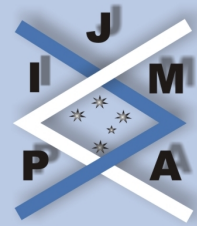
$$\begin{aligned} \|Q(x) - Q'(x)\| &= \frac{1}{4^n} \|Q(2^n x) - Q'(2^n x)\| \\ &\leq \frac{1}{4^n} (\|Q(2^n x) - f(2^n x)\| + \|f(2^n x) - Q'(2^n x)\|) \\ &\leq \frac{1}{4} \sum_{i=0}^{\infty} \frac{\psi(2^{i+n}, 0)}{4^{i+n}} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

Therefore Q is unique. This completes the proof of the theorem. \square

From Theorem 2.1, we obtain the following corollaries concerning the stability of the equation (1.2).

Corollary 2.4. *Let X be a real normed space and Y a Banach space. Let ϵ, p, q be real numbers such that $\epsilon \geq 0, q > 0$ and either $p, q < 2$ or $p, q > 2$. Suppose that a function $f : X \rightarrow Y$ satisfies*

$$(2.9) \quad \|Df(x, y)\| \leq \epsilon (\|x\|^p + \|y\|^q)$$



for all $x, y \in X$. Then there exists one and only one quadratic function $Q : X \rightarrow Y$ which satisfies (1.2) and the inequality

$$(2.10) \quad \|f(x) - Q(x)\| \leq \frac{\epsilon}{2\|4 - 2^p\|} \|x\|^p$$

for all $x \in X$. The function Q is defined in (2.4). Furthermore, if $f(tx)$ is continuous for all $t \in \mathbb{R}$ and $x \in X$ then, $f(tx) = t^2 f(x)$.

Proof. Taking $\psi(x, y) = \epsilon (\|x\|^p + \|y\|^q)$ and applying Theorem 2.1, the equation (2.3) give rise to equation (2.10) which proves Corollary 2.4. \square

Corollary 2.5. Let X be a real normed space and Y be a Banach space. Let ϵ be real number. If a function $f : X \rightarrow Y$ satisfies

$$(2.11) \quad \|Df(x, y)\| \leq \epsilon$$

for all $x, y \in X$, then there exists one and only one quadratic function $Q : X \rightarrow Y$ which satisfies (1.2) and the inequality

$$(2.12) \quad \|f(x) - Q(x)\| \leq \frac{\epsilon}{4}$$

for all $x \in X$. The function Q is defined in (2.4). Furthermore, if $f(tx)$ is continuous for all $t \in \mathbb{R}$ and $x \in X$ then, $f(tx) = t^2 f(x)$.

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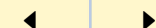
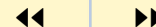
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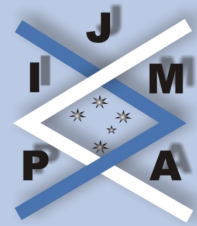
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