THE GENERALIZED HYERS-ULAM-RASSIAS STABILITY OF A QUADRATIC FUNCTIONAL EQUATION

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Abstract: In this paper, we investigate the generalized Hyers - Ulam - Rassias stability of a new quadratic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+4f(x)-2f(y).$$

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Contents

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1. Introduction

The problem of the stability of functional equations was originally stated by S.M.Ulam [20]. In 1941 D.H. Hyers [10] proved the stability of the linear functional equation for the case when the groups G_1 and G_2 are Banach spaces. In 1950, T. Aoki discussed the Hyers-Ulam stability theorem in [2]. His result was further generalized and rediscovered by Th.M. Rassias [17] in 1978 . The stability problem for functional equations have been extensively investigated by a number of mathematicians [5], [8], [9], [12] – [16], [19].

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

and therefore the equation (1.1) is called the quadratic functional equation.

The Hyers - Ulam stability theorem for the quadratic functional equation (1.1) was proved by F. Skof [19] for the functions $f : E_1 \to E_2$ where E_1 is a normed space and E_2 a Banach space. The result of Skof is still true if the relevant domain E_1 is replaced by an Abelian group and this was dealt with by P.W.Cholewa [6]. S.Czerwik [7] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation (1.1). This result was further generalized by Th.M. Rassais [18], C. Borelli and G.L. Forti [4].

In this paper, we discuss a new quadratic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 4f(x) - 2f(y). \tag{1.2}$$

The generalized Hyers-Ulam-Rassias stability of the equation (1.2) is dealt with here. As a result of the paper, we have a much better possible upper bound for (1.2) than S. Czerwik and Skof-Cholewa.

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2. Hyers-Ulam-Rassias stability of (1.2)

In this section, let X be a real vector space and let Y be a Banach space. We will investigate the Hyers-Ulam-Rassias stability problem for the functional equation (1.2). Define

$$Df(x,y) = f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 4f(x) + 2f(y).$$

Now we state some theorems which will be useful in proving our results.

Theorem 2.1 ([7]). *If a function $f : G \to Y$, where G is an abelian group and Y a Banach space, satisfies the inequality*

$$||f(x+y) + f(x-y) - 2f(x) - 2f(y)|| \leq \epsilon \left(||x||^p + ||y||^q\right)$$

for $p \neq 2$ and for all $x, y \in G$, then there exists a unique quadratic function Q such that

$$\|f(x) - Q(x)\| \leq \frac{\epsilon||x||^p}{|4-2^p|} + \frac{||f(0)||}{3}$$

for all $x \in G$.

Theorem 2.2 ([6]). *If a function $f : G \to Y$, where G is an abelian group and Y is a Banach space, satisfies the inequality*

$$||f(x+y) + f(x-y) - 2f(x) - 2f(y)|| \leq \epsilon$$

for all $x, y \in G$, then there exists a unique quadratic function Q such that

$$\|f(x) - Q(x)\| \leq \frac{\epsilon}{2}$$

for all $x \in G$, and for all $x \in G - 0$, and $||f(0)|| = 0$.

Generalized Hyers-Ulam-Rassias Stability
K. Ravi, R. Murali and M. Arunkumar
vol. 9, iss. 1, art. 20, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Theorem 2.3. *Let $\psi : X^2 \to \mathbb{R}^+$ be a function such that*

$$\sum_{i=0}^{\infty} \frac{\psi(2^i x, 0)}{4^i} \quad \text{converges and} \quad \lim_{n\to\infty} \frac{\psi(2^n x, 2^n y)}{4^n} = 0 \tag{2.1}$$

for all $x, y \in X$. If a function $f : X \to Y$ satisfies

$$||Df(x,y)|| \le \psi(x,y) \tag{2.2}$$

for all $x, y \in X$, then there exists one and only one quadratic function $Q : X \to Y$ which satisfies equation (1.2) and the inequality

$$||f(x) - Q(x)|| \le \frac{1}{8} \sum_{i=0}^{\infty} \frac{\psi(2^i x, 0)}{4^i} \tag{2.3}$$

for all $x \in X$. The function Q is defined by

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \tag{2.4}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (1.2), we get $f(0) = 0$. Putting $y = 0$ in (2.2) and dividing by 8, we have

$$\left\| f(x) - \frac{f(2x)}{4} \right\| \le \frac{1}{8}\psi(x,0) \tag{2.5}$$

for all $x \in X$. Replacing x by $2x$ in (2.5) and dividing by 4 and summing the resulting inequality with (2.5), we get

$$\left\| f(x) - \frac{f(2x)}{4} \right\| \le \frac{1}{8}\left[\psi(x,0) + \frac{\psi(2x,0)}{4} \right] \tag{2.6}$$

journal of inequalities in pure and applied mathematics
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for all $x \in X$. Using induction on a positive integer n we obtain that

$$\left\| f(x) - \frac{f(2^n x)}{4^n} \right\| \le \frac{1}{8} \sum_{i=0}^{n-1} \frac{\psi(2^i x, 0)}{4^i} \le \frac{1}{8} \sum_{i=0}^{\infty} \frac{\psi(2^i x, 0)}{4^i} \tag{2.7}$$

for all $x \in X$.

Now, for $m, n > 0$

$$\begin{aligned} \left\| \frac{f(2^m x)}{4^m} - \frac{f(2^n)}{4^n} \right\| &\le \left\| \frac{f(2^{m+n-n} x)}{4^{m+n-n}} - \frac{f(2^n x)}{4^n} \right\| \\ &\le \frac{1}{4^n} \left\| \frac{f(2^{m-n} 2^n x)}{4^{m-n}} - f(2^n x) \right\| \\ &\le \frac{1}{8} \sum_{i=0}^{n-1} \frac{\psi(2^{i+n} x, 0)}{4^{i+n}} \\ &\le \frac{1}{8} \sum_{i=0}^{\infty} \frac{\psi(2^{i+n} x, 0)}{4^{i+n}}. \end{aligned} \tag{2.8}$$

Since the right-hand side of the inequality (2.8) tends to 0 as n tends to infinity, the sequence $\left\{ \frac{f(2^n x)}{4^n} \right\}$ is a Cauchy sequence. Therefore, we may define $Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ for all $x \in X$. Letting $n \to \infty$ in (2.7), we arrive at (2.3).

Next, we have to show that Q satisfies (1.2). Replacing x, y by $2^n x, 2^n y$ in (2.2) and dividing by 4^n, it then follows that

$$\begin{aligned} &\frac{1}{4^n} \| f\left(2^n(2x+y)\right) + f\left(2^n(2x-y)\right) \\ &\quad - 2f\left(2^n(x+y)\right) - 2f\left(2^n(x-y)\right) - 4f(2^n x) + 2f(2^n y) \| \le \frac{1}{4^n} \psi(2^n x, 2^n y). \end{aligned}$$

journal of inequalities in pure and applied mathematics
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Taking the limit as $n \to \infty$, using (2.1) and (2.4), we see that

$$||Q(2x+y) + Q(2x-y) - 2Q(x+y) - 2Q(x-y) - 4Q(x) + 2Q(y)|| \leq 0$$

which gives

$$Q(2x+y) + Q(2x-y) = 2Q(x+y) + 2Q(x-y) + 4Q(x) - 2Q(y).$$

Therefore, we have that Q satisfies (1.2) for all $x, y \in X$. To prove the uniqueness of the quadratic function Q, let us assume that there exists a quadratic function $Q' : X \to Y$ which satisfies (1.2) and the inequality (2.3). But we have $Q(2^n x) = 4^n Q(x)$ and $Q'(2^n x) = 4^n Q'(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Hence it follows from (2.3) that

$$\begin{aligned} ||Q(x) - Q'(x)|| &= \frac{1}{4^n} ||Q(2^n x) - Q'(2^n x)|| \\ &\leq \frac{1}{4^n} \left(||Q(2^n x) - f(2^n x)|| + ||f(2^n x) - Q'(2^n x)|| \right) \\ &\leq \frac{1}{4} \sum_{i=0}^{\infty} \frac{\psi(2^{i+n}, 0)}{4^{i+n}} \quad \to 0 \text{ as } n \to \infty \end{aligned}$$

Therefore Q is unique. This completes the proof of the theorem. □

From Theorem 2.1, we obtain the following corollaries concerning the stability of the equation (1.2).

Corollary 2.4. *Let X be a real normed space and Y a Banach space. Let ϵ, p, q be real numbers such that $\epsilon \geq 0, q > 0$ and either $p, q < 2$ or $p, q > 2$. Suppose that a function $f : X \to Y$ satisfies*

(2.9) $$||Df(x,y)|| \leq \epsilon \left(||x||^p + ||y||^q \right)$$

Generalized Hyers-Ulam-Rassias Stability
K. Ravi, R. Murali
and M. Arunkumar
vol. 9, iss. 1, art. 20, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

for all $x, y \in X$. Then there exists one and only one quadratic function $Q : X \to Y$ which satisfies (1.2) and the inequality

$$(2.10) \qquad ||f(x) - Q(x)| \leq \frac{\epsilon}{2||4 - 2^p||}||x||^p$$

for all $x \in X$. The function Q is defined in (2.4). Furthermore, if $f(tx)$ is continuous for all $t \in \mathbb{R}$ and $x \in X$ then, $f(tx) = t^2 f(x)$.

Proof. Taking $\psi(x, y) = \epsilon\left(||x||^p + ||y||^q\right)$ and applying Theorem 2.1, the equation (2.3) give rise to equation (2.10) which proves Corollary 2.4. □

Corollary 2.5. *Let X be a real normed space and Y be a Banach space. Let ϵ be real number. If a function $f : X \to Y$ satisfies*

$$(2.11) \qquad ||Df(x, y)|| \leq \epsilon$$

for all $x, y \in X$, then there exists one and only one quadratic function $Q : X \to Y$ which satisfies (1.2) and the inequality

$$(2.12) \qquad ||f(x) - Q(x)| \leq \frac{\epsilon}{4}$$

for all $x \in X$. The function Q is defined in (2.4). Furthermore, if $f(tx)$ is continuous for all $t \in \mathbb{R}$ and $x \in X$ then, $f(tx) = t^2 f(x)$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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