

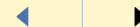
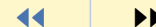
# ON SOME INEQUALITIES FOR $p$ -NORMS



Some Inequalities for  $p$ -Norms  
U.S. Kirmaci, M. Klaričić Bakula,  
M. E. Özdemir and J. Pečarić  
vol. 9, iss. 1, art. 27, 2008

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*Abstract:* In this paper we establish several new inequalities including  $p$ -norms for functions whose absolute values aroused to the  $p$ -th power are convex functions.

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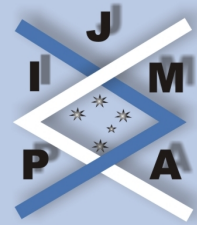
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# 1. Introduction

Integral inequalities have become a major tool in the analysis of integral equations, so it is not surprising that many of them appear in the literature (see for example [2], [5], [3] and [1]).

One of the most important inequalities in analysis is the integral Hölder's inequality which is stated as follows (for this variant see [3, p. 106]).

**Theorem A.** *Let  $p, q \in \mathbb{R} \setminus \{0\}$  be such that  $\frac{1}{p} + \frac{1}{q} = 1$  and let  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $a < b$ , be such that  $|f(x)|^p$  and  $|g(x)|^q$  are integrable on  $[a, b]$ . If  $p, q > 0$ , then*

$$(1.1) \quad \int_a^b |f(x)g(x)| dx \leq \left( \int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int_a^b |g(x)|^q dx \right)^{\frac{1}{q}}.$$

*If  $p < 0$  and additionally  $f([a, b]) \subseteq \mathbb{R} \setminus \{0\}$ , or  $q < 0$  and  $g([a, b]) \subseteq \mathbb{R} \setminus \{0\}$ , then the inequality in (1.1) is reversed.*

The Hermite-Hadamard inequalities for convex functions is also well known. This double inequality is stated as follows (see for example [3, p. 10]): Let  $f$  be a convex function on  $[a, b] \subset \mathbb{R}$ , where  $a \neq b$ . Then

$$(1.2) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

To prove our main result we need comparison inequalities between the power

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means defined by

$$M_n^{[r]}(\mathbf{x}; \mathbf{p}) = \begin{cases} \left( \frac{1}{P_n} \sum_{i=1}^n p_i x_i^r \right)^{\frac{1}{r}}, & r \neq -\infty, 0, \infty; \\ \left( \prod_{i=1}^n x_i^{p_i} \right)^{\frac{1}{P_n}}, & r = 0; \\ \min(x_1, \dots, x_n), & r = -\infty; \\ \max(x_1, \dots, x_n), & r = \infty, \end{cases}$$

where  $\mathbf{x}, \mathbf{p}$  are positive  $n$ -tuples and  $P_n = \sum_{i=1}^n p_i$ . It is well known that for such means the following inequality holds:

$$(1.3) \quad M_n^{[r]}(\mathbf{x}; \mathbf{p}) \leq M_n^{[s]}(\mathbf{x}; \mathbf{p})$$

whenever  $r < s$  (see for example [3, p. 15]).

In this paper we also use the following result (see [5, p. 152]):

**Theorem B.** Let  $\xi \in [a, b]^n$ ,  $0 < a < b$ , and  $\mathbf{p} \in [0, \infty)^n$  be two  $n$ -tuples such that

$$\sum_{i=1}^n p_i \xi_i \in [a, b], \quad \sum_{i=1}^n p_i \xi_i \geq \xi_j, \quad j = 1, 2, \dots, n.$$

If  $f : [a, b] \rightarrow \mathbb{R}$  is such that the function  $f(x)/x$  is decreasing, then

$$(1.4) \quad f\left(\sum_{i=1}^n p_i \xi_i\right) \leq \sum_{i=1}^n p_i f(\xi_i).$$

If  $f(x)/x$  is increasing, then the inequality in (1.4) is reversed.

Our goal is to establish several new inequalities for functions whose absolute values raised to some real powers are convex functions.



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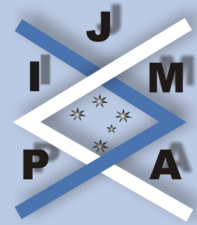
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## 2. Results

In the literature, the following definition is well known.

Let  $f : [a, b] \rightarrow \mathbb{R}$  and  $p \in \mathbb{R}^+$ . The  $p$ -norm of the function  $f$  on  $[a, b]$  is defined by

$$\|f\|_p = \begin{cases} \left( \int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}, & 0 < p < \infty; \\ \sup |f(x)|, & p = \infty, \end{cases}$$

and  $L^p([a, b])$  is the set of all functions  $f : [a, b] \rightarrow \mathbb{R}$  such that  $\|f\|_p < \infty$ .

Observe that if  $|f|^p$  is convex (or concave) on  $[a, b]$  it is also integrable on  $[a, b]$ , hence  $0 \leq \|f\|_p < \infty$ , that is,  $f$  belongs to  $L^p([a, b])$ .

Although  $p$ -norms are not defined for  $p < 0$ , for the sake of the simplicity we will use the same notation  $\|f\|_p$  when  $p \in \mathbb{R} \setminus \{0\}$ .

In order to prove our results we need the following two lemmas.

**Lemma 2.1.** *Let  $x$  and  $p$  be two  $n$ -tuples such that*

$$(2.1) \quad x_i > 0, p_i \geq 1, \quad i = 1, 2, \dots, n.$$

*If  $r < s < 0$  or  $0 < r < s$ , then*

$$(2.2) \quad \left( \sum_{i=1}^n p_i x_i^s \right)^{\frac{1}{s}} \leq \left( \sum_{i=1}^n p_i x_i^r \right)^{\frac{1}{r}},$$

*and if  $r < 0 < s$ , then*

$$\left( \sum_{i=1}^n p_i x_i^r \right)^{\frac{1}{r}} \leq \left( \sum_{i=1}^n p_i x_i^s \right)^{\frac{1}{s}}.$$

*If the  $n$ -tuple  $x$  is only nonnegative, then (2.2) holds whenever  $0 < r < s$ .*

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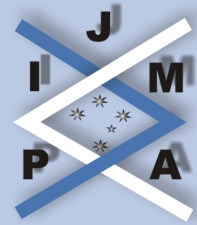
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*Proof.* Suppose that  $x$  and  $p$  are such that the inequalities in (2.1) hold. It can be easily seen that in this case for any  $q \in \mathbb{R}$

$$\sum_{i=1}^n p_i x_i^q \geq x_j^q > 0, \quad j = 1, 2, \dots, n.$$

To prove the lemma we must consider three cases: (i)  $r < s < 0$ , (ii)  $0 < r < s$  and (iii)  $r < 0 < s$ . In case (i) we define the function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  by  $f(x) = x^{\frac{s}{r}}$ . Since in this case we have  $(s - r) / r < 0$ , the function

$$f(x) / x = x^{\frac{s}{r}-1} = x^{\frac{s-r}{r}}$$

is decreasing. Applying Theorem B on  $f$ ,  $\xi = (x_1^r, \dots, x_n^r)$  and  $p$  we obtain

$$\left( \sum_{i=1}^n p_i x_i^r \right)^{\frac{s}{r}} \leq \sum_{i=1}^n p_i (x_i^r)^{\frac{s}{r}} = \sum_{i=1}^n p_i x_i^s,$$

i.e.,

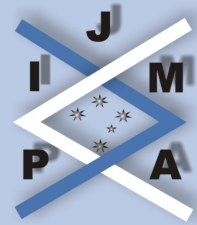
$$\left( \sum_{i=1}^n p_i x_i^r \right)^{\frac{1}{r}} \geq \left( \sum_{i=1}^n p_i x_i^s \right)^{\frac{1}{s}}$$

since  $s$  is negative.

In case (ii) for the same  $f$  as in (i) we have  $(s - r) / r > 0$ , so similarly as before from Theorem B we obtain

$$\left( \sum_{i=1}^n p_i x_i^r \right)^{\frac{s}{r}} \geq \sum_{i=1}^n p_i (x_i^r)^{\frac{s}{r}} = \sum_{i=1}^n p_i x_i^s,$$

and since  $s$  is positive, (2.2) immediately follows.



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And in the end, in case (iii) we have  $(s - r) / r < 0$ , so using again Theorem B we obtain (2.2) reversed. □

*Remark 1.* In this paper we will use Lemma 2.1 only in a special case when all weights are equal to 1. Then for  $r < s < 0$  or  $0 < r < s$ , (2.2) becomes

$$(2.3) \quad \left( \sum_{i=1}^n x_i^s \right)^{\frac{1}{s}} \leq \left( \sum_{i=1}^n x_i^r \right)^{\frac{1}{r}}$$

and for  $r < 0 < s$ ,

$$\left( \sum_{i=1}^n x_i^s \right)^{\frac{1}{s}} \geq \left( \sum_{i=1}^n x_i^r \right)^{\frac{1}{r}}.$$

In the rest of the paper we denote

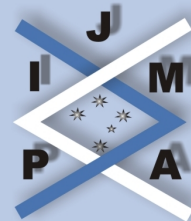
$$C_p = \begin{cases} 2^{-\frac{1}{p}}, & p \leq -1 \text{ or } p \geq 1; \\ 2, & -1 < p < 0; \\ 2^{-1}, & 0 < p < 1; \end{cases} \quad \tilde{C}_p = \begin{cases} 2, & p \leq -1; \\ 2^{-\frac{1}{p}}, & -1 < p < 1, p \neq 0; \\ 2^{-1}, & p \geq 1. \end{cases}$$

**Lemma 2.2.** Let  $f : [a, b] \rightarrow \mathbb{R}$ ,  $a < b$ . If  $|f|^p$  is convex on  $[a, b]$  for some  $p > 0$ , then

$$\left| f \left( \frac{a+b}{2} \right) \right| \leq (b-a)^{-\frac{1}{p}} \|f\|_p \leq \left( \frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}} \leq C_p (|f(a)| + |f(b)|),$$

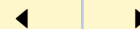
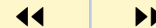
and if  $|f|^p$  is concave on  $[a, b]$ , then

$$\tilde{C}_p (|f(a)| + |f(b)|) \leq \left( \frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}} \leq (b-a)^{-\frac{1}{p}} \|f\|_p \leq \left| f \left( \frac{a+b}{2} \right) \right|.$$



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*Proof.* Suppose first that  $|f|^p$  is convex on  $[a, b]$  for some  $p > 0$ . We have

$$\|f\|_p = \left( \int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} = (b-a)^{\frac{1}{p}} \left( \frac{1}{b-a} \int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}.$$

From (1.2) we obtain

$$(2.4) \quad \left| f\left(\frac{a+b}{2}\right) \right|^p \leq \frac{1}{b-a} \int_a^b |f(x)|^p dx \leq \frac{|f(a)|^p + |f(b)|^p}{2},$$

hence

$$\left| f\left(\frac{a+b}{2}\right) \right| \leq (b-a)^{-\frac{1}{p}} \|f\|_p \leq \left( \frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}}.$$

Now we must consider two cases. If  $p \geq 1$  we can use (2.3) to obtain

$$(|f(a)|^p + |f(b)|^p)^{\frac{1}{p}} \leq |f(a)| + |f(b)|,$$

hence

$$(2.5) \quad \left( \frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}} \leq C_p (|f(a)| + |f(b)|),$$

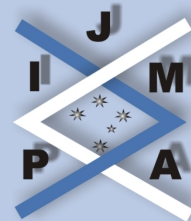
where  $C_p = 2^{-\frac{1}{p}}$ .

In the other case, when  $0 < p < 1$ , from (1.3) we have

$$\left( \frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}} \leq \frac{|f(a)| + |f(b)|}{2},$$

so again we obtain (2.5), where  $C_p = 2^{-1}$ . This completes the proof for  $|f|^p$  convex.





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Suppose now that  $|f|^p$  is concave on  $[a, b]$  for some  $p > 0$ . In that case  $-|f|^p$  is convex on  $[a, b]$ , hence (1.2) implies

$$\frac{|f(a)|^p + |f(b)|^p}{2} \leq \frac{1}{b-a} \int_a^b |f(x)|^p dx \leq \left| f\left(\frac{a+b}{2}\right) \right|^p.$$

If  $p \geq 1$  from (1.3) we obtain

$$\left( \frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}} \geq \frac{|f(a)| + |f(b)|}{2},$$

hence

$$\left( \frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}} \geq \tilde{C}_p (|f(a)| + |f(b)|),$$

where  $\tilde{C}_p = 2^{-1}$ .

In the other case, when  $0 < p < 1$ , from (2.3) we have

$$\left( |f(a)|^p + |f(b)|^p \right)^{\frac{1}{p}} \geq |f(a)| + |f(b)|,$$

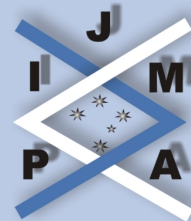
hence

$$\left( \frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}} \geq \tilde{C}_p (|f(a)| + |f(b)|),$$

where  $\tilde{C}_p = 2^{-\frac{1}{p}}$ . This completes the proof.  $\square$

**Lemma 2.3.** Let  $f : [a, b] \rightarrow \mathbb{R} \setminus \{0\}$ ,  $a < b$ . If  $|f|^p$  is convex on  $[a, b]$  for some  $p < 0$ , then

$$C_p \frac{|f(a)f(b)|}{|f(a)| + |f(b)|} \leq \left( \frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}} \leq (b-a)^{-\frac{1}{p}} \|f\|_p \leq \left| f\left(\frac{a+b}{2}\right) \right|$$



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and if  $|f|^p$  is concave on  $[a, b]$ , then

$$\left| f\left(\frac{a+b}{2}\right) \right| \leq (b-a)^{-\frac{1}{p}} \|f\|_p \leq \left( \frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}} \leq \tilde{C}_p \frac{|f(a)f(b)|}{|f(a)| + |f(b)|}.$$

*Proof.* Suppose that  $|f|^p$  is convex on  $[a, b]$  for some  $p < 0$ . From (2.4), using the fact that  $p < 0$ , we obtain

$$\left( \frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}} \leq (b-a)^{-\frac{1}{p}} \|f\|_p \leq \left| f\left(\frac{a+b}{2}\right) \right|.$$

Again we consider two cases. If  $-1 < p < 0$ , then from (1.3) we have

$$\left( \frac{|f(a)|^{-1} + |f(b)|^{-1}}{2} \right)^{-1} \leq \left( \frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}},$$

hence

$$C_p \frac{|f(a)f(b)|}{|f(a)| + |f(b)|} \leq \left( \frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}},$$

where  $C_p = 2$ .

In the other case, when  $p \leq -1$ , from (2.3) we have

$$\left( |f(a)|^{-1} + |f(b)|^{-1} \right)^{-1} \leq \left( |f(a)|^p + |f(b)|^p \right)^{\frac{1}{p}},$$

hence

$$C_p \frac{|f(a)f(b)|}{|f(a)| + |f(b)|} \leq \left( \frac{|f(a)|^p + |f(b)|^p}{2} \right)^{\frac{1}{p}},$$

where  $C_p = 2^{-\frac{1}{p}}$ .

In the other case, when  $|f|^p$  is concave on  $[a, b]$  for some  $p < 0$ , the proof is similar.  $\square$



**Theorem 2.4.** Let  $p, q > 0$  and let  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $a < b$ , be such that

$$(2.6) \quad m (|g(a)| + |g(b)|) \leq |f(a)| + |f(b)| \leq M (|g(a)| + |g(b)|)$$

for some  $0 < m \leq M$ .

If  $|f|^p$  and  $|g|^q$  are convex on  $[a, b]$ , then

$$(2.7) \quad \|f\|_p + \|g\|_q \leq \left[ \frac{M}{M+1} C_p (b-a)^{\frac{1}{p}} + \frac{1}{m+1} C_q (b-a)^{\frac{1}{q}} \right] K(f, g),$$

where

$$K(f, g) = |f(a)| + |f(b)| + |g(a)| + |g(b)|.$$

If  $|f|^p$  and  $|g|^q$  are concave on  $[a, b]$ , then

$$(2.8) \quad \|f\|_p + \|g\|_q \geq \left[ \frac{m}{m+1} \tilde{C}_p (b-a)^{\frac{1}{p}} + \frac{1}{M+1} \tilde{C}_q (b-a)^{\frac{1}{q}} \right] K(f, g).$$

*Proof.* Suppose that  $|f|^p$  and  $|g|^q$  are convex on  $[a, b]$  for some fixed  $p, q > 0$ . From Lemma 2.2 we have that

$$\begin{aligned} & \|f\|_p + \|g\|_q \\ & \leq \left( \frac{b-a}{2} \right)^{\frac{1}{p}} (|f(a)|^p + |f(b)|^p)^{\frac{1}{p}} + \left( \frac{b-a}{2} \right)^{\frac{1}{q}} (|g(a)|^q + |g(b)|^q)^{\frac{1}{q}} \\ (2.9) \quad & \leq C_p (b-a)^{\frac{1}{p}} (|f(a)| + |f(b)|) + C_q (b-a)^{\frac{1}{q}} (|g(a)| + |g(b)|). \end{aligned}$$

Using (2.6) we can write

$$|f(a)| + |f(b)| \leq M (|f(a)| + |f(b)| + |g(a)| + |g(b)|) - M (|f(a)| + |f(b)|),$$

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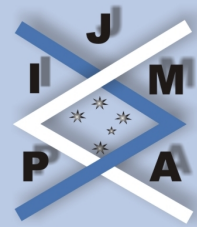
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i.e.,

$$(2.10) \quad |f(a)| + |f(b)| \leq \frac{M}{M+1} (|f(a)| + |f(b)| + |g(a)| + |g(b)|) \\ = \frac{M}{M+1} K(f, g),$$

and analogously

$$(2.11) \quad |g(a)| + |g(b)| \leq \frac{1}{m+1} K(f, g).$$

Combining (2.10) and (2.11) with (2.9) we obtain (2.7).

Suppose now that  $|f|^p$  and  $|g|^q$  are concave on  $[a, b]$  for some fixed  $p, q > 0$ . From Lemma 2.2 we have that

$$\|f\|_p + \|g\|_q \geq \tilde{C}_p (b-a)^{\frac{1}{p}} (|f(a)| + |f(b)|) + \tilde{C}_q (b-a)^{\frac{1}{q}} (|g(a)| + |g(b)|).$$

Using again (2.6) we can write

$$|f(a)| + |f(b)| \geq m (|f(a)| + |f(b)| + |g(a)| + |g(b)|) - m (|f(a)| + |f(b)|),$$

i.e.,

$$|f(a)| + |f(b)| \geq \frac{m}{m+1} K(f, g),$$

and analogously

$$|g(a)| + |g(b)| \geq \frac{1}{M+1} K(f, g),$$

from which (2.8) easily follows.  $\square$

*Remark 2.* A similar type of condition as in (2.6) was used in [1, Theorem 1.1] where a variant of the reversed Minkowski's integral inequality for  $p > 1$  was proved.

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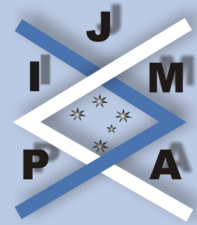
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**Theorem 2.5.** Let  $p, q < 0$  and let  $f, g : [a, b] \rightarrow \mathbb{R} \setminus \{0\}$ ,  $a < b$ , be such that

$$m \frac{|g(a)g(b)|}{|g(a)| + |g(b)|} \leq \frac{|f(a)f(b)|}{|f(a)| + |f(b)|} \leq M \frac{|g(a)g(b)|}{|g(a)| + |g(b)|}$$

for some  $0 < m \leq M$ .

If  $|f|^p$  and  $|g|^q$  are concave on  $[a, b]$ , then

$$\|f\|_p + \|g\|_q \leq \left[ \frac{M}{M+1} \tilde{C}_p (b-a)^{\frac{1}{p}} + \frac{1}{m+1} \tilde{C}_q (b-a)^{\frac{1}{q}} \right] H(f, g),$$

where

$$H(f, g) = \frac{|f(a)f(b)|}{|f(a)| + |f(b)|} + \frac{|g(a)g(b)|}{|g(a)| + |g(b)|}.$$

If  $|f|^p$  and  $|g|^q$  are convex on  $[a, b]$ , then

$$\|f\|_p + \|g\|_q \geq \left[ \frac{m}{m+1} C_p (b-a)^{\frac{1}{p}} + \frac{1}{M+1} C_q (b-a)^{\frac{1}{q}} \right] H(f, g).$$

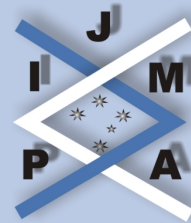
*Proof.* Similar to that of Theorem 2.4. □

**Theorem 2.6.** Let  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $a < b$ , be such that  $|f|^p$  and  $|g|^q$  are convex on  $[a, b]$  for some fixed  $p, q > 1$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ . Then

$$\begin{aligned} \left| \int_a^b f(x)g(x) dx \right| &\leq \frac{b-a}{2} (|f(a)|^p + |f(b)|^p)^{\frac{1}{p}} (|g(a)|^q + |g(b)|^q)^{\frac{1}{q}} \\ &\leq \frac{b-a}{2} [M(f, g) + N(f, g)], \end{aligned}$$

where

$$M(f, g) = |f(a)||g(a)| + |f(b)||g(b)|, \quad N(f, g) = |f(a)||g(b)| + |f(b)||g(a)|.$$



*Proof.* First note that since  $|f|^p$  and  $|g|^q$  are convex on  $[a, b]$  we have  $f \in L^p([a, b])$  and  $g \in L^q([a, b])$ , and since  $\frac{1}{p} + \frac{1}{q} = 1$  we know that  $fg \in L^1([a, b])$ , that is,  $fg$  is integrable on  $[a, b]$ . Using Hölder's integral inequality (1.1) we obtain

$$\left| \int_a^b f(x)g(x) dx \right| \leq \int_a^b |f(x)g(x)| dx \leq \|f\|_p \|g\|_q.$$

From Lemma 2.3 we have that

$$\|f\|_p \leq \left( \frac{b-a}{2} \right)^{\frac{1}{p}} (|f(a)|^p + |f(b)|^p)^{\frac{1}{p}} \leq \left( \frac{b-a}{2} \right)^{\frac{1}{p}} (|f(a)| + |f(b)|)$$

and

$$\|g\|_q \leq \left( \frac{b-a}{2} \right)^{\frac{1}{q}} (|g(a)|^q + |g(b)|^q)^{\frac{1}{q}} \leq \left( \frac{b-a}{2} \right)^{\frac{1}{q}} (|g(a)| + |g(b)|),$$

hence

$$\begin{aligned} \left| \int_a^b f(x)g(x) dx \right| &\leq \frac{b-a}{2} (|f(a)|^p + |f(b)|^p)^{\frac{1}{p}} (|g(a)|^q + |g(b)|^q)^{\frac{1}{q}} \\ &\leq \frac{b-a}{2} (|f(a)| + |f(b)|) (|g(a)| + |g(b)|) \\ &= \frac{b-a}{2} [M(f, g) + N(f, g)]. \end{aligned}$$

□

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