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APPROXIMATION OF A MIXED FUNCTIONAL EQUATION IN QUASI-BANACH SPACES

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ABSTRACT. In this paper we establish the general solution of the functional equation

$$f(2x+y)+f(x+2y)=6f(x+y)+f(2x)+f(2y)-5[f(x)+f(y)]$$

and investigate its generalized Hyers-Ulam stability in quasi-Banach spaces. The concept of Hyers-Ulam-Rassias stability originated from Th.M. Rassias' stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.

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1. INTRODUCTION AND PRELIMINARIES

In 1940, S.M. Ulam [30] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let $(G_1,*)$ be a group and let $(G_2,\diamond,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta(\epsilon)>0$ such that if a function $h:G_1\to G_2$ satisfies the inequality

$$d(h(x*y),h(x)\diamond h(y))<\delta$$

for all $x,y\in G_1$, then there is a homomorphism $H:G_1\to G_2$ with

$$d(h(x),H(x))<\epsilon$$

for all $x\in G_1$?

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In 1941, D.H. Hyers [9] considered the case of approximately additive functions $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive function satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

T. Aoki [2] and Th.M. Rassias [27] provided a generalization of Hyers' theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded.

Theorem 1.1 (Th.M. Rassias)**.** *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. Also, if for each $x \in E$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

The inequality (1.1) has provided much influence in the development of what is now known as *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. P. Găvruta in [7] provided a further generalization of Th.M. Rassias' theorem. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [4], [6], [8], [11], [13], [15] – [26]). We also refer the readers to the books [1], [5], [10], [14] and [28].

Jun and Kim [12] introduced the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.3}$$

and they established the general solution and the generalized Hyers-Ulam stability problem for the functional equation (1.3). They proved that a function $f : E_1 \to E_2$ satisfies the functional equation (1.3) if and only if there exists a function $B : E_1 \times E_1 \times E_1 \to E_2$ such that $f(x) = B(x, x, x)$ for all $x \in E_1$, and B is symmetric for each fixed one variable and additive for each fixed two variables. The function B is given by

$$B(x, y, z) = \frac{1}{24}[f(x+y+z) + f(x-y-z) - f(x+y-z) - f(x-y+z)]$$

for all $x, y, z \in E_1$.

A. Najati and G.Z. Eskandani [25] established the general solution and investigated the generalized Hyers-Ulam stability of the following functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 2f(2x) - 2f(x)$$

in quasi-Banach spaces.

In this paper, we deal with the following functional equation derived from cubic and additive functions:

$$f(2x+y)+f(x+2y)=6f(x+y)+f(2x)+f(2y)-5[f(x)+f(y)]. \tag{1.4}$$

It is easy to see that the function $f(x)=ax^3+cx$ is a solution of the functional equation (1.4).

The main purpose of this paper is to establish the general solution of (1.4) and investigate its generalized Hyers-Ulam stability.

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1 ([3, 29])**.** Let X be a real linear space. A *quasi-norm* is a real-valued function on X satisfying the following:

(i) $\|x\|\geq 0$ for all $x\in X$ and $\|x\|=0$ if and only if $x=0$;
(ii) $\|\lambda x\|=|\lambda|\|x\|$ for all $\lambda\in\mathbb{R}$ and all $x\in X$;
(iii) There is a constant $K\geq 1$ such that $\|x+y\|\leq K(\|x\|+\|y\|)$ for all $x,y\in X$.

It follows from condition (iii) that

$$\left\|\sum_{i=1}^{2n}x_i\right\|\leq K^n\sum_{i=1}^{2n}\|x_i\|,\qquad \left\|\sum_{i=1}^{2n+1}x_i\right\|\leq K^{n+1}\sum_{i=1}^{2n+1}\|x_i\|$$

for all integers $n\geq 1$ and all $x_1,x_2,\dots,x_{2n+1}\in X$.

The pair $(X,\|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a p*-norm* $(0<p\leq 1)$ if

$$\|x+y\|^p\leq\|x\|^p+\|y\|^p$$

for all $x,y\in X$. In this case, a quasi-Banach space is called a p*-Banach space.*

By the Aoki-Rolewicz theorem [29] (see also [3]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

2. SOLUTIONS OF EQ. (1.4)

Throughout this section, X and Y will be real vector spaces. Before proceeding to the proof of Theorem 2.3 which is the main result in this section, we need the following two lemmas.

Lemma 2.1. *If a function $f:X\to Y$ satisfies (1.4), then the function $g:X\to Y$ defined by $g(x)=f(2x)-8f(x)$ is additive.*

Proof. Let $f:X\to Y$ satisfy the functional equation (1.4). Letting $x=y=0$ in (1.4), we get that $f(0)=0$. Replacing y by $2y$ in (1.4), we get

$$f(2x+2y)+f(x+4y)=6f(x+2y)+f(2x)+f(4y)-5[f(x)+f(2y)] \tag{2.1}$$

for all $x,y\in X$. Replacing y by x and x by y in (2.1), we have

$$f(2x+2y)+f(4x+y)=6f(2x+y)+f(4x)+f(2y)-5[f(2x)+f(y)] \tag{2.2}$$

for all $x,y\in X$. Adding (2.1) to (2.2) and using (1.4), we have

$$\begin{aligned}2f(2x+2y)&+f(4x+y)+f(x+4y)\\&=36f(x+y)+f(4x)+f(4y)+2[f(2x)+f(2y)]-35[f(x)+f(y)]\end{aligned} \tag{2.3}$$

for all $x, y \in X$. Replacing y by $-x$ in (2.3), we get

$$f(3x) + f(-3x) = f(4x) + f(-4x) + 2[f(2x) + f(-2x)] - 35[f(x) + f(-x)] \tag{2.4}$$

for all $x \in X$. Letting $y = x$ in (1.4), we get

$$f(3x) = 4f(2x) - 5f(x) \tag{2.5}$$

for all $x \in X$. Letting $y = -x$ in (1.4), we have

$$f(2x) + f(-2x) = 6[f(x) + f(-x)] \tag{2.6}$$

for all $x \in X$. It follows from (2.4), (2.5) and (2.6) that $f(-x) = -f(x)$ for all $x \in X$, i.e., f is odd. Replacing x by $x + y$ and y by $-y$ in (1.4) and using the oddness of f, we have

$$f(2x + y) + f(x - y) = 6f(x) + f(2x + 2y) - f(2y) - 5[f(x + y) - f(y)] \tag{2.7}$$

for all $x, y \in X$. Replacing y by x and y by x in (2.7), we get

$$f(x + 2y) - f(x - y) = 6f(y) + f(2x + 2y) - f(2x) - 5[f(x + y) - f(x)] \tag{2.8}$$

for all $x, y \in X$. Adding (2.7) to (2.8), we have

$$f(2x+y)+f(x+2y) = 2f(2x+2y) - f(2x) - f(2y) - 10f(x+y) + 11[f(x)+f(y)] \tag{2.9}$$

for all $x, y \in X$. It follows from (1.4) and (2.9) that

$$f(2x + 2y) - 8f(x + y) = f(2x) + f(2y) - 8[f(x) + f(y)] \tag{2.10}$$

for all $x, y \in X$, which means that the function $g : X \to Y$ is additive. □

Lemma 2.2. *If a function $f : X \to Y$ satisfies the functional equation (1.4), then the function $h : X \to Y$ defined by $h(x) = f(2x) - 2f(x)$ is cubic .*

Proof. Let $g : X \to Y$ be a function defined by $g(x) = f(2x) - 8f(x)$ for all $x \in X$. By Lemma 2.1 and its proof, the function f is odd and the function g is additive. It is clear that

$$h(x) = g(x) + 6f(x), \quad f(2x) = g(x) + 8f(x) \tag{2.11}$$

for all $x \in X$. Replacing x by $x - y$ in (1.4), we have

$$f(2x - y) + f(x + y) = 6f(x) + f(2x - 2y) + f(2y) - 5[f(x - y) + f(y)] \tag{2.12}$$

for all $x, y \in X$. Replacing y by $-y$ in(2.12), we have

$$f(2x + y) + f(x - y) = 6f(x) + f(2x + 2y) - f(2y) - 5[f(x + y) - f(y)] \tag{2.13}$$

for all $x, y \in X$. Adding (2.12) to (2.13), we get

$$\begin{aligned} f(2x - y) + f(2x + y) \\ = 12f(x) + f(2x + 2y) + f(2x - 2y) - 6[f(x + y) + f(x - y)] \end{aligned} \tag{2.14}$$

for all $x, y \in X$. Since g is additive, it follows from (2.11) and (2.14) that

$$h(2x + y) + h(2x - y) = 2[h(x + y) + h(x - y)] + 12h(x)$$

for all $x, y \in X$. So the function h is cubic. □

Theorem 2.3. *A function $f : X \to Y$ satisfies (1.4) if and only if there exist functions $C : X \times X \times X \to Y$ and $A : X \to Y$ such that*

$$f(x) = C(x, x, x) + A(x)$$

for all $x \in X$, where the function C is symmetric for each fixed one variable and is additive for fixed two variables and the function A is additive.

Proof. We first assume that the function $f : X \to Y$ satisfies (1.4). Let $g, h : X \to Y$ be functions defined by

$$g(x) := f(2x) - 8f(x), \qquad h(x) := f(2x) - 2f(x)$$

for all $x \in X$. By Lemmas 2.1 and 2.2, we achieve that the functions g and h are additive and cubic, respectively, and

$$f(x) = \frac{1}{6}[h(x) - g(x)] \tag{2.15}$$

for all $x \in X$. Therefore by Theorem 2.1 of [12] there exists a function $C : X \times X \times X \to Y$ such that $h(x) = 6C(x, x, x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables. So

$$f(x) = C(x, x, x) + A(x)$$

for all $x \in X$, where $A(x) = -\frac{1}{6}g(x)$ for all $x \in X$.

Conversely, let

$$f(x) = C(x, x, x) + A(x)$$

for all $x \in X$, where the function C is symmetric for each fixed one variable and additive for fixed two variables and the function A is additive. By a simple computation one can show that the functions $x \mapsto C(x, x, x)$ and A satisfy the functional equation (1.4). So the function f satisfies (1.4). □

3. GENERALIZED HYERS-ULAM STABILITY OF EQ. (1.4)

Throughout this section, assume that X is a quasi-normed space with quasi-norm $\|\cdot\|_X$ and that Y is a p-Banach space with p-norm $\|\cdot\|_Y$. Let K be the modulus of concavity of $\|\cdot\|_Y$.

In this section, using an idea of Găvruta [7] we prove the stability of the functional equation (1.4) in the spirit of Hyers, Ulam and Rassias. For convenience, we use the following abbreviation for a given function $f : X \to Y$:

$$Df(x, y) := f(2x + y) + f(x + 2y) - 6f(x + y) - f(2x) - f(2y) + 5[f(x) + f(y)]$$

for all $x, y \in X$.

We will use the following lemma in this section.

Lemma 3.1 ([23]). *Let $0 \le p \le 1$ and let $x_1, x_2, \ldots, x_n$ be non-negative real numbers. Then*

$$\left(\sum_{i=1}^{n} x_i\right)^p \le \sum_{i=1}^{n} x_i^p. \tag{3.1}$$

Theorem 3.2. *Let $\varphi : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y) = 0 \tag{3.2}$$

for all $x, y \in X$, and

$$M(x, y) := \sum_{i=0}^{\infty} \frac{1}{2^{ip}}\varphi^p(2^i x, 2^i y) < \infty \tag{3.3}$$

for all $x \in X$ and all $y \in \{0, x, -x/2\}$. Suppose that a function $f : X \to Y$ satisfies the inequalities

$$\|Df(x, y)\|_Y \le \varphi(x, y), \tag{3.4}$$

(3.5) $$\|f(x)+f(-x)\|_Y \leq \varphi(x,0)$$

for all $x, y \in X$. *Then the limit*

$$A(x)=\lim_{n\to\infty}\frac{f(2^{n+1}x)-8f(2^n x)}{2^n}$$

exists for all $x \in X$, *and the function* $A : X \to Y$ *is a unique additive function satisfying*

(3.6) $$\|f(2x)-8f(x)-A(x)+f(0)\|_Y \leq \frac{K}{2}[\widetilde{\varphi}(x)]^{\frac{1}{p}}$$

for all $x \in X$, *where*

$$\widetilde{\varphi}(x)=K^{2p}M(2x,-x)+\frac{K^{2p}}{2^p}M(x,x)+K^pM(2x,0)+5^pM(x,0).$$

Proof. Letting $y = x$ in (3.4), we have

(3.7) $$\|f(3x)-4f(2x)+5f(x)\|_Y \leq \frac{1}{2}\varphi(x,x)$$

for all $x \in X$. Replacing x by $2x$ and y by $-x$ in (3.4), we have

(3.8) $$\|f(3x)-f(4x)+5f(2x)-f(-2x)-6f(x)+5f(-x)+f(0)\|_Y \leq \varphi(2x,-x).$$

Using (3.5), (3.7) and (3.8), we have

(3.9) $$\|g(2x)-2g(x)-f(0)\|_Y \leq \phi(x)$$

for all $x \in X$, where

$$\phi(x)=K\left[K^2\varphi(2x,-x)+\frac{K^2}{2}\varphi(x,x)+K\varphi(2x,0)+5\varphi(x,0)\right]$$

and $g(x) = f(2x) - 8f(x)$. By Lemma 3.1 and (3.3), we infer that

(3.10) $$\sum_{i=0}^{\infty}\frac{1}{2^{ip}}\phi^p(2^i x)<\infty$$

for all $x \in X$. Replacing x by $2^n x$ in (3.9) and dividing both sides of (3.9) by 2^{n+1}, we get

(3.11) $$\left\|\frac{1}{2^{n+1}}g(2^{n+1}x)-\frac{1}{2^n}g(2^n x)-\frac{1}{2^{n+1}}f(0)\right\|_Y \leq \frac{1}{2^{n+1}}\phi(2^n x)$$

for all $x \in X$ and all non-negative integers n. Since Y is a p-Banach space, we have

(3.12) $$\begin{aligned}\left\|\frac{1}{2^{n+1}}g(2^{n+1}x)-\frac{1}{2^m}g(2^m x)-\sum_{i=m}^{n}\frac{1}{2^{i+1}}f(0)\right\|_Y^p &\leq \sum_{i=m}^{n}\left\|\frac{1}{2^{i+1}}g(2^{i+1}x)-\frac{1}{2^i}g(2^i x)-\frac{1}{2^{i+1}}f(0)\right\|_Y^p\\ &\leq \frac{1}{2^p}\sum_{i=m}^{n}\frac{1}{2^{ip}}\phi^p(2^i x)\end{aligned}$$

for all $x \in X$ and all non-negative integers n and m with $n \geq m$. Therefore we conclude from (3.10) and (3.12) that the sequence $\left\{\frac{1}{2^n}g(2^n x)\right\}$ is a Cauchy sequence in Y for all $x \in X$. Since

Y is complete, the sequence $\left\{\frac{1}{2^n}g(2^nx)\right\}$ converges in Y for all $x \in X$. So we can define the function $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} g(2^n x) \tag{3.13}$$

for all $x \in X$. Letting $m = 0$ and passing the limit when $n \to \infty$ in (3.12), we get (3.6). Now, we show that A is an additive function. It follows from (3.10), (3.11) and (3.13) that

$$\begin{aligned}
&\|A(2x) - 2A(x)\|_Y \\
&\quad = \lim_{n\to\infty} \left\| \frac{1}{2^n} g(2^{n+1}x) - \frac{1}{2^{n-1}} g(2^n x) \right\|_Y \\
&\quad \le 2K \lim_{n\to\infty} \left(\left\| \frac{1}{2^{n+1}} g(2^{n+1}x) - \frac{1}{2^n} g(2^n x) - \frac{1}{2^{n+1}} f(0) \right\|_Y + \frac{1}{2^{n+1}} \|f(0)\|_Y \right) \\
&\quad \le \lim_{n\to\infty} \frac{K}{2^n} \phi(2^n x) = 0
\end{aligned}$$

for all $x \in X$. So

$$A(2x) = 2A(x) \tag{3.14}$$

for all $x \in X$. On the other hand, it follows from (3.2), (3.4) and (3.13) that

$$\begin{aligned}
\|DA(x,y)\|_Y &= \lim_{n\to\infty} \frac{1}{2^n} \|Dg(2^n x, 2^n y)\|_Y \\
&\le \lim_{n\to\infty} \frac{K}{2^n} \left\{ \left\| Df(2^{n+1}x, 2^{n+1}y) \right\|_Y + 8 \|Df(2^n x, 2^n y)\|_Y \right\} \\
&\le \lim_{n\to\infty} \frac{K}{2^n} \left[\varphi(2^{n+1}x, 2^{n+1}y) + 8\varphi(2^n x, 2^n y) \right] = 0
\end{aligned}$$

for all $x, y \in X$. Hence the function A satisfies (1.4). So by Lemma 2.1, the function $x \mapsto A(2x) - 8A(x)$ is additive. Therefore (3.14) implies that the function A is additive.

To prove the uniqueness of A, let $T : X \to Y$ be another additive function satisfying (3.6). It follows from (3.3) that

$$\lim_{n\to\infty} \frac{1}{2^{np}} M(2^n x, 2^n y) = \lim_{n\to\infty} \sum_{i=n}^{\infty} \frac{1}{2^{ip}} \varphi^p(2^i x, 2^i y) = 0$$

for all $x \in X$ and all $y \in \{0, x, -x/2\}$. Hence $\lim_{n\to\infty} \frac{1}{2^{np}} \widetilde{\varphi}(2^n x) = 0$ for all $x \in X$. So it follows from (3.6) and (3.13) that

$$\begin{aligned}
\|A(x) - T(x)\|_Y^p &= \lim_{n\to\infty} \frac{1}{2^{np}} \|g(2^n x) - T(2^n x) + f(0)\|_Y^p \\
&\le \frac{K^p}{2^p} \lim_{n\to\infty} \frac{1}{2^{np}} \widetilde{\varphi}(2^n x) = 0
\end{aligned}$$

for all $x \in X$. So $A = T$. □

Corollary 3.3. *Let θ be non-negative real number . Suppose that a function $f : X \to Y$ satisfies the inequalities*

$$\|Df(x,y)\|_Y \le \theta, \qquad \|f(x) + f(-x)\|_Y \le \theta \tag{3.15}$$

for all $x, y \in X$. Then there exists a unique additive function $A : X \to Y$ satisfying

$$\|f(2x) - 8f(x) - A(x)\|_Y \le \frac{K^2\theta}{2} \left\{ \frac{(2K^2)^p + (2K)^p + K^{2p} + 10^p}{2^p - 1} \right\}^{\frac{1}{p}} + \frac{K\theta}{4}$$

for all $x \in X$.

Proof. It follows from (3.15) that $\|f(0)\|_Y \le \theta/4$. So the result follows from Theorem 3.2. $\square$

Theorem 3.4. *Let $\varphi : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$$

for all $x, y \in X$, and

$$M(x,y) := \sum_{i=1}^{\infty} 2^{ip} \varphi^p \left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty \tag{3.16}$$

for all $x \in X$ and all $y \in \{0, x, -x/2\}$. Suppose that a function $f : X \to Y$ satisfies the inequalities

$$\|Df(x,y)\|_Y \le \varphi(x,y), \qquad \|f(x) + f(-x)\|_Y \le \varphi(x,0)$$

for all $x, y \in X$. Then the limit

$$A(x) = \lim_{n\to\infty} 2^n \left[f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right]$$

exists for all $x \in X$ and the function $A : X \to Y$ is a unique additive function satisfying

$$\|f(2x) - 8f(x) - A(x)\|_Y \le \frac{K}{2}[\widetilde{\varphi}(x)]^{\frac{1}{p}} \tag{3.17}$$

for all $x \in X$, where

$$\widetilde{\varphi}(x) = K^{2p} M(2x, -x) + \frac{K^{2p}}{2^p} M(x,x) + K^p M(2x, 0) + 5^p M(x, 0).$$

Proof. It follows from (3.16) that $\varphi(0,0) = 0$ and so $f(0) = 0$. We introduce the same definitions for $g : X \to Y$ and $\phi(x)$ as in the proof of Theorem 3.2. Similar to the proof of Theorem 3.2, we have

$$\|g(2x) - 2g(x)\|_Y \le \phi(x) \tag{3.18}$$

for all $x \in X$. By Lemma 3.1 and (3.16), we infer that

$$\sum_{i=1}^{\infty} 2^{ip} \phi^p \left(\frac{x}{2^i}\right) < \infty \tag{3.19}$$

for all $x \in X$. Replacing x by $\frac{x}{2^{n+1}}$ in (3.18) and multiplying both sides of (3.18) by 2^n, we get

$$\left\|2^{n+1} g\left(\frac{x}{2^{n+1}}\right) - 2^n g\left(\frac{x}{2^n}\right)\right\|_Y \le 2^n \phi\left(\frac{x}{2^{n+1}}\right)$$

for all $x \in X$ and all non-negative integers n. Since Y is a p-Banach space, we have

$$\begin{aligned} \left\|2^{n+1} g\left(\frac{x}{2^{n+1}}\right) - 2^m g\left(\frac{x}{2^m}\right)\right\|_Y^p &\le \sum_{i=m}^{n} \left\|2^{i+1} g\left(\frac{x}{2^{i+1}}\right) - 2^i g\left(\frac{x}{2^i}\right)\right\|_Y^p \\ &\le \sum_{i=m}^{n} 2^{ip} \phi^p \left(\frac{x}{2^{i+1}}\right) \end{aligned} \tag{3.20}$$

for all $x \in X$ and all non-negative integers n and m with $n \ge m$. Therefore we conclude from (3.19) and (3.20) that the sequence $\{2^n g(x/2^n)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{2^n g(x/2^n)\}$ converges in Y for all $x \in X$. So we can define the function $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n g\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Letting $m = 0$ and passing the limit when $n \to \infty$ in (3.20) and applying Lemma 3.1, we get (3.17).

The rest of the proof is similar to the proof of Theorem 3.2 and we omit the details. □

Corollary 3.5. *Let θ, r, s be non-negative real numbers such that $r, s > 1$ or $0 < r, s < 1$. Suppose that a function $f : X \to Y$ satisfies the inequalities*

(3.21) $$\|Df(x,y)\|_Y \le \theta(\|x\|_X^r + \|y\|_X^s), \quad \|f(x) + f(-x)\|_Y \le \theta\|x\|_X^r$$

for all $x, y \in X$. Then there exists a unique additive function $A : X \to Y$ satisfying

$$\|f(2x) - 8f(x) - A(x)\|_Y \le \frac{K\theta}{2}\left\{\frac{(2^{r+1}K^2)^p + K^{2p} + (2^{r+1}K)^p + 10^p}{|2^p - 2^{rp}|}\|x\|_X^{rp} + \frac{(2K^2)^p + K^{2p}}{|2^p - 2^{sp}|}\|x\|_X^{sp}\right\}^{\frac{1}{p}}$$

for all $x \in X$.

Proof. It follows from (3.21) that $f(0) = 0$. Hence the result follows from Theorems 3.2 and 3.4. □

Corollary 3.6. *Let $\theta \ge 0$ and $r, s > 0$ be real numbers such that $\lambda := r + s \neq 1$. Suppose that an odd function $f : X \to Y$ satisfies the inequality*

(3.22) $$\|Df(x,y)\|_Y \le \theta\|x\|_X^r\|y\|_Y^s$$

for all $x, y \in X$. Then there exists a unique additive function $A : X \to Y$ satisfying

$$\|f(2x) - 8f(x) - A(x)\|_Y \le \frac{K^3\theta}{2}\left\{\frac{1 + 2^{p(r+1)}}{|2^p - 2^{\lambda p}|}\right\}^{\frac{1}{p}}\|x\|_X^\lambda$$

for all $x \in X$.

Proof. $f(0) = 0$, since f is odd. Hence the result follows from Theorems 3.2 and 3.4. □

Theorem 3.7. *Let $\psi : X \times X \to [0, \infty)$ be a function such that*

(3.23) $$\lim_{n\to\infty} \frac{1}{8^n}\psi(2^n x, 2^n y) = 0$$

for all $x, y \in X$, and

(3.24) $$M(x,y) := \sum_{i=0}^{\infty} \frac{1}{8^{ip}}\psi^p(2^i x, 2^i y) < \infty$$

for all $x \in X$ and all $y \in \{0, x, -x/2\}$. Suppose that a function $f : X \to Y$ satisfies the inequalities

(3.25) $$\|Df(x,y)\|_Y \le \psi(x,y), \qquad \|f(x) + f(-x)\|_Y \le \psi(x,0)$$

for all $x, y \in X$. Then the limit

$$C(x) = \lim_{n\to\infty} \frac{1}{8^n}[f(2^{n+1}x) - 2f(2^n x)]$$

exists for all $x \in X$, and $C : X \to Y$ is a unique cubic function satisfying

(3.26) $$\left\|f(2x) - 2f(x) - C(x) + \frac{1}{7}f(0)\right\|_Y \le \frac{K}{8}[\widetilde{\psi}(x)]^{\frac{1}{p}}$$

for all $x \in X$, where

$$\widetilde{\psi}(x) := K^{2p}M(2x, -x) + \frac{K^{2p}}{2^p}M(x,x) + K^p M(2x, 0) + 5^p M(x, 0).$$

Proof. Similar to the proof of Theorem 3.2, we have

$$\|f(4x) - 10f(2x) + 16f(x) - f(0)\|_Y \leq \phi(x) \tag{3.27}$$

for all $x \in X$, where

$$\phi(x) = K\left[K^2\psi(2x, -x) + \frac{K^2}{2}\psi(x, x) + K\psi(2x, 0) + 5\psi(x, 0)\right].$$

Let $h : X \to Y$ be a function defined by $h(x) = f(2x) - 2f(x)$. Hence (3.27) means

$$\|h(2x) - 8h(x) - f(0)\|_Y \leq \phi(x) \tag{3.28}$$

for all $x \in X$. By Lemma 3.1 and (3.24), we infer that

$$\sum_{i=0}^{\infty} \frac{1}{8^{ip}}\phi^p(2^i x) < \infty \tag{3.29}$$

for all $x \in X$. Replacing x by $2^n x$ in (3.28) and dividing both sides of (3.28) by 8^{n+1}, we get

$$\left\|\frac{1}{8^{n+1}}h(2^{n+1}x) - \frac{1}{8^n}h(2^n x) - \frac{1}{8^{n+1}}f(0)\right\|_Y \leq \frac{1}{8^{n+1}}\phi(2^n x) \tag{3.30}$$

for all $x \in X$ and all non-negative integers n. Since Y is a p-Banach space, we have

$$\begin{aligned} \left\|\frac{1}{8^{n+1}}h(2^{n+1}x) - \frac{1}{8^m}h(2^m x) - \sum_{i=m}^{n}\frac{1}{8^{i+1}}f(0)\right\|_Y^p &\leq \sum_{i=m}^{n}\left\|\frac{1}{8^{i+1}}h(2^{i+1}x) - \frac{1}{8^i}h(2^i x) - \frac{1}{8^{i+1}}f(0)\right\|_Y^p \\ &\leq \frac{1}{8^p}\sum_{i=m}^{n}\frac{1}{8^{ip}}\phi^p(2^i x) \end{aligned} \tag{3.31}$$

for all $x \in X$ and all non-negative integers n and m with $n \geq m$. Therefore we conclude from (3.29) and (3.31) that the sequence $\left\{\frac{1}{8^n}h(2^n x)\right\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\left\{\frac{1}{8^n}h(2^n x)\right\}$ converges for all $x \in X$. So we can define the function $C : X \to Y$ by:

$$C(x) = \lim_{n\to\infty}\frac{1}{8^n}h(2^n x) \tag{3.32}$$

for all $x \in X$. Letting $m = 0$ and passing the limit when $n \to \infty$ in (3.31), we get (3.26). Now, we show that the function C is cubic. It follows from (3.29), (3.30) and (3.32) that

$$\begin{aligned} &\|C(2x) - 8C(x)\|_Y \\ &\quad = \lim_{n\to\infty}\left\|\frac{1}{8^n}h(2^{n+1}x) - \frac{1}{8^{n-1}}h(2^n x)\right\|_Y \\ &\quad \leq 8K\lim_{n\to\infty}\left(\left\|\frac{1}{8^{n+1}}h(2^{n+1}x) - \frac{1}{8^n}h(2^n x) - \frac{1}{8^{n+1}}f(0)\right\|_Y + \frac{1}{8^{n+1}}\|f(0)\|_Y\right) \\ &\quad \leq \lim_{n\to\infty}\frac{K}{8^n}\phi(2^n x) = 0 \end{aligned}$$

for all $x \in X$. Therefore we have

$$C(2x) = 8C(x) \tag{3.33}$$

for all $x \in X$. On the other hand, it follows from (3.23), (3.25) and (3.32) that

$$\begin{aligned}\|DC(x,y)\|_Y &= \lim_{n\to\infty} \frac{1}{8^n}\big\|Dh(2^n x, 2^n y)\big\|_Y \\ &= \lim_{n\to\infty} \frac{1}{8^n}\left\{\big\|Df(2^{n+1}x, 2^{n+1}y) - 2Df(2^n x, 2^n y)\big\|_Y\right\} \\ &\le \lim_{n\to\infty} \frac{K}{8^n}\left\{\big\|Df(2^{n+1}x, 2^{n+1}y)\big\|_Y + 2\big\|Df(2^n x, 2^n y)\big\|_Y\right\} \\ &\le \lim_{n\to\infty} \frac{K}{8^n}\left[\psi(2^{n+1}x, 2^{n+1}y) + 2\psi(2^n x, 2^n y)\right] = 0\end{aligned}$$

for all $x, y \in X$. Hence the function C satisfies (1.4). So by Lemma 2.2, the function $x \mapsto C(2x) - 2C(x)$ is cubic. Hence (3.33) implies that the function C is cubic. To prove the uniqueness of C, let $T : X \to Y$ be another cubic function satisfying (3.26). It follows from (3.24) that

$$\lim_{n\to\infty} \frac{1}{8^{np}} M(2^n x, 2^n y) = \lim_{n\to\infty} \sum_{i=n}^{\infty} \frac{1}{8^{ip}} \psi^p(2^i x, 2^i y) = 0$$

for all $x \in X$ and $y \in \{0, x, -x/2\}$. Hence $\lim_{n\to\infty} \frac{1}{8^{np}} \widetilde{\psi}(2^n x) = 0$ for all $x \in X$. So it follows from (3.26) and (3.32) that

$$\begin{aligned}\|C(x) - T(x)\|_Y^p &= \lim_{n\to\infty} \frac{1}{8^{np}}\Big\|h(2^n x) - T(2^n x) + \frac{1}{7}f(0)\Big\|_Y^p \\ &\le \frac{K^p}{8^p} \lim_{n\to\infty} \frac{1}{8^{np}} \widetilde{\psi}(2^n x) = 0\end{aligned}$$

for all $x \in X$. So $C = T$. □

Corollary 3.8. *Let θ be non-negative real number. Suppose that a function $f : X \to Y$ satisfies the inequalities (3.15). Then there exists a unique cubic function $C : X \to Y$ satisfying*

$$\|f(2x) - 2f(x) - C(x)\|_Y \le \frac{K^2\theta}{2}\left\{\frac{(2K^2)^p + (2K)^p + K^{2p} + 10^p}{8^p - 1}\right\}^{\frac{1}{p}} + \frac{K\theta}{28}$$

for all $x \in X$.

Proof. We get from (3.15) that $\|f(0)\| \le \theta/4$. So the result follows from Theorem 3.7. □

Theorem 3.9. *Let $\psi : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} 8^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$$

for all $x, y \in X$, and

$$M(x,y) := \sum_{i=1}^{\infty} 8^{ip} \psi^p\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty \tag{3.34}$$

for all $x \in X$ and all $y \in \{0, x, -x/2\}$. Suppose that a function $f : X \to Y$ satisfies the inequalities

$$\|Df(x,y)\|_Y \le \psi(x,y), \qquad \|f(x) + f(-x)\|_Y \le \psi(x,0)$$

for all $x, y \in X$. Then the limit

$$C(x) = \lim_{n\to\infty} 8^n \left[f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right]$$

exists for all $x \in X$ and the function $C : X \to Y$ is a unique cubic function satisfying

$$\|f(2x) - 2f(x) - C(x)\|_Y \le \frac{K}{8}[\widetilde{\psi}(x)]^{\frac{1}{p}} \tag{3.35}$$

for all $x \in X$, *where*

$$\widetilde{\psi}(x) = K^{2p}M(2x,-x) + \frac{K^{2p}}{2^p}M(x,x) + K^pM(2x,0) + 5^pM(x,0).$$

Proof. It follows from (3.34) that $\psi(0,0) = 0$ and so $f(0) = 0$. We introduce the same definitions for $h : X \to Y$ and $\phi(x)$ as in the proof of Theorem 3.7. Similar to the proof of Theorem 3.7, we have

$$\|h(2x) - 8h(x)\|_Y \le \phi(x) \tag{3.36}$$

for all $x \in X$. By Lemma 3.1 and (3.34), we infer that

$$\sum_{i=1}^{\infty} 8^{ip}\phi^p\left(\frac{x}{2^i}\right) < \infty \tag{3.37}$$

for all $x \in X$. Replacing x by $\frac{x}{2^{n+1}}$ in (3.36) and multiplying both sides of (3.36) to 8^n, we get

$$\left\|8^{n+1}h\left(\frac{x}{2^{n+1}}\right) - 8^nh\left(\frac{x}{2^n}\right)\right\|_Y \le 8^n\phi\left(\frac{x}{2^{n+1}}\right)$$

for all $x \in X$ and all non-negative integers n. Since Y is a p-Banach space, we have

$$\begin{aligned}\left\|8^{n+1}h\left(\frac{x}{2^{n+1}}\right) - 8^mh\left(\frac{x}{2^m}\right)\right\|_Y^p &\le \sum_{i=m}^{n}\left\|8^{i+1}h\left(\frac{x}{2^{i+1}}\right) - 8^ih\left(\frac{x}{2^i}\right)\right\|_Y^p \\ &\le \sum_{i=m}^{n} 8^{ip}\phi^p\left(\frac{x}{2^{i+1}}\right)\end{aligned} \tag{3.38}$$

for all $x \in X$ and all non-negative integers n and m with $n \ge m$. Therefor we conclude from (3.37) and (3.38) that the sequence $\{8^nh(x/2^n)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{8^nh(x/2^n)\}$ converges in Y for all $x \in X$. So we can define the function $C : X \to Y$ by

$$C(x) := \lim_{n\to\infty} 8^nh\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Letting $m = 0$ and passing the limit when $n \to \infty$ in (3.38) and applying Lemma 3.1, we get (3.35).

The rest of the proof is similar to the proof of Theorem 3.7 and we omit the details. □

Corollary 3.10. *Let* θ, r, s *be non-negative real numbers such that* $r, s > 3$ *or* $0 < r, s < 3$. *Suppose that a function* $f : X \to Y$ *satisfies the inequalities (3.21). Then there exists a unique cubic function* $C : X \to Y$ *satisfying*

$$\begin{aligned}&\|f(2x) - 2f(x) - C(x)\|_Y \\ &\qquad \le \frac{K\theta}{2}\left\{\frac{(2^{r+1}K^2)^p + K^{2p} + (2^{r+1}K^2)^p + 10^p}{|8^p - 2^{rp}|}\|x\|_X^{rp} + \frac{(2K^2)^p + K^{2p}}{|8^p - 2^{sp}|}\|x\|_X^{sp}\right\}^{\frac{1}{p}}\end{aligned}$$

for all $x \in X$.

Proof. It follows from (3.21) that $f(0) = 0$. Hence the result follows from Theorems 3.7 and 3.9. □

Corollary 3.11. *Let* θ *and* $r, s > 0$ *be non-negative real numbers such that* $\lambda := r + s \neq 3$. *Suppose that an odd function* $f : X \to Y$ *satisfies the inequality (3.22). Then there exists a unique cubic function* $C : X \to Y$ *satisfying*

$$\|f(2x) - 2f(x) - C(x)\|_Y \le \frac{K^3\theta}{2}\left\{\frac{1 + 2^{(r+1)p}}{|8^p - 2^{\lambda p}|}\right\}^{\frac{1}{p}}\|x\|_X^{\lambda}$$

for all $x \in X$.

Proof. $f(0) = 0$, since f is odd. Hence the result follows from Theorems 3.7 and 3.9. □

Theorem 3.12. *Let* $\varphi : X \times X \to [0, \infty)$ *be a function such that*

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y) = 0$$

for all $x, y \in X$, *and*

$$M_a(x, y) := \sum_{i=0}^{\infty} \frac{1}{2^{ip}}\varphi^p(2^i x, 2^i y) < \infty$$

for all $x \in X$ *and all* $y \in \{0, x, -x/2\}$. *Suppose that a function* $f : X \to Y$ *satisfies the inequalities*

$$\|Df(x, y)\|_Y \le \varphi(x, y) \qquad \|f(x) + f(-x)\|_Y \le \varphi(x, 0)$$

for all $x, y \in X$. *Then there exist a unique additive function* $A : X \to Y$ *and a unique cubic function* $C : X \to Y$ *such that*

$$\left\|f(x) - A(x) - C(x) - \frac{1}{7}f(0)\right\|_Y \le \frac{K^2}{48}\left\{4[\widetilde{\varphi_a}(x)]^{\frac{1}{p}} + [\widetilde{\varphi}_c(x)]^{\frac{1}{p}}\right\} \tag{3.39}$$

for all $x \in X$, *where*

$$M_c(x, y) := \sum_{i=0}^{\infty} \frac{1}{8^{ip}}\varphi^p(2^i x, 2^i y),$$
$$\widetilde{\varphi}_c(x) := K^{2p}M_c(2x, -x) + \frac{K^{2p}}{2^p}M_c(x, x) + K^p M_c(2x, 0) + 5^p M_c(x, 0),$$
$$\widetilde{\varphi_a}(x) := K^{2p}M_a(2x, -x) + \frac{K^{2p}}{2^p}M_a(x, x) + K^p M_a(2x, 0) + 5^p M_a(x, 0).$$

Proof. By Theorems 3.2 and 3.7, there exists an additive function $A_0 : X \to Y$ and a cubic function $C_0 : X \to Y$ such that

$$\|A_0(x) - f(2x) + 8f(x) - f(0)\|_Y \le \frac{K}{2}[\widetilde{\varphi_a}(x)]^{\frac{1}{p}},$$
$$\|C_0(x) - f(2x) + 2f(x) - \frac{1}{7}f(0)\|_Y \le \frac{K}{8}[\widetilde{\varphi}_c(x)]^{\frac{1}{p}}$$

for all $x \in X$. Therefore it follows from the last inequalities that

$$\left\|f(x) + \frac{1}{6}A_0(x) - \frac{1}{6}C_0(x) - \frac{1}{7}f(0)\right\|_Y \le \frac{K^2}{48}\left\{4[\widetilde{\varphi_a}(x)]^{\frac{1}{p}} + [\widetilde{\varphi}_c(x)]^{\frac{1}{p}}\right\}$$

for all $x \in X$. So we obtain (3.39) by letting $A(x) = -\frac{1}{6}A_0(x)$ and $C(x) = \frac{1}{6}C_0(x)$ for all $x \in X$.

To prove the uniqueness of A and C, let $A_1, C_1 : X \to Y$ be further additive and cubic functions satisfying (3.39). Let $A' = A - A_1$ and $C' = C - C_1$. Then

$$\begin{aligned} \|A'(x) + C'(x)\|_Y &\le K\left[\left\|f(x) - A(x) - C(x) - \frac{1}{7}f(0)\right\|_Y + \left\|f(x) - A_1(x) - C_1(x) - \frac{1}{7}f(0)\right\|_Y\right] \\ &\le \frac{K^3}{24}\left\{4[\widetilde{\varphi_a}(x)]^{\frac{1}{p}} + [\widetilde{\varphi}_c(x)]^{\frac{1}{p}}\right\} \end{aligned} \tag{3.40}$$

for all $x \in X$. Since

$$\lim_{n\to\infty} \frac{1}{8^{np}} \widetilde{\varphi_c}(2^n x) = \lim_{n\to\infty} \frac{1}{2^{np}} \widetilde{\varphi_a}(2^n x) = 0$$

for all $x \in X$, then (3.40) implies that

$$\lim_{n\to\infty} \frac{1}{8^n} \|A'(2^n x) + C'(2^n x)\|_Y = 0$$

for all $x \in X$. Since A' is additive and C' is cubic, we get $C' = 0$. So it follows from (3.40) that

$$\|A'(x)\|_Y \le \frac{5K^3}{24} [\widetilde{\varphi_a}(x)]^{\frac{1}{p}}$$

for all $x \in X$. Therefore $A' = 0$. □

Corollary 3.13. *Let θ be a non-negative real number. Suppose that a function $f : X \to Y$ satisfies the inequalities (3.15). Then there exist a unique additive function $A : X \to Y$ and a unique cubic function $C : X \to Y$ satisfying*

$$\|f(x) - A(x) - C(x)\|_Y \le \frac{K}{6}(\delta_a + \delta_c)$$

for all $x \in X$, where

$$\delta_a = \frac{K^2\theta}{2} \left\{ \frac{(2K^2)^p + (2K)^p + k^{2p} + 10^p}{2^p - 1} \right\}^{\frac{1}{p}} + \frac{K\theta}{4},$$

$$\delta_c = \frac{K^2\theta}{2} \left\{ \frac{(2K^2)^p + (2K)^p + K^{2p} + 10^p}{8^p - 1} \right\}^{\frac{1}{p}} + \frac{K\theta}{28}.$$

Theorem 3.14. *Let $\psi : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} 8^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$$

for all $x, y \in X$, and

$$M_c(x, y) := \sum_{i=1}^{\infty} 8^{ip} \psi^p \left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty$$

for all $x \in X$ and all $y \in \{0, x, -x/2\}$. Suppose that a function $f : X \to Y$ satisfies the inequalities

$$\|Df(x, y)\|_Y \le \psi(x, y) \qquad \|f(x) + f(-x)\|_Y \le \psi(x, 0)$$

for all $x, y \in X$. Then there exist a unique additive function $A : X \to Y$ and a unique cubic function $C : X \to Y$ such that

$$\|f(x) - A(x) - C(x)\|_Y \le \frac{K^2}{48} \left\{ 4[\widetilde{\psi_a}(x)]^{\frac{1}{p}} + [\widetilde{\psi_c}(x)]^{\frac{1}{p}} \right\} \tag{3.41}$$

for all $x \in X$, where

$$M_a(x, y) := \sum_{i=1}^{\infty} 2^{ip} \psi^p \left(\frac{x}{2^i}, \frac{y}{2^i}\right),$$

$$\widetilde{\psi_c}(x) := K^{2p} M_c(2x, -x) + \frac{K^{2p}}{2^p} M_c(x, x) + K^p M_c(2x, 0) + 5^p M_c(x, 0),$$

$$\widetilde{\psi_a}(x) := K^{2p} M_a(2x, -x) + \frac{K^{2p}}{2^p} M_a(x, x) + K^p M_a(2x, 0) + 5^p M_a(x, 0).$$

Proof. Applying Theorems 3.4 and 3.9, we get (3.41). □

Corollary 3.15. *Let θ, r, s be non-negative real numbers such that $r, s > 3$ or $0 < r, s < 1$. Suppose that a function $f : X \to Y$ satisfies the inequalities (3.21). Then there exist a unique additive function $A : X \to Y$ and a unique cubic function $C : X \to Y$ such that*

$$\|f(x) - A(x) - C(x)\|_Y \leq \frac{K^2\theta}{12}[\delta_a(x) + \delta_c(x)] \tag{3.42}$$

for all $x \in X$, where

$$\delta_a(x) = \left\{ \frac{(2^{r+1}K^2)^p + K^{2p} + (2^{r+1}K^2)^p + 10^p}{|2^p - 2^{rp}|}\|x\|_X^{rp} + \frac{(2K^2)^p + K^{2p}}{|2^p - 2^{sp}|}\|x\|_X^{sp} \right\}^{\frac{1}{p}},$$

$$\delta_c(x) = \left\{ \frac{(2^{r+1}K^2)^p + K^{2p} + (2^{r+1}K^2)^p + 10^p}{|8^p - 2^{rp}|}\|x\|_X^{rp} + \frac{(2K^2)^p + K^{2p}}{|8^p - 2^{sp}|}\|x\|_X^{sp} \right\}^{\frac{1}{p}}.$$

Corollary 3.16. *Let $\theta \geq 0$ and $r, s > 0$ be real numbers such that $\lambda := r+s \in (0,1)\cup(3,+\infty)$. Suppose that an odd function $f : X \to Y$ satisfies the inequality (3.22). Then there exist a unique additive function $A : X \to Y$ and a unique cubic function $C : X \to Y$ such that*

$$\|f(x) - A(x) - C(x)\|_Y \leq \frac{K^4\theta}{12}\left[\left\{ \frac{1 + 2^{p(r+1)}}{|2^p - 2^{\lambda p}|} \right\}^{\frac{1}{p}} + \left\{ \frac{1 + 2^{p(r+1)}}{|8^p - 2^{\lambda p}|} \right\}^{\frac{1}{p}} \right] \|x\|_X^{\lambda} \tag{3.43}$$

for all $x \in X$.

Theorem 3.17. *Let $\varphi : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{8^n}\varphi(2^n x, 2^n y) = 0, \quad \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$$

for all $x, y \in X$, and

$$M_a(x, y) := \sum_{i=1}^{\infty} 2^{ip}\varphi^p\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \quad M_c(x, y) := \sum_{i=0}^{\infty} \frac{1}{8^{ip}}\varphi^p(2^i x, 2^i y) < \infty$$

for all $x \in X$ and all $y \in \{0, x, -x/2\}$. Suppose that a function $f : X \to Y$ satisfies the inequalities

$$\|Df(x, y)\|_Y \leq \varphi(x, y) \qquad \|f(x) + f(-x)\|_Y \leq \varphi(x, 0)$$

for all $x, y \in X$. Then there exist a unique additive function $A : X \to Y$ and a unique cubic function $C : X \to Y$ such that

$$\|f(x) - A(x) - C(x)\|_Y \leq \frac{K^2}{48}\left\{ 4[\widetilde{\varphi_a}(x)]^{\frac{1}{p}} + [\widetilde{\varphi}_c(x)]^{\frac{1}{p}} \right\}$$

for all $x \in X$, where

$$\widetilde{\varphi}_c(x) := K^{2p}M_c(2x, -x) + \frac{K^{2p}}{2^p}M_c(x, x) + K^p M_c(2x, 0) + 5^p M_c(x, 0),$$

$$\widetilde{\varphi_a}(x) := K^{2p}M_a(2x, -x) + \frac{K^{2p}}{2^p}M_a(x, x) + K^p M_a(2x, 0) + 5^p M_a(x, 0).$$

Proof. By the assumption, we get $f(0) = 0$. So the result follows from Theorem 3.4 and Theorem 3.7. ☐

Corollary 3.18. *Let θ, r, s be non-negative real numbers such that $1 < r, s < 3$. Suppose that a function $f : X \to Y$ satisfies the inequalities (3.21) for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ satisfying (3.42).*

Proof. It follows from (3.21) that $f(0) = 0$. Hence the result follows from Corollaries 3.5 and 3.10. ☐

Corollary 3.19. *Let θ, r, s be non-negative real numbers such that $1 < \lambda := r+s < 3$. Suppose that an odd function $f : X \to Y$ satisfies the inequality (3.22) for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ satisfying (3.43).*

Proof. $f(0) = 0$, since f is odd. Hence the result follows from Corollaries 3.6 and 3.11. □

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