FEJÉR INEQUALITIES FOR WRIGHT-CONVEX FUNCTIONS

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Abstract: In this paper, we establish several inequalities of Fejér type for Wright-convex functions.

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1. Introduction

If $f : [a, b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as the Hermite-Hadamard inequality ([5]).

In [4], Fejér established the following weighted generalization of the inequality (1.1):

Theorem A. *If $f : [a, b] \to \mathbb{R}$ is a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \leq \int_a^b f(x)\,p(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx \tag{1.2}$$

holds, where $p : [a, b] \to \mathbb{R}$ is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$.

In recent years there have been many extensions, generalizations, applications and similar results of the inequalities (1.1) and (1.2) see [1] – [8], [10] – [16].

In [2], Dragomir established the following theorem which is a refinement of the first inequality of (1.1).

Theorem B. *If $f : [a, b] \to \mathbb{R}$ is a convex function, and H is defined on $[0, 1]$ by*

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx,$$

then H is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have

$$f\left(\frac{a+b}{2}\right) = H(0) \leq H(t) \leq H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.3}$$

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In [11], Yang and Hong established the following theorem which is a refinement of the second inequality of (1.1):

Theorem C. *If $f : [a,b] \to \mathbb{R}$ is a convex function, and F is defined on $[0,1]$ by*

$$F(t) = \frac{1}{2(b-a)} \int_a^b \left[f\left(\left(\frac{1+t}{2} \right) a + \left(\frac{1-t}{2} \right) x \right) + f\left(\left(\frac{1+t}{2} \right) b + \left(\frac{1-t}{2} \right) x \right) \right] dx,$$

then F is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have

$$\frac{1}{b-a} \int_a^b f(x)\, dx = F(0) \le F(t) \le F(1) = \frac{f(a)+f(b)}{2}. \tag{1.4}$$

We recall the definition of a Wright-convex function:

Definition 1.1 *([9, p. 223]).* We say that $f : [a,b] \to \mathbb{R}$ is a Wright-convex function, if, for all $x, y+\delta \in [a,b]$ with $x < y$ and $\delta \ge 0$, we have

$$f(x+\delta) + f(y) \le f(y+\delta) + f(x). \tag{1.5}$$

Let $C([a,b])$ be the set of all convex functions on $[a,b]$ and $W([a,b])$ be the set of all Wright-convex functions on $[a,b]$. Then $C([a,b]) \subsetneqq W([a,b])$. That is, a convex function must be a Wright-convex function but the converse is not true. (see [9, p. 224]).

In [10], Tseng, Yang and Dragomir established the following theorems for Wright-convex functions related to the inequality (1.1), Theorem A and Theorem B:

Theorem D. *Let $f \in W([a,b]) \cap L_1[a,b]$. Then the inequality (1.1) holds.*

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Theorem E. *Let $f \in W([a,b]) \cap L_1[a,b]$ and let H be defined as in Theorem B. Then $H \in W([0,1])$ is increasing on $[0,1]$, and the inequality (1.3) holds for all $t \in [0,1]$.*

Theorem F. *Let $f \in W([a,b]) \cap L_1[a,b]$ and let F be defined as in Theorem C. Then $F \in W([0,1])$ is increasing on $[0,1]$, and the inequality (1.4) holds for all $t \in [0,1]$.*

In [12], Yang and Tseng established the following theorem which refines the inequality (1.2):

Theorem G ([12, Remark 6]). *Let f and p be defined as in Theorem A. If P, Q are defined on $[0,1]$ by*

$$P(t) = \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) p(x)\,dx \qquad (t \in (0,1)) \tag{1.6}$$

and

$$\begin{aligned} Q(t) = \int_a^b \frac{1}{2}\Bigg[& f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) p\left(\frac{x+a}{2}\right) \\ & + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right) p\left(\frac{x+b}{2}\right)\Bigg] dx \qquad (t \in (0,1)), \end{aligned} \tag{1.7}$$

then P, Q are convex and increasing on $[0,1]$ and, for all $t \in [0,1]$,

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx = P(0) \le P(t) \le P(1) = \int_a^b f(x)\,p(x)\,dx \tag{1.8}$$

and

$$\int_a^b f(x)\,p(x)\,dx = Q(0) \le Q(t) \le Q(1) = \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx. \tag{1.9}$$

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In this paper, we establish some results about Theorem A and Theorem G for Wright-convex functions which are weighted generalizations of Theorem D, E and F.

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2. Main Results

In order to prove our main theorems, we need the following lemma [10]:

Lemma 2.1. *If* $f : [a,b] \to \mathbb{R}$, *then the following statements are equivalent:*

1. $f \in W([a,b])$;
2. *for all* $s,t,u,v \in [a,b]$ *with* $s \le t \le u \le v$ *and* $t+u = s+v$, *we have*

$$f(t) + f(u) \le f(s) + f(v). \tag{2.1}$$

Theorem 2.2. *Let* $f \in W([a,b]) \cap L_1[a,b]$ *and let* $p : [a,b] \to \mathbb{R}$ *be nonnegative, integrable, and symmetric about* $x = \frac{a+b}{2}$. *Then the inequality* (1.2) *holds.*

Proof. For the inequality (2.1) and the assumptions that p is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$, we have

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right) \int_a^b p(x)\,dx \\
&= \int_a^{\frac{a+b}{2}} f\left(\frac{a+b}{2}\right) p(x)\,dx + \int_a^{\frac{a+b}{2}} f\left(\frac{a+b}{2}\right) p(a+b-x)\,dx \\
&= \int_a^{\frac{a+b}{2}} \left[f\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}\right)\right] p(x)\,dx \\
&\le \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)]\, p(x)\,dx \qquad \left(x \le \frac{a+b}{2} \le \frac{a+b}{2} \le a+b-x\right)
\end{aligned}
$$

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$$= \int_a^{\frac{a+b}{2}} f(x)\, p(x)\, dx + \int_{\frac{a+b}{2}}^b f(x)\, p(x)\, dx$$
$$= \int_a^b f(x)\, p(x)\, dx,$$

and

$$\frac{f(a)+f(b)}{2} \int_a^b p(x)\, dx$$
$$= \int_a^{\frac{a+b}{2}} \left[\frac{f(a)+f(b)}{2}\right] p(x)\, dx + \int_a^{\frac{a+b}{2}} \left[\frac{f(a)+f(b)}{2}\right] p(a+b-x)\, dx$$
$$= \int_a^{\frac{a+b}{2}} [f(a)+f(b)]\, p(x)\, dx$$
$$\geq \int_a^{\frac{a+b}{2}} [f(x)+f(a+b-x)]\, p(x)\, dx \qquad (a \leq x \leq a+b-x \leq b)$$
$$= \int_a^{\frac{a+b}{2}} f(x)\, p(x)\, dx + \int_{\frac{a+b}{2}}^b f(x)\, p(x)\, dx = \int_a^b f(x)\, p(x)\, dx.$$

This completes the proof. ∎

Remark 1. If we set $p(x) \equiv 1$ $(x \in [a,b])$ in Theorem 2.2, then Theorem 2.2 generalizes Theorem D.

Remark 2. From $C([a,b]) \subsetneqq W([a,b])$, Theorem 2.2 generalizes Theorem A.

Theorem 2.3. *Let f and p be defined as in Theorem 2.2 and let P be defined as in (1.6). Then $P \in W([0,1])$ is increasing on $[0,1]$, and the inequality (1.8) holds for all $t \in [0,1]$.*

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Proof. If $s,t,u,v \in [0,1]$ and $s \le t \le u \le v$, $t+u = s+v$, then for $x \in \left[a, \frac{a+b}{2}\right]$ we have

$$\begin{aligned} b &\ge sx + (1-s)\frac{a+b}{2} \ge tx + (1-t)\frac{a+b}{2} \\ &\ge ux + (1-u)\frac{a+b}{2} \ge vx + (1-v)\frac{a+b}{2} \ge a \end{aligned}$$

and if $x \in \left[\frac{a+b}{2}, b\right]$, then

$$\begin{aligned} a &\le sx + (1-s)\frac{a+b}{2} \le tx + (1-t)\frac{a+b}{2} \\ &\le ux + (1-u)\frac{a+b}{2} \le vx + (1-v)\frac{a+b}{2} \le b, \end{aligned}$$

where

$$\begin{aligned} &\left[tx + (1-t)\frac{a+b}{2}\right] + \left[ux + (1-u)\frac{a+b}{2}\right] \\ &\qquad = \left[sx + (1-s)\frac{a+b}{2}\right] + \left[vx + (1-v)\frac{a+b}{2}\right]. \end{aligned}$$

By the inequality (2.1), we have

$$\begin{aligned} f\left(tx + (1-t)\frac{a+b}{2}\right) &+ f\left(ux + (1-u)\frac{a+b}{2}\right) \\ &\le f\left(sx + (1-s)\frac{a+b}{2}\right) + f\left(vx + (1-v)\frac{a+b}{2}\right) \end{aligned} \tag{2.2}$$

for all $x \in [a,b]$. Now, using the inequality (2.2) and p is nonnegative on $[a,b]$, we

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have

$$\left[f\left(tx+(1-t)\frac{a+b}{2}\right)+f\left(ux+(1-u)\frac{a+b}{2}\right)\right]p(x) \le \left[f\left(sx+(1-s)\frac{a+b}{2}\right)+f\left(vx+(1-v)\frac{a+b}{2}\right)\right]p(x) \tag{2.3}$$

for all $x \in [a,b]$. Integrating the inequality (2.3) over x on $[a,b]$, we have

$$P(t)+P(u) \le P(s)+P(v).$$

Hence $P \in W([0,1])$.

Next, if $0 \le s \le t \le 1$ and $x \in \left[a, \frac{a+b}{2}\right]$, then

$$\begin{aligned} tx+(1-t)\frac{a+b}{2} &\le sx+(1-s)\frac{a+b}{2} \\ &\le s(a+b-x)+(1-s)\frac{a+b}{2} \\ &\le t(a+b-x)+(1-t)\frac{a+b}{2}, \end{aligned}$$

where

$$\begin{aligned} &\left[sx+(1-s)\frac{a+b}{2}\right]+\left[s(a+b-x)+(1-s)\frac{a+b}{2}\right] \\ &\qquad = \left[tx+(1-t)\frac{a+b}{2}\right]+\left[t(a+b-x)+(1-t)\frac{a+b}{2}\right]. \end{aligned}$$

By the inequality (2.1) and the assumptions that p is nonnegative, integrable, and

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symmetric about $x = \frac{a+b}{2}$, we have

$$\begin{aligned}
P(s) &= \int_a^b f\left(sx + (1-s)\frac{a+b}{2}\right) p(x)\, dx \\
&= \int_a^{\frac{a+b}{2}} f\left(sx + (1-s)\frac{a+b}{2}\right) p(x)\, dx \\
&\qquad + \int_a^{\frac{a+b}{2}} f\left(s(a+b-x) + (1-s)\frac{a+b}{2}\right) p(a+b-x)\, dx \\
&= \int_a^{\frac{a+b}{2}} \left[f\left(sx + (1-s)\frac{a+b}{2}\right) + f\left(s(a+b-x) + (1-s)\frac{a+b}{2}\right)\right] p(x)\, dx \\
&\leq \int_a^{\frac{a+b}{2}} \left[f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right)\right] p(x)\, dx \\
&= \int_a^{\frac{a+b}{2}} f\left(tx + (1-t)\frac{a+b}{2}\right) p(x)\, dx \\
&\qquad + \int_a^{\frac{a+b}{2}} f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) p(a+b-x)\, dx \\
&= \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) p(x)\, dx = P(t).
\end{aligned}$$

Thus, P is increasing on $[0,1]$, and the inequality (1.8) holds for all $t \in [0,1]$.

This completes the proof. ∎

Remark 3. If we set $p(x) \equiv 1$ $(x \in [a,b])$ in Theorem 2.3, then Theorem 2.2 generalizes Theorem E.

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Theorem 2.4. *Let f and p be defined as in Theorem 2.2 and let Q be defined as in (1.7). Then $Q \in W([0,1])$ is increasing on $[0,1]$, and the inequality (1.9) holds for all $t \in [0,1]$.*

Proof. If $s,t,u,v \in [0,1]$ and $s \le t \le u \le v$, $t+u = s+v$, then for all $x \in [a,b]$ we have

$$\begin{aligned} a &\le \left(\frac{1+v}{2}\right) a + \left(\frac{1-v}{2}\right) x \le \left(\frac{1+u}{2}\right) a + \left(\frac{1-u}{2}\right) x \\ &\le \left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) x \le \left(\frac{1+s}{2}\right) a + \left(\frac{1-s}{2}\right) x \le b \end{aligned}$$

and

$$\begin{aligned} a &\le \left(\frac{1+s}{2}\right) b + \left(\frac{1-s}{2}\right) x \le \left(\frac{1+t}{2}\right) b + \left(\frac{1-t}{2}\right) x \\ &\le \left(\frac{1+u}{2}\right) b + \left(\frac{1-u}{2}\right) x \le \left(\frac{1+v}{2}\right) b + \left(\frac{1-v}{2}\right) x \le b, \end{aligned}$$

where

$$\begin{aligned} &\left[\left(\frac{1+u}{2}\right) a + \left(\frac{1-u}{2}\right) x\right] + \left[\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) x\right] \\ &\qquad = \left[\left(\frac{1+v}{2}\right) a + \left(\frac{1-v}{2}\right) x\right] + \left[\left(\frac{1+s}{2}\right) a + \left(\frac{1-s}{2}\right) x\right] \end{aligned}$$

and

$$\begin{aligned} &\left[\left(\frac{1+t}{2}\right) b + \left(\frac{1-t}{2}\right) x\right] + \left[\left(\frac{1+u}{2}\right) b + \left(\frac{1-u}{2}\right) x\right] \\ &\qquad = \left[\left(\frac{1+s}{2}\right) b + \left(\frac{1-s}{2}\right) x\right] + \left[\left(\frac{1+v}{2}\right) b + \left(\frac{1-v}{2}\right) x\right]. \end{aligned}$$

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By the inequality (2.1), we have

$$f\left(\left(\frac{1+u}{2}\right)a+\left(\frac{1-u}{2}\right)x\right)+f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right) \leq f\left(\left(\frac{1+v}{2}\right)a+\left(\frac{1-v}{2}\right)x\right)+f\left(\left(\frac{1+s}{2}\right)a+\left(\frac{1-s}{2}\right)x\right) \tag{2.4}$$

and

$$f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)+f\left(\left(\frac{1+u}{2}\right)b+\left(\frac{1-u}{2}\right)x\right) \leq f\left(\left(\frac{1+s}{2}\right)b+\left(\frac{1-s}{2}\right)x\right)+f\left(\left(\frac{1+v}{2}\right)b+\left(\frac{1-v}{2}\right)x\right) \tag{2.5}$$

for all $x \in [a,b]$. Now, using the inequality (2.4), (2.5) and the assumptions that p is nonnegative on $[a,b]$, we have

$$\begin{aligned} &\frac{1}{2}f\left(\left(\frac{1+u}{2}\right)a+\left(\frac{1-u}{2}\right)x\right)p\left(\frac{x+a}{2}\right) \\ &\quad+\frac{1}{2}f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)p\left(\frac{x+a}{2}\right) \\ &\quad+\frac{1}{2}f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)p\left(\frac{x+b}{2}\right) \\ &\quad+\frac{1}{2}f\left(\left(\frac{1+u}{2}\right)b+\left(\frac{1-u}{2}\right)x\right)p\left(\frac{x+b}{2}\right) \end{aligned} \tag{2.6}$$

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$$
\begin{aligned}
&\leq \frac{1}{2} f\left(\left(\frac{1+v}{2}\right) a+\left(\frac{1-v}{2}\right) x\right) p\left(\frac{x+a}{2}\right) \\
&\quad+\frac{1}{2} f\left(\left(\frac{1+s}{2}\right) a+\left(\frac{1-s}{2}\right) x\right) p\left(\frac{x+a}{2}\right) \\
&\quad+\frac{1}{2} f\left(\left(\frac{1+s}{2}\right) b+\left(\frac{1-s}{2}\right) x\right) p\left(\frac{x+b}{2}\right) \\
&\quad+\frac{1}{2} f\left(\left(\frac{1+v}{2}\right) b+\left(\frac{1-v}{2}\right) x\right) p\left(\frac{x+b}{2}\right)
\end{aligned}
$$

Integrating the inequality (2.6) over x on $[a, b]$, we have

$$Q(t)+Q(u) \leq Q(s)+Q(v).$$

Hence $Q \in W([0,1])$.

Next, if $0 \leq s \leq t \leq 1$ and $x \in [a, b]$, then

$$
\begin{aligned}
\left(\frac{1+t}{2}\right) a+\left(\frac{1-t}{2}\right) x &\leq \left(\frac{1+s}{2}\right) a+\left(\frac{1-s}{2}\right) x \\
&\leq \left(\frac{1+s}{2}\right) b+\left(\frac{1-s}{2}\right)(a+b-x) \\
&\leq \left(\frac{1+t}{2}\right) b+\left(\frac{1-t}{2}\right)(a+b-x)
\end{aligned}
$$

and

$$\left(\frac{1+t}{2}\right) a+\left(\frac{1-t}{2}\right)(a+b-x) \leq \left(\frac{1+s}{2}\right) a+\left(\frac{1-s}{2}\right)(a+b-x)$$

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$$\le \left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x \le \left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x,$$

where

$$\left[\left(\frac{1+s}{2}a\right) + \left(\frac{1-s}{2}\right)x\right] + \left[\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)(a+b-x)\right]$$
$$= \left[\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right] + \left[\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)(a+b-x)\right],$$

and

$$\left[\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)(a+b-x)\right] + \left[\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right]$$
$$= \left[\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)(a+b-x)\right] + \left[\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right].$$

By the inequality (2.1) and the assumptions that p is nonnegative and symmetric about $x = \frac{a+b}{2}$, we have

$$(2.7)\quad f\left(\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x\right)p\left(\frac{x+a}{2}\right)$$
$$+ f\left(\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)(a+b-x)\right)p\left(\frac{2a+b-x}{2}\right)$$
$$+ f\left(\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)(a+b-x)\right)p\left(\frac{a+2b-x}{2}\right)$$
$$+ f\left(\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right)p\left(\frac{x+b}{2}\right)$$

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$$
\begin{aligned}
&= \left[f\left(\left(\frac{1+s}{2}\right) a + \left(\frac{1-s}{2}\right) x \right) \right. \\
&\qquad \left. + f\left(\left(\frac{1+s}{2}\right) a + \left(\frac{1-s}{2}\right) (a+b-x) \right) \right] p\left(\frac{x+a}{2}\right) \\
&\qquad + \left[f\left(\left(\frac{1+s}{2}\right) b + \left(\frac{1-s}{2}\right) (a+b-x) \right) \right. \\
&\qquad \left. + f\left(\left(\frac{1+s}{2}\right) b + \left(\frac{1-s}{2}\right) x \right) \right] p\left(\frac{x+b}{2}\right) \\
&\le \left[f\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) x \right) \right. \\
&\qquad \left. + f\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) (a+b-x) \right) \right] p\left(\frac{x+a}{2}\right) \\
&\qquad + \left[f\left(\left(\frac{1+t}{2}\right) b + \left(\frac{1-t}{2}\right) (a+b-x) \right) \right. \\
&\qquad \left. + f\left(\left(\frac{1+t}{2}\right) b + \left(\frac{1-t}{2}\right) x \right) \right] p\left(\frac{x+b}{2}\right) \\
&= f\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) x \right) p\left(\frac{x+a}{2}\right) \\
&\qquad + f\left(\left(\frac{1+t}{2}\right) b + \left(\frac{1-t}{2}\right) (a+b-x) \right) p\left(\frac{2a+b-x}{2}\right) \\
&\qquad + f\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) (a+b-x) \right) p\left(\frac{a+2b-x}{2}\right) \\
&\qquad + f\left(\left(\frac{1+t}{2}\right) b + \left(\frac{1-t}{2}\right) x \right) p\left(\frac{x+b}{2}\right).
\end{aligned}
$$

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Integrating the inequality (2.7) over x on $[a,b]$, we have

$$4Q(s) \leq 4Q(t)$$

Hence Q is increasing on $[0,1]$, and the inequality (1.9) holds for all $t \in [0,1]$.

This completes the proof. ∎

Remark 4. If we set $p(x) \equiv 1$ $(x \in [a,b])$ in Theorem 2.4, then Theorem 2.2 generalizes Theorem F.

Remark 5. From $C([a,b]) \subsetneqq W([a,b])$, Theorem 2.3 and Theorem 2.4 generalize Theorem C.

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