

FEJÉR INEQUALITIES FOR WRIGHT-CONVEX FUNCTIONS

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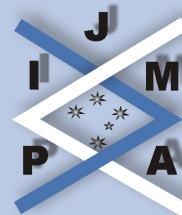
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Abstract: In this paper, we establish several inequalities of Fejér type for Wright-convex functions.



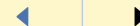
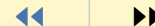
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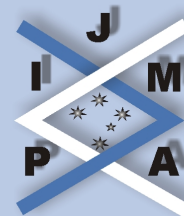
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1. Introduction

If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, then

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

is known as the Hermite-Hadamard inequality ([5]).

In [4], Fejér established the following weighted generalization of the inequality (1.1):

Theorem A. *If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, then the inequality*

$$(1.2) \quad f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \leq \int_a^b f(x) p(x) dx \leq \frac{f(a) + f(b)}{2} \int_a^b p(x) dx$$

holds, where $p : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$.

In recent years there have been many extensions, generalizations, applications and similar results of the inequalities (1.1) and (1.2) see [1] – [8], [10] – [16].

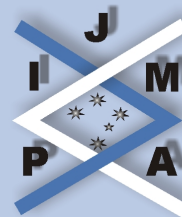
In [2], Dragomir established the following theorem which is a refinement of the first inequality of (1.1).

Theorem B. *If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, and H is defined on $[0, 1]$ by*

$$H(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx,$$

then H is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have

$$(1.3) \quad f\left(\frac{a+b}{2}\right) = H(0) \leq H(t) \leq H(1) = \frac{1}{b-a} \int_a^b f(x) dx.$$



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In [11], Yang and Hong established the following theorem which is a refinement of the second inequality of (1.1):

Theorem C. *If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, and F is defined on $[0, 1]$ by*

$$F(t) = \frac{1}{2(b-a)} \int_a^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right) \right] dx,$$

then F is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have

$$(1.4) \quad \frac{1}{b-a} \int_a^b f(x) dx = F(0) \leq F(t) \leq F(1) = \frac{f(a) + f(b)}{2}.$$

We recall the definition of a Wright-convex function:

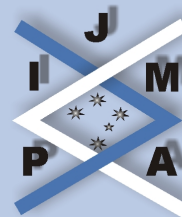
Definition 1.1 ([9, p. 223]). We say that $f : [a, b] \rightarrow \mathbb{R}$ is a Wright-convex function, if, for all $x, y + \delta \in [a, b]$ with $x < y$ and $\delta \geq 0$, we have

$$(1.5) \quad f(x + \delta) + f(y) \leq f(y + \delta) + f(x).$$

Let $C([a, b])$ be the set of all convex functions on $[a, b]$ and $W([a, b])$ be the set of all Wright-convex functions on $[a, b]$. Then $C([a, b]) \subsetneq W([a, b])$. That is, a convex function must be a Wright-convex function but the converse is not true. (see [9, p. 224]).

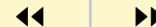
In [10], Tseng, Yang and Dragomir established the following theorems for Wright-convex functions related to the inequality (1.1), Theorem A and Theorem B:

Theorem D. *Let $f \in W([a, b]) \cap L_1[a, b]$. Then the inequality (1.1) holds.*



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Theorem E. Let $f \in W([a, b]) \cap L_1[a, b]$ and let H be defined as in Theorem B. Then $H \in W([0, 1])$ is increasing on $[0, 1]$, and the inequality (1.3) holds for all $t \in [0, 1]$.

Theorem F. Let $f \in W([a, b]) \cap L_1[a, b]$ and let F be defined as in Theorem C. Then $F \in W([0, 1])$ is increasing on $[0, 1]$, and the inequality (1.4) holds for all $t \in [0, 1]$.

In [12], Yang and Tseng established the following theorem which refines the inequality (1.2):

Theorem G ([12, Remark 6]). Let f and p be defined as in Theorem A. If P, Q are defined on $[0, 1]$ by

$$(1.6) \quad P(t) = \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) p(x) dx \quad (t \in (0, 1))$$

and

$$(1.7) \quad Q(t) = \int_a^b \frac{1}{2} \left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) p\left(\frac{x+a}{2}\right) + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right) p\left(\frac{x+b}{2}\right) \right] dx \quad (t \in (0, 1)),$$

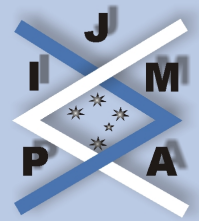
then P, Q are convex and increasing on $[0, 1]$ and, for all $t \in [0, 1]$,

$$(1.8) \quad f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx = P(0) \leq P(t) \leq P(1) = \int_a^b f(x) p(x) dx$$

and

$$(1.9) \quad \int_a^b f(x) p(x) dx = Q(0) \leq Q(t) \leq Q(1) = \frac{f(a) + f(b)}{2} \int_a^b p(x) dx.$$

In this paper, we establish some results about Theorem **A** and Theorem **G** for Wright-convex functions which are weighted generalizations of Theorem **D**, **E** and **F**.



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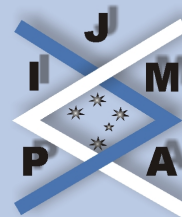
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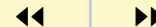
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2. Main Results

In order to prove our main theorems, we need the following lemma [10]:

Lemma 2.1. *If $f : [a, b] \rightarrow \mathbb{R}$, then the following statements are equivalent:*

1. $f \in W([a, b])$;
2. for all $s, t, u, v \in [a, b]$ with $s \leq t \leq u \leq v$ and $t + u = s + v$, we have

$$(2.1) \quad f(t) + f(u) \leq f(s) + f(v).$$

Theorem 2.2. *Let $f \in W([a, b]) \cap L_1[a, b]$ and let $p : [a, b] \rightarrow \mathbb{R}$ be nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$. Then the inequality (1.2) holds.*

Proof. For the inequality (2.1) and the assumptions that p is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$, we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \\ &= \int_a^{\frac{a+b}{2}} f\left(\frac{a+b}{2}\right) p(x) dx + \int_a^{\frac{a+b}{2}} f\left(\frac{a+b}{2}\right) p(a+b-x) dx \\ &= \int_a^{\frac{a+b}{2}} \left[f\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}\right) \right] p(x) dx \\ &\leq \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)] p(x) dx \quad \left(x \leq \frac{a+b}{2} \leq \frac{a+b}{2} \leq a+b-x \right) \end{aligned}$$

$$\begin{aligned}
&= \int_a^{\frac{a+b}{2}} f(x) p(x) dx + \int_{\frac{a+b}{2}}^b f(x) p(x) dx \\
&= \int_a^b f(x) p(x) dx,
\end{aligned}$$

and

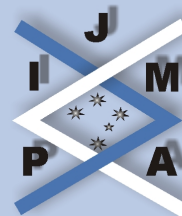
$$\begin{aligned}
&\frac{f(a) + f(b)}{2} \int_a^b p(x) dx \\
&= \int_a^{\frac{a+b}{2}} \left[\frac{f(a) + f(b)}{2} \right] p(x) dx + \int_{\frac{a+b}{2}}^b \left[\frac{f(a) + f(b)}{2} \right] p(a+b-x) dx \\
&= \int_a^{\frac{a+b}{2}} [f(a) + f(b)] p(x) dx \\
&\geq \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)] p(x) dx \quad (a \leq x \leq a+b-x \leq b) \\
&= \int_a^{\frac{a+b}{2}} f(x) p(x) dx + \int_{\frac{a+b}{2}}^b f(x) p(x) dx = \int_a^b f(x) p(x) dx.
\end{aligned}$$

This completes the proof. ■

Remark 1. If we set $p(x) \equiv 1$ ($x \in [a, b]$) in Theorem 2.2, then Theorem 2.2 generalizes Theorem D.

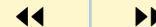
Remark 2. From $C([a, b]) \subsetneq W([a, b])$, Theorem 2.2 generalizes Theorem A.

Theorem 2.3. Let f and p be defined as in Theorem 2.2 and let P be defined as in (1.6). Then $P \in W([0, 1])$ is increasing on $[0, 1]$, and the inequality (1.8) holds for all $t \in [0, 1]$.



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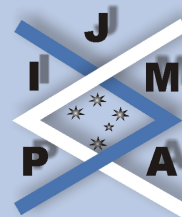


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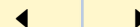
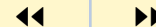
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Proof. If $s, t, u, v \in [0, 1]$ and $s \leq t \leq u \leq v, t + u = s + v$, then for $x \in [a, \frac{a+b}{2}]$ we have

$$\begin{aligned} b &\geq sx + (1-s) \frac{a+b}{2} \geq tx + (1-t) \frac{a+b}{2} \\ &\geq ux + (1-u) \frac{a+b}{2} \geq vx + (1-v) \frac{a+b}{2} \geq a \end{aligned}$$

and if $x \in [\frac{a+b}{2}, b]$, then

$$\begin{aligned} a &\leq sx + (1-s) \frac{a+b}{2} \leq tx + (1-t) \frac{a+b}{2} \\ &\leq ux + (1-u) \frac{a+b}{2} \leq vx + (1-v) \frac{a+b}{2} \leq b, \end{aligned}$$

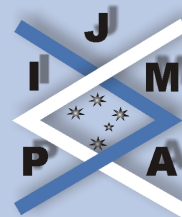
where

$$\begin{aligned} \left[tx + (1-t) \frac{a+b}{2} \right] + \left[ux + (1-u) \frac{a+b}{2} \right] \\ = \left[sx + (1-s) \frac{a+b}{2} \right] + \left[vx + (1-v) \frac{a+b}{2} \right]. \end{aligned}$$

By the inequality (2.1), we have

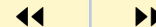
$$\begin{aligned} (2.2) \quad f \left(tx + (1-t) \frac{a+b}{2} \right) + f \left(ux + (1-u) \frac{a+b}{2} \right) \\ \leq f \left(sx + (1-s) \frac{a+b}{2} \right) + f \left(vx + (1-v) \frac{a+b}{2} \right) \end{aligned}$$

for all $x \in [a, b]$. Now, using the inequality (2.2) and p is nonnegative on $[a, b]$, we



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have

$$(2.3) \quad \left[f \left(tx + (1-t) \frac{a+b}{2} \right) + f \left(ux + (1-u) \frac{a+b}{2} \right) \right] p(x) \\ \leq \left[f \left(sx + (1-s) \frac{a+b}{2} \right) + f \left(vx + (1-v) \frac{a+b}{2} \right) \right] p(x)$$

for all $x \in [a, b]$. Integrating the inequality (2.3) over x on $[a, b]$, we have

$$P(t) + P(u) \leq P(s) + P(v).$$

Hence $P \in W([0, 1])$.

Next, if $0 \leq s \leq t \leq 1$ and $x \in [a, \frac{a+b}{2}]$, then

$$tx + (1-t) \frac{a+b}{2} \leq sx + (1-s) \frac{a+b}{2} \\ \leq s(a+b-x) + (1-s) \frac{a+b}{2} \\ \leq t(a+b-x) + (1-t) \frac{a+b}{2},$$

where

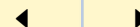
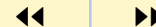
$$\left[sx + (1-s) \frac{a+b}{2} \right] + \left[s(a+b-x) + (1-s) \frac{a+b}{2} \right] \\ = \left[tx + (1-t) \frac{a+b}{2} \right] + \left[t(a+b-x) + (1-t) \frac{a+b}{2} \right].$$

By the inequality (2.1) and the assumptions that p is nonnegative, integrable, and



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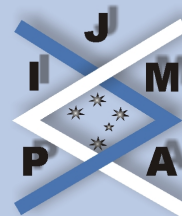
symmetric about $x = \frac{a+b}{2}$, we have

$$\begin{aligned} P(s) &= \int_a^b f\left(sx + (1-s)\frac{a+b}{2}\right) p(x) dx \\ &= \int_a^{\frac{a+b}{2}} f\left(sx + (1-s)\frac{a+b}{2}\right) p(x) dx \\ &\quad + \int_a^{\frac{a+b}{2}} f\left(s(a+b-x) + (1-s)\frac{a+b}{2}\right) p(a+b-x) dx \\ &= \int_a^{\frac{a+b}{2}} \left[f\left(sx + (1-s)\frac{a+b}{2}\right) + f\left(s(a+b-x) + (1-s)\frac{a+b}{2}\right) \right] p(x) dx \\ &\leq \int_a^{\frac{a+b}{2}} \left[f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) \right] p(x) dx \\ &= \int_a^{\frac{a+b}{2}} f\left(tx + (1-t)\frac{a+b}{2}\right) p(x) dx \\ &\quad + \int_a^{\frac{a+b}{2}} f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) p(a+b-x) dx \\ &= \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) p(x) dx = P(t). \end{aligned}$$

Thus, P is increasing on $[0, 1]$, and the inequality (1.8) holds for all $t \in [0, 1]$.

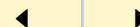
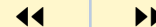
This completes the proof. ■

Remark 3. If we set $p(x) \equiv 1$ ($x \in [a, b]$) in Theorem 2.3, then Theorem 2.2 generalizes Theorem E.



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Theorem 2.4. Let f and p be defined as in Theorem 2.2 and let Q be defined as in (1.7). Then $Q \in W([0, 1])$ is increasing on $[0, 1]$, and the inequality (1.9) holds for all $t \in [0, 1]$.

Proof. If $s, t, u, v \in [0, 1]$ and $s \leq t \leq u \leq v, t + u = s + v$, then for all $x \in [a, b]$ we have

$$\begin{aligned} a &\leq \left(\frac{1+v}{2}\right)a + \left(\frac{1-v}{2}\right)x \leq \left(\frac{1+u}{2}\right)a + \left(\frac{1-u}{2}\right)x \\ &\leq \left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x \leq \left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x \leq b \end{aligned}$$

and

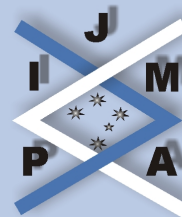
$$\begin{aligned} a &\leq \left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x \leq \left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x \\ &\leq \left(\frac{1+u}{2}\right)b + \left(\frac{1-u}{2}\right)x \leq \left(\frac{1+v}{2}\right)b + \left(\frac{1-v}{2}\right)x \leq b, \end{aligned}$$

where

$$\begin{aligned} &\left[\left(\frac{1+u}{2}\right)a + \left(\frac{1-u}{2}\right)x\right] + \left[\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right] \\ &= \left[\left(\frac{1+v}{2}\right)a + \left(\frac{1-v}{2}\right)x\right] + \left[\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x\right] \end{aligned}$$

and

$$\begin{aligned} &\left[\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right] + \left[\left(\frac{1+u}{2}\right)b + \left(\frac{1-u}{2}\right)x\right] \\ &= \left[\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right] + \left[\left(\frac{1+v}{2}\right)b + \left(\frac{1-v}{2}\right)x\right]. \end{aligned}$$



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By the inequality (2.1), we have

$$(2.4) \quad f\left(\left(\frac{1+u}{2}\right)a + \left(\frac{1-u}{2}\right)x\right) + f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) \\ \leq f\left(\left(\frac{1+v}{2}\right)a + \left(\frac{1-v}{2}\right)x\right) + f\left(\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x\right)$$

and

$$(2.5) \quad f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right) + f\left(\left(\frac{1+u}{2}\right)b + \left(\frac{1-u}{2}\right)x\right) \\ \leq f\left(\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right) + f\left(\left(\frac{1+v}{2}\right)b + \left(\frac{1-v}{2}\right)x\right)$$

for all $x \in [a, b]$. Now, using the inequality (2.4), (2.5) and the assumptions that p is nonnegative on $[a, b]$, we have

$$(2.6) \quad \frac{1}{2}f\left(\left(\frac{1+u}{2}\right)a + \left(\frac{1-u}{2}\right)x\right)p\left(\frac{x+a}{2}\right) \\ + \frac{1}{2}f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right)p\left(\frac{x+a}{2}\right) \\ + \frac{1}{2}f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right)p\left(\frac{x+b}{2}\right) \\ + \frac{1}{2}f\left(\left(\frac{1+u}{2}\right)b + \left(\frac{1-u}{2}\right)x\right)p\left(\frac{x+b}{2}\right)$$

$$\begin{aligned}
&\leq \frac{1}{2}f\left(\left(\frac{1+v}{2}\right)a + \left(\frac{1-v}{2}\right)x\right)p\left(\frac{x+a}{2}\right) \\
&\quad + \frac{1}{2}f\left(\left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x\right)p\left(\frac{x+a}{2}\right) \\
&\quad + \frac{1}{2}f\left(\left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x\right)p\left(\frac{x+b}{2}\right) \\
&\quad + \frac{1}{2}f\left(\left(\frac{1+v}{2}\right)b + \left(\frac{1-v}{2}\right)x\right)p\left(\frac{x+b}{2}\right)
\end{aligned}$$

Integrating the inequality (2.6) over x on $[a, b]$, we have

$$Q(t) + Q(u) \leq Q(s) + Q(v).$$

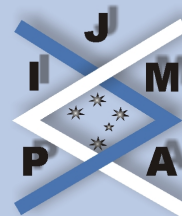
Hence $Q \in W([0, 1])$.

Next, if $0 \leq s \leq t \leq 1$ and $x \in [a, b]$, then

$$\begin{aligned}
\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x &\leq \left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x \\
&\leq \left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)(a+b-x) \\
&\leq \left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)(a+b-x)
\end{aligned}$$

and

$$\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)(a+b-x) \leq \left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)(a+b-x)$$



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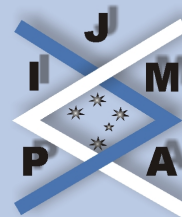


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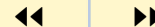
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$$\leq \left(\frac{1+s}{2}\right)b + \left(\frac{1-s}{2}\right)x \leq \left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x,$$

where

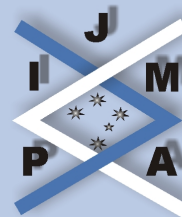
$$\begin{aligned} & \left[\left(\frac{1+s}{2}a\right) + \left(\frac{1-s}{2}x\right) \right] + \left[\left(\frac{1+s}{2}b\right) + \left(\frac{1-s}{2}(a+b-x)\right) \right] \\ &= \left[\left(\frac{1+t}{2}a\right) + \left(\frac{1-t}{2}x\right) \right] + \left[\left(\frac{1+t}{2}b\right) + \left(\frac{1-t}{2}(a+b-x)\right) \right], \end{aligned}$$

and

$$\begin{aligned} & \left[\left(\frac{1+s}{2}a\right) + \left(\frac{1-s}{2}(a+b-x)\right) \right] + \left[\left(\frac{1+s}{2}b\right) + \left(\frac{1-s}{2}x\right) \right] \\ &= \left[\left(\frac{1+t}{2}a\right) + \left(\frac{1-t}{2}(a+b-x)\right) \right] + \left[\left(\frac{1+t}{2}b\right) + \left(\frac{1-t}{2}x\right) \right]. \end{aligned}$$

By the inequality (2.1) and the assumptions that p is nonnegative and symmetric about $x = \frac{a+b}{2}$, we have

$$\begin{aligned} (2.7) \quad & f\left(\left(\frac{1+s}{2}a\right) + \left(\frac{1-s}{2}x\right)\right)p\left(\frac{x+a}{2}\right) \\ & + f\left(\left(\frac{1+s}{2}b\right) + \left(\frac{1-s}{2}(a+b-x)\right)\right)p\left(\frac{2a+b-x}{2}\right) \\ & + f\left(\left(\frac{1+s}{2}a\right) + \left(\frac{1-s}{2}(a+b-x)\right)\right)p\left(\frac{a+2b-x}{2}\right) \\ & + f\left(\left(\frac{1+s}{2}b\right) + \left(\frac{1-s}{2}x\right)\right)p\left(\frac{x+b}{2}\right) \end{aligned}$$



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$$\begin{aligned} &= \left[f \left(\left(\frac{1+s}{2} \right) a + \left(\frac{1-s}{2} \right) x \right) \right. \\ &\quad \left. + f \left(\left(\frac{1+s}{2} \right) a + \left(\frac{1-s}{2} \right) (a+b-x) \right) \right] p \left(\frac{x+a}{2} \right) \\ &\quad + \left[f \left(\left(\frac{1+s}{2} \right) b + \left(\frac{1-s}{2} \right) (a+b-x) \right) \right. \\ &\quad \left. + f \left(\left(\frac{1+s}{2} \right) b + \left(\frac{1-s}{2} \right) x \right) \right] p \left(\frac{x+b}{2} \right) \\ &\leq \left[f \left(\left(\frac{1+t}{2} \right) a + \left(\frac{1-t}{2} \right) x \right) \right. \\ &\quad \left. + f \left(\left(\frac{1+t}{2} \right) a + \left(\frac{1-t}{2} \right) (a+b-x) \right) \right] p \left(\frac{x+a}{2} \right) \\ &\quad + \left[f \left(\left(\frac{1+t}{2} \right) b + \left(\frac{1-t}{2} \right) (a+b-x) \right) \right. \\ &\quad \left. + f \left(\left(\frac{1+t}{2} \right) b + \left(\frac{1-t}{2} \right) x \right) \right] p \left(\frac{x+b}{2} \right) \\ &= f \left(\left(\frac{1+t}{2} \right) a + \left(\frac{1-t}{2} \right) x \right) p \left(\frac{x+a}{2} \right) \\ &\quad + f \left(\left(\frac{1+t}{2} \right) b + \left(\frac{1-t}{2} \right) (a+b-x) \right) p \left(\frac{2a+b-x}{2} \right) \\ &\quad + f \left(\left(\frac{1+t}{2} \right) a + \left(\frac{1-t}{2} \right) (a+b-x) \right) p \left(\frac{a+2b-x}{2} \right) \\ &\quad + f \left(\left(\frac{1+t}{2} \right) b + \left(\frac{1-t}{2} \right) x \right) p \left(\frac{x+b}{2} \right). \end{aligned}$$

Integrating the inequality (2.7) over x on $[a, b]$, we have

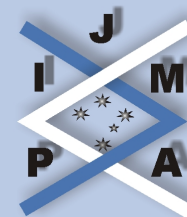
$$4Q(s) \leq 4Q(t)$$

Hence Q is increasing on $[0, 1]$, and the inequality (1.9) holds for all $t \in [0, 1]$.

This completes the proof. ■

Remark 4. If we set $p(x) \equiv 1$ ($x \in [a, b]$) in Theorem 2.4, then Theorem 2.2 generalizes Theorem F.

Remark 5. From $C([a, b]) \subsetneq W([a, b])$, Theorem 2.3 and Theorem 2.4 generalize Theorem C.



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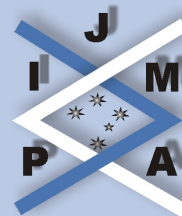
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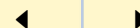
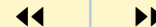
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