



SOME GENERALIZED INEQUALITIES INVOLVING THE q -GAMMA FUNCTION

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Received: 06 August, 2008

Accepted: 14 May, 2009

Communicated by: J. Sándor

2000 AMS Sub. Class.: 33B15.

Key words: q -Gamma Function.

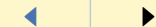
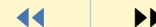
Abstract: In this paper we establish some generalized double inequalities involving the q -gamma function.

**Inequalities Involving the
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vol. 10, iss. 4, art. 120, 2009

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1. Introduction and Preliminary Results

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$, by

$$(1.1) \quad \Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt,$$

and the Psi (or digamma) function is defined by

$$(1.2) \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \quad (x > 0).$$

The q -psi function is defined for $0 < q < 1$, by

$$(1.3) \quad \psi_q(x) = \frac{d}{dx} \log \Gamma_q(x),$$

where the q -gamma function $\Gamma_q(x)$ is defined by ($0 < q < 1$)

$$(1.4) \quad \Gamma_q(x) = (1-q)^{1-x} \prod_{i=1}^{\infty} \frac{1-q^i}{1-q^{x+i}}.$$

Many properties of the q -gamma function were derived by Askey [2]. The explicit form of the q -psi function $\psi_q(x)$ is

$$(1.5) \quad \psi_q(x) = -\log(1-q) + \log q \sum_{i=0}^{\infty} \frac{q^{x+i}}{1-q^{x+i}}.$$

In particular

$$\lim_{q \rightarrow 1^-} \Gamma_q(x) = \Gamma(x) \quad \text{and} \quad \lim_{q \rightarrow 1^-} \psi_q(x) = \psi(x).$$

For the gamma function Alsina and Thomas [1] proved the following double inequality:



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Theorem 1.1. For all $x \in [0, 1]$, and all nonnegative integers n , the following double inequality holds true

$$(1.6) \quad \frac{1}{n!} \leq \frac{[\Gamma(1+x)]^n}{\Gamma(1+nx)} \leq 1.$$

Sándor [4] and Shabani [5] proved the following generalizations of (1.6) given by Theorem 1.2 and Theorem 1.3 respectively.

Theorem 1.2. For all $a \geq 1$ and all $x \in [0, 1]$, one has

$$(1.7) \quad \frac{1}{\Gamma(1+a)} \leq \frac{[\Gamma(1+x)]^a}{\Gamma(1+ax)} \leq 1.$$

Theorem 1.3. Let $a \geq b > 0$, c, d be positive real numbers such that $bc \geq ad > 0$ and $\psi(b+ax) > 0$, where $x \in [0, 1]$. Then the following double inequality holds:

$$(1.8) \quad \frac{[\Gamma(a)]^c}{[\Gamma(b)]^d} \leq \frac{[\Gamma(a+bx)]^c}{[\Gamma(b+ax)]^d} \leq [\Gamma(a+b)]^{c-d}.$$

Recently, Mansour [3] extended above gamma function inequalities to the case of $\Gamma_q(x)$, given by Theorem 1.4, below:

Theorem 1.4. Let $x \in [0, 1]$ and $q \in (0, 1)$. If $a \geq b > 0$, c, d are positive real numbers with $bc \geq ad > 0$ and $\psi_q(b+ax) > 0$, then

$$(1.9) \quad \frac{[\Gamma_q(a)]^c}{[\Gamma_q(b)]^d} \leq \frac{[\Gamma_q(a+bx)]^c}{[\Gamma_q(b+ax)]^d} \leq [\Gamma_q(a+b)]^{c-d}.$$

In our investigation we shall require the following lemmas:



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Lemma 1.5. Let $q \in (0, 1)$, $\alpha > 0$ and a, b be any two positive real numbers such that $a \geq b$. Then

$$(1.10) \quad \psi_q(a\alpha + bx) \geq \psi_q(b\alpha + ax) \quad x \in [0, \alpha],$$

and

$$(1.11) \quad \psi_q(a\alpha + bx) \leq \psi_q(b\alpha + ax) \quad x \in [\alpha, \infty).$$

Proof. By using (1.5), we have

$$\begin{aligned} \psi_q(a\alpha + bx) - \psi_q(b\alpha + ax) &= \log q \sum_{i=0}^{\infty} \left(\frac{q^{a\alpha+bx+i}}{1 - q^{a\alpha+bx+i}} - \frac{q^{b\alpha+ax+i}}{1 - q^{b\alpha+ax+i}} \right) \\ &= \log q \sum_{i=0}^{\infty} \frac{q^i (q^{a\alpha+bx} - q^{b\alpha+ax})}{(1 - q^{a\alpha+bx+i})(1 - q^{b\alpha+ax+i})} \\ &= \log q \sum_{i=0}^{\infty} \frac{q^{b(x+\alpha)+i} (q^{(a-b)\alpha} - q^{(a-b)x})}{(1 - q^{a\alpha+bx+i})(1 - q^{b\alpha+ax+i})}. \end{aligned}$$

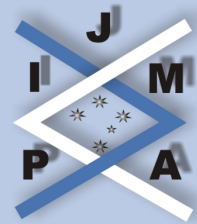
Since for $0 < q < 1$, we have $\log q < 0$. In addition, for $a \geq b$, $x \in [0, \alpha]$, we get $(1 - q^{a\alpha+bx+i}) > 0$, $(1 - q^{b\alpha+ax+i}) > 0$ and $q^{(a-b)\alpha} \leq q^{(a-b)x}$. Hence

$$\psi_q(a\alpha + bx) \geq \psi_q(b\alpha + ax) \quad x \in [0, \alpha].$$

Furthermore, for $a \geq b$ and $x \in [\alpha, \infty)$, we have $(1 - q^{a\alpha+bx+i}) > 0$, $(1 - q^{b\alpha+ax+i}) > 0$ and $q^{(a-b)\alpha} \geq q^{(a-b)x}$. Hence

$$\psi_q(a\alpha + bx) \leq \psi_q(b\alpha + ax) \quad x \in [\alpha, \infty).$$

which completes the proof. \square



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Lemma 1.6. Let $x \in [0, \alpha]$, $\alpha > 0$ and $q \in (0, 1)$. If a, b, c, d are positive real numbers such that $a \geq b$ and $[bc \geq ad, \psi_q(b\alpha + ax) > 0]$ or $[bc \leq ad, \psi_q(a\alpha + bx) < 0]$, we have

$$(1.12) \quad bc\psi_q(a\alpha + bx) - ad\psi_q(b\alpha + ax) \geq 0.$$

Proof. Since $bc \geq ad$ and $\psi_q(b\alpha + ax) > 0$, then using (1.10), we obtain

$$\begin{aligned} ad\psi_q(b\alpha + ax) &\leq bc\psi_q(b\alpha + ax) \\ &\leq bc\psi_q(a\alpha + bx). \end{aligned}$$

Similarly, when $bc \leq ad$ and $\psi_q(a\alpha + bx) < 0$, we have

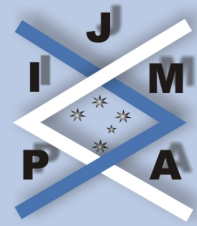
$$bc\psi_q(a\alpha + bx) \geq ad\psi_q(a\alpha + bx) \geq ad\psi_q(b\alpha + ax).$$

This proves Lemma 1.6. □

Similarly, using (1.11) and a similar proof to that above, we have the following lemma:

Lemma 1.7. Let $q \in (0, 1)$ and $x \in [\alpha, \infty)$, $\alpha > 0$. If a, b, c, d are positive real numbers such that $a \geq b$ and $[bc \geq ad, \psi_q(b\alpha + ax) < 0]$ or $[bc \leq ad, \psi_q(a\alpha + bx) < 0]$, we have

$$(1.13) \quad bc\psi_q(a\alpha + bx) - ad\psi_q(b\alpha + ax) \leq 0.$$



2. Main Results

In this section we will establish some generalized double inequalities involving the q - gamma function.

Theorem 2.1. For all $q \in (0, 1)$, $x \in [0, \alpha]$, $\alpha > 0$ and positive real numbers a, b, c, d such that $a \geq b$ and $[bc \geq ad, \psi_q(b\alpha + ax) > 0]$ or $[bc \leq ad, \psi_q(a\alpha + bx) < 0]$, we have

$$(2.1) \quad \frac{[\Gamma_q(a\alpha)]^c}{[\Gamma_q(b\alpha)]^d} \leq \frac{[\Gamma_q(a\alpha + bx)]^c}{[\Gamma_q(b\alpha + ax)]^d} \leq [\Gamma_q\{(a + b)\alpha\}]^{c-d}.$$

Proof. Let

$$(2.2) \quad f(x) = \frac{[\Gamma_q(a\alpha + bx)]^c}{[\Gamma_q(b\alpha + ax)]^d},$$

and assume that $g(x)$ is a function defined by $g(x) = \log f(x)$. Then

$$g(x) = c \log \Gamma_q(a\alpha + bx) - d \log \Gamma_q(b\alpha + ax),$$

so

$$\begin{aligned} g'(x) &= bc \frac{\Gamma'_q(a\alpha + bx)}{\Gamma_q(a\alpha + bx)} - ad \frac{\Gamma'_q(b\alpha + ax)}{\Gamma_q(b\alpha + ax)} \\ &= bc\psi_q(a\alpha + bx) - ad\psi_q(b\alpha + ax). \end{aligned}$$

Thus using Lemma 1.6, we have $g'(x) \geq 0$. This means that $g(x)$ is an increasing function in $[0, \alpha]$, which implies that the function $f(x)$ is also an increasing function in $[0, \alpha]$, so that

$$f(0) \leq f(x) \leq f(\alpha), \quad x \in [0, \alpha],$$

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and this is equivalent to

$$\frac{[\Gamma_q(a\alpha)]^c}{[\Gamma_q(b\alpha)]^d} \leq \frac{[\Gamma_q(a\alpha + bx)]^c}{[\Gamma_q(b\alpha + ax)]^d} \leq [\Gamma_q\{(a + b)\alpha\}]^{c-d}.$$

This completes the proof of Theorem 2.1. □

Theorem 2.2. For all $q \in (0, 1)$, $x \in [\alpha, \infty)$, $\alpha > 0$ and positive real numbers a, b, c, d such that $a \geq b$ and $[bc \geq ad, \psi_q(b\alpha + ax) < 0]$ or $[bc \leq ad, \psi_q(a\alpha + bx) > 0]$, we have

$$(2.3) \quad \frac{[\Gamma_q(a\alpha + bx)]^c}{[\Gamma_q(b\alpha + ax)]^d} \leq [\Gamma_q(a + b)\alpha]^{c-d}$$

and

$$(2.4) \quad \frac{[\Gamma_q(a\alpha + bx)]^c}{[\Gamma_q(b\alpha + ax)]^d} \leq \frac{[\Gamma_q(a\alpha + by)]^c}{[\Gamma_q(b\alpha + ay)]^d}, \quad \alpha < y < x.$$

Proof. Applying Lemma 1.7 and an argument similar to that of Theorem 2.1, we see that the function $f(x)$ defined by (2.2) is a decreasing function. Therefore we have

$$f(x) \leq f(\alpha), \quad x \in [\alpha, \infty),$$

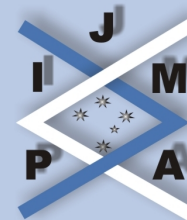
which gives the desired result. □

Remark 1.

- (i) Taking $\alpha = 1$, Theorem 2.1 and Theorem 2.2 yield the results obtained by Mansour [3].
- (ii) Taking $\alpha = 1$ and $q \rightarrow 1^-$, Theorem 2.1 and Theorem 2.2 yield the results obtained by Shabani [5].

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