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ON AN OPEN PROBLEM OF INTEGRAL INEQUALITIES

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ABSTRACT. In this paper, some generalized integral inequalities which originate from an open problem posed in [F. Qi, Several integral inequalities, *J. Inequal. Pure Appl. Math.*, **1**(2) (2000), Art. 19] are established.

Key words and phrases: Integral inequality, Cauchy mean value theorem, Mathematical induction.

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In the paper [3] Qi proved the following result:

Theorem 1 ([3, Proposition 1.1]). *Let $f(x)$ be continuous on $[a,b]$, differentiable on (a,b) and $f(a)=0$. If $f'(x)\geq 1$, then*

$$\int_a^b [f(x)]^3 dx \geq \left[\int_a^b f(x)dx\right]^2. \tag{1}$$

If $0\leq f'(x)\leq 1$, then inequality (1) is reversed.

Qi extended this result to a more general case (see [3]), and obtained the following inequality (2).

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Theorem 2 ([3, Proposition 1.3]). *Let n be a positive integer. Suppose $f(x)$ has continuous derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a) \geq 0$ for $i = 0,1,2,\ldots,n-1$. If $f^{(n)}(x) \geq n!$, then*

$$\int_a^b [f(x)]^{n+2}dx \geq \left[\int_a^b f(x)dx\right]^{n+1}. \tag{2}$$

Qi then proposed an open problem: Under what conditions is the inequality (2) still true if n is replaced by any positive number p?

Some results on this open problem can be found in [1] and [2].

Recently, Chen and Kimball [1] claimed to have given an answer to Qi's open problem as follows.

Claim 1 ([1, Theorem 3]). *Let p be a positive number and $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = 0$. If $\left[f^{\frac{1}{p}}\right]'(x) \geq (p+1)^{\frac{1}{p}-1}$ for $x \in (a,b)$, then*

$$\int_a^b [f(x)]^{p+2}dx \geq \left[\int_a^b f(x)dx\right]^{p+1}. \tag{3}$$

If $0 \leq \left[f^{\frac{1}{p}}\right]'(x) \leq (p+1)^{\frac{1}{p}-1}$ for $x \in (a,b)$, then inequality (3) is reversed.

As a matter of fact, Claim 1 is not true. To see this, choose $f(x) = -2\sqrt{x}$, $p = \frac{1}{2}$, $a = 0$, $b = 1$. It is easy to check that the conditions of Claim 1 are satisfied, but that inequality (3) does not hold. The error in the proof of [1, Theorem 3] is the statement that if $f^{\frac{1}{p}}(x)$ is a non-decreasing function, then $f(x) \geq 0$ for all $x \in (a,b]$, which is *not* true as our example above shows . If one adds $f(x) \geq 0$ for all $x \in (a,b]$ to the hypotheses, then Claim 1 becomes a valid theorem.

Pecaric and Pejkovic [2, Theorem 2] proved the following result which gives an answer to the above open problem.

Theorem 3 ([2, Theorem 2]). *Let p be a positive number and let $f(x)$ be continuous on $[a,b]$, differentiable on (a,b), and satisfy $f(a) \geq 0$. If $f'(x) \geq p(x-a)^{p-1}$ for $x \in (a,b)$, then inequality (3) holds.*

In the present paper we give new answers to Qi's problem and some new results concerning the integral inequality (3) and its reversed form, which extend related results in the references. The following result is a generalization of Theorems 3, 4 and 5 in [1], Proposition 1.1 in [3], and Theorem 2 in [4].

Theorem 4. *Let k be a non-negative integer and let p be a positive number such that $p > k$. Suppose that $f(x)$ has a derivative of the $(k+1)$-th order on the interval (a,b) such that $f^{(k)}(x)$ is continuous on $[a,b]$, $f(x)$ is non-negative on $[a,b]$ and $f^{(i)}(a) = 0$ for $i = 0,1,2,\ldots,k$.*

(i) *If*

$$\left(\left(f^{(k)}\right)^{\frac{1}{p-k}}\right)'(x) \geq \left(\frac{k!\binom{p}{k}}{(p+1)^{p-1}}\right)^{\frac{1}{p-k}}, \quad x \in (a,b),$$

then (3) holds.

(ii) *If*

$$0 \leq \left(\left(f^{(k)}\right)^{\frac{1}{p-k}}\right)'(x) \leq \left(\frac{k!\binom{p}{k}}{(p+1)^{p-1}}\right)^{\frac{1}{p-k}}, \quad x \in (a,b),$$

then inequality (3) is reversed.

Proof. First notice that if $f \equiv 0$ on $[a,b]$, then Theorem 4 is trivial. Suppose that f is not identically 0 on $[a,b]$. If $\int_a^x f(s)ds = 0$ for some $x \in (a,b]$ then $f(s) = 0$ for all $s \in [a,x]$, because $f(x)$ is non-negative on $[a,b]$. So we can assume that $\int_a^x f(s)ds > 0$ for all $x \in (a,b]$. (Otherwise, we can find $a_1 \in (a,b)$ such that $\int_a^x f(s)ds = 0$ for $x \in [a,a_1]$ and $\int_a^x f(s)ds > 0$ for $x \in (a_1,b)$ and hence we only need to consider f on $[a_1,b]$).

(i) Suppose that

$$\left(\left(f^{(k)}\right)^{\frac{1}{p-k}}\right)'(x) \geq \left(\frac{k!\binom{p}{k}}{(p+1)^{p-1}}\right)^{\frac{1}{p-k}}, \quad x \in (a,b).$$

(1) $k = 0 < p$. By Cauchy's mean value theorem (CMVT) (that is, the statement that for h,g differentiable on (a,b) and continuous on $[a,b]$ there exists a $\xi \in (a,b)$ such that $h'(\xi)(g(b)-g(a)) = g'(\xi)(h(b)-h(a))$), by using CMVT twice, there exist $a < b_2 < b_1 < b$ such that

$$\begin{aligned}\frac{\int_a^b (f(x))^{p+2}dx}{\left(\int_a^b f(x)dx\right)^{p+1}} &= \frac{(f(b_1))^{p+1}}{(p+1)\left(\int_a^{b_1} f(x)dx\right)^p} \\ &= \frac{1}{p+1}\left(\frac{(f(b_1))^{\frac{p+1}{p}}}{\int_a^{b_1} f(x)dx}\right)^p \\ &= \frac{(p+1)^{p-1}}{p^p}\frac{(f'(b_2))^p}{(f(b_2))^{p-1}} \geq 1,\end{aligned}$$

since $\left(f^{\frac{1}{p}}\right)'(x) \geq (p+1)^{\frac{1}{p}-1}$ for $x \in (a,b)$.

So (i) is true for $k = 0$.

(2) $k = 1 < p$. By using CMVT three times, there exist $a < b_3 < b_2 < b_1 < b$ such that

$$\begin{aligned}\frac{\int_a^b (f(x))^{p+2}dx}{\left(\int_a^b f(x)dx\right)^{p+1}} &= \frac{(f(b_1))^{p+1}}{(p+1)\left(\int_a^{b_1} f(x)dx\right)^p} \\ &= \frac{1}{p+1}\left(\frac{(f(b_1))^{\frac{p+1}{p}}}{\int_a^{b_1} f(x)dx}\right)^p \\ &= \frac{(p+1)^{p-1}}{p^p}\left(\frac{(f'(b_2))^{\frac{p}{p-1}}}{f(b_2)}\right)^{p-1} \\ &= \frac{(p+1)^{p-1}}{p^p}\left(\frac{p}{p-1}\right)^{p-1}\frac{(f''(b_3))^{p-1}}{(f'(b_3))^{p-2}} \\ &\geq 1,\end{aligned}$$

since

$$\left((f')^{\frac{1}{p-1}}\right)'(x) \geq \left(\frac{p}{(p+1)^{p-1}}\right)^{\frac{1}{p-1}}, \quad x \in (a,b).$$

So (i) is true for $k = 1 < p$.

(3) $1 < k < p$. By using CMVT $(k+2)$ times and mathematical induction, there exist $a < b_{k+2} < \cdots < b_1 < b$ such that

$$\begin{aligned}\frac{\int_a^b (f(x))^{p+2}dx}{(\int_a^b f(x)dx)^{p+1}} &= \frac{1}{p+1}\frac{(f(b_1))^{p+1}}{(\int_a^{b_1} f(x)dx)^p} \\ &= \frac{(p+1)^{p-1}}{p^p}\frac{(f'(b_2))^p}{(f(b_2))^{p-1}} \\ &= \cdots \\ &= \frac{(p+1)^{p-1}}{p(p-1)\cdots(p-k+1)(p-k)^{p-k}}\frac{(f^{(k+1)}(b_{k+2}))^{p-k}}{(f^{(k)}(b_{k+2}))^{p-k-1}} \\ &\geq 1,\end{aligned}$$

since

$$\left(\left(f^{(k)}\right)^{\frac{1}{p-k}}\right)'(x) \geq \left(\frac{k!\binom{p}{k}}{(p+1)^{p-1}}\right)^{\frac{1}{p-k}}, \quad x \in (a,b).$$

So (i) is true for $0 \leq k < p$.

(ii) The proof of the second part is similar so we omit the details. This completes the proof of Theorem 4. □

Remark 5. If $p = 1$ and $k = 0$, then Theorem 4 is reduced to Proposition 1.1 in [3]. If $p = k+1$, then

$$\left(\frac{k!\binom{p}{k}}{(p+1)^{p-1}}\right)^{\frac{1}{p-k}} = \frac{(k+1)!}{(k+2)^k}.$$

In [4] we proved the conjecture in [1] (i.e., Theorem 2 in [4]). It is obvious that Theorem 4 is a generalization of Proposition 1.1 in [3], Theorems 3, 4 and 5 in [1], and Theorem 2 in [4].

The following result is a generalization of Theorem 2 in [2] and Proposition 1.3 in [3].

Theorem 6. *Let k be a non-negative integer and let p be a positive number such that $p > k$. Suppose that $f(x)$ has a derivative of the $(k+1)$-th order on the interval (a,b) such that $f^{(k)}(x)$ is continuous on $[a,b]$, $f^{(i)}(a) = 0$ for $i = 0,1,2,\ldots,k-1$, and $f^{(k)}(a) \geq 0$.*
If

$$f^{(k+1)}(x) \geq \frac{(k+1)!\binom{p}{k+1}(p-k+1)^{p-k-1}}{(p+1)^{p-1}}(x-a)^{p-k-1}$$

for $x \in (a,b)$, then inequality (3) holds.

Proof. As in the proof of Theorem 4 we can assume that $\int_a^x f(s)ds > 0$ for all $x \in (a,b]$.

(1) $k = 0$. By using CMVT three times, there exist $a < b_3 < b_2 < b_1 < b$ such that

$$\begin{aligned}\frac{\int_a^b (f(x))^{p+2}dx}{\left(\int_a^b f(x)dx\right)^{p+1}} &= \frac{(f(b_1))^{p+1}}{(p+1)\left(\int_a^{b_1} f(x)dx\right)^p} \\ &\geq \frac{(f(b_2))^{p-1}f'(b_2)}{p\left(\int_a^{b_2} f(x)dx\right)^{p-1}} \\ &\geq \frac{(f(b_2))^{p-1}(b_2-a)^{p-1}}{\left(\int_a^{b_2} f(x)dx\right)^{p-1}},\end{aligned}$$

since $f'(x) \geq p(x-a)^{p-1}$ for $x \in (a,b)$. Then

$$\frac{(f(b_2))^{p-1}(b_2-a)^{p-1}}{\left(\int_a^{b_2} f(x)dx\right)^{p-1}} = \left(\frac{f(b_2)(b_2-a)}{\int_a^{b_2} f(x)dx}\right)^{p-1}$$
$$= \left(1 + \frac{f'(b_3)(b_3-a)}{f(b_3)}\right)^{p-1} \geq 1.$$

So Theorem 6 is true for $k = 0$.

(2) $k = 1 < p$. By using CMVT four times, there exist $a < b_4 < b_3 < b_2 < b_1 < b$ such that

$$\frac{\int_a^b (f(x))^{p+2}dx}{\left(\int_a^b f(x)dx\right)^{p+1}} = \frac{(f(b_1))^{p+1}}{(p+1)\left(\int_a^{b_1} f(x)dx\right)^{p}}$$
$$= \frac{1}{p+1}\left(\frac{(f(b_1))^{\frac{p+1}{p}}}{\int_a^{b_1} f(x)dx}\right)^{p}$$
$$= \frac{(p+1)^{p-1}}{p^p}\frac{(f'(b_2))^p}{\left(\int_a^{b_2} f'(x)dx\right)^{p-1}}$$
$$\geq \frac{(p+1)^{p-1}}{p^{p-1}}\frac{1}{p-1}\frac{(f'(b_3))^{p-2}f''(b_3)}{\left(\int_a^{b_3} f'(x)dx\right)^{p-2}}$$
$$\geq \left(\frac{f'(b_3)(b_3-a)}{\int_a^{b_3} f'(x)dx}\right)^{p-2},$$

since

$$f''(x) \geq (p-1)\left(\frac{p}{p+1}\right)^{p-1}(x-a)^{p-2}, \quad x \in (a,b).$$

Then

$$\left(\frac{f'(b_3)(b_3-a)}{\int_a^{b_3} f'(x)dx}\right)^{p-2} = \left(\frac{f(b_2)(b_2-a)}{\int_a^{b_2} f(x)dx}\right)^{p-1} = \left(1 + \frac{f''(b_4)(b_4-a)}{f'(b_4)}\right)^{p-2} \geq 1.$$

So Theorem 6 is true for $k = 1$.

(3) $1 < k < p$. By using CMVT $(k+3)$ times, there exist $a < b_{k+3} < \cdots < b_1 < b$ such that

$$\frac{\int_a^b (f(x))^{p+2}dx}{\left(\int_a^b f(x)dx\right)^{p+1}} = \frac{(f(b_1))^{p+1}}{(p+1)\left(\int_a^{b_1} f(x)dx\right)^{p}}$$
$$= \frac{(p+1)^{p-1}}{p^p}\frac{(f'(b_2))^p}{\left(\int_a^{b_2} f'(x)dx\right)^{p-1}}$$
$$= \cdots$$
$$= \frac{(p+1)^{p-1}}{p(p-1)\cdots(p-k+2)(p-k+1)^{p-k+1}} \cdot \frac{(f^{(k)}(b_{k+1}))^{p-k+1}}{\left(\int_a^{b_{k+1}} f^{(k)}(x)dx\right)^{p-k}}$$

$$\geq \frac{(p+1)^{p-1}}{p(p-1)\cdots(p-k+2)(p-k+1)^{p-k}(p-k)} \times \frac{(f^{(k)}(b_{k+2}))^{p-k-1} f^{(k+1)}(b_{k+2})}{\left(\int_a^{b_{k+2}} f^{(k)}(x)dx\right)^{p-k-1}}$$

$$\geq \left(\frac{f^{(k)}(b_{k+2})(b_{k+2}-a)}{\int_a^{b_{k+2}} f^k(x)dx}\right)^{p-k-1},$$

since

$$f^{(k+1)}(x) \geq \frac{(k+1)!\binom{p}{k+1}(p-k+1)^{p-k-1}}{(p+1)^{p-1}}(x-a)^{p-k-1}, \quad x \in (a,b).$$

Then

$$\left(\frac{f^{(k)}(b_{k+2})(b_{k+2}-a)}{\int_a^{b_{k+2}} f^k(x)dx}\right)^{p-k-1} = \left(1 + \frac{f^{(k+1)}(b_{k+3})(b_{k+3}-a)}{f^{(k)}(b_{k+3})}\right)^{p-k-1} \geq 1.$$

This completes the proof of Theorem 6.

□

Remark 7. If $k = 0$, Theorem 6 is reduced to Theorem 2 in [2]. If $k = 0$ and $p = n$, then

$$\frac{(k+1)!\binom{p}{k+1}(p-k+1)^{p-k-1}}{(p+1)^{p-1}}(x-a)^{p-k-1} = n(x-a)^{n-1}.$$

It follows that Proposition 1.3 in [3] is a corollary of Theorem 6. In fact, let f satisfy the conditions of Theorem 2. Since $f^{(n)}(x) \geq n!$, successively integrating $n-1$ times over $[a,x]$, we have $f'(x) \geq n(x-a)^{n-1}$ for $x \in (a,b)$. Hence, Theorem 6 is a generalization of Proposition 1.3 in [3] and Theorem 2 in [2].

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