journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 8 (2007), Issue 1, Article 11, 6 pp.

REDHEFFER TYPE INEQUALITY FOR BESSEL FUNCTIONS

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Received 23 August, 2006; accepted 09 February, 2007
Communicated by F. Qi

ABSTRACT. In this short note, by using mathematical induction and infinite product representations of the functions $\mathcal{J}_p : \mathbb{R} \to (-\infty, 1]$ and $\mathcal{I}_p : \mathbb{R} \to [1, \infty)$, defined by

$$\mathcal{J}_p(x) = 2^p\Gamma(p+1)x^{-p}J_p(x) \ \text{ and } \ \mathcal{I}_p(x) = 2^p\Gamma(p+1)x^{-p}I_p(x),$$

an extension of Redheffer's inequality for the function $\mathcal{J}_p$ and a Redheffer-type inequality for the function $\mathcal{I}_p$ are established. Here J_p and I_p, denotes the Bessel function and modified Bessel function, while Γ stands for the Euler gamma function. At the end of this work a lower bound for the Γ function is deduced, using Euler's infinite product formula. Our main motivation to write this note is the publication of C.P. Chen, J.W. Zhao and F. Qi [2], which we wish to complement.

Key words and phrases: Bessel functions, Modified Bessel functions, Redheffer's inequality.

2000 *Mathematics Subject Classification.* 33C10, 26D05.

1. INTRODUCTION AND PRELIMINARIES

The following inequality

$$\frac{\sin x}{x} \geq \frac{\pi^2 - x^2}{\pi^2 + x^2}, \quad \text{for all } \ x \in \mathbb{R} \tag{1.1}$$

is known in literature as Redheffer's inequality [4, 5]. Motivated by this inequality recently C.P. Chen, J.W. Zhao and F. Qi [2] (see also the survey article of F.Qi [3]) using mathematical induction and infinite product representation of $\cos x$, $\sinh x$ and $\cosh x$ established the following Redheffer-type inequalities:

$$\cos x \geq \frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}, \quad \text{and} \quad \cosh x \leq \frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}, \quad \text{for all } \ |x| \leq \frac{\pi}{2}. \tag{1.2}$$

Research partially supported by the Institute of Mathematics, University of Debrecen, Hungary. The author is grateful to Professor Szilárd András and to the referee for their instructive comments which were helpful.

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Moreover, the authors found the hyperbolic analogue of inequality (1.1), by showing that

$$\frac{\sinh x}{x} \leq \frac{\pi^2 + x^2}{\pi^2 - x^2} \text{ for all } |x| < \pi. \tag{1.3}$$

As we mentioned above, the proofs of inequalities (1.2) and (1.3) by C.P. Chen, J. W. Zhao and F. Qi are based on the following representations [1, p. 75 and 85] of $\cos x$, $\sinh x$ and $\cosh x$:

$$\cos x = \prod_{n\geq 1} \left[1 - \frac{4x^2}{(2n-1)^2\pi^2}\right], \qquad \cosh x = \prod_{n\geq 1} \left[1 + \frac{4x^2}{(2n-1)^2\pi^2}\right], \tag{1.4}$$

and

$$\frac{\sinh x}{x} = \prod_{n\geq 1} \left(1 + \frac{x^2}{n^2\pi^2}\right) \tag{1.5}$$

respectively. In this paper our aim is to show that the idea of using mathematical induction and infinite product representation is also fruitful for Bessel functions as well as for the Γ function.

2. An Extension of Redheffer's Inequality and its Hyperbolic Analogue

Our first main result reads as follows.

Theorem 2.1. *Let us consider the functions* $\mathcal{J}_p : \mathbb{R} \to (-\infty, 1]$ *and* $\mathcal{I}_p : \mathbb{R} \to [1, \infty)$, *defined by the relations*

$$\mathcal{J}_p(x) = 2^p\Gamma(p+1)x^{-p}J_p(x) \text{ and } \mathcal{I}_p(x) = 2^p\Gamma(p+1)x^{-p}I_p(x),$$

where J_p *and* I_p *are the well-known Bessel function, and modified Bessel function respectively. Furthermore suppose that* $p > -1$ *and let* $j_{p,n}$ *be the* n*-th positive zero of the Bessel function* J_p*. If* $\Delta_p(n) := j_{p,n+1}^2 - j_{p,1}j_{p,n} - j_{p,n}j_{p,n+1} \geq 0$, *where* $n = 1, 2, \ldots$, *then the following inequalities hold*

$$\mathcal{J}_p(x) \geq \frac{j_{p,1}^2 - x^2}{j_{p,1}^2 + x^2}, \text{ for all } |x| \leq \alpha_p := \min_{n\geq 1, p>-1} \left\{ j_{p,1}, \sqrt{\Delta_p(n)} \right\}, \tag{2.1}$$

$$\mathcal{I}_p(x) \leq \frac{j_{p,1}^2 + x^2}{j_{p,1}^2 - x^2}, \text{ for all } |x| < j_{p,1}. \tag{2.2}$$

Remark 2.2. For later use it is worth mentioning that in particular for $p = -1/2$ and $p = 1/2$ respectively the functions $\mathcal{J}_p$ and $\mathcal{I}_p$ reduce to some elementary functions [1, p. 438 and 443], such as

$$\begin{aligned} \mathcal{J}_{-1/2}(x) &= \sqrt{\pi/2} \cdot x^{1/2} J_{-1/2}(x) = \cos x, \\ \mathcal{J}_{1/2}(x) &= \sqrt{\pi/2} \cdot x^{-1/2} J_{1/2}(x) = \frac{\sin x}{x}, \end{aligned} \tag{2.3}$$

with their hyperbolic analogs

$$\begin{aligned} \mathcal{I}_{-1/2}(x) &= \sqrt{\pi/2} \cdot x^{1/2} I_{-1/2}(x) = \cosh x, \\ \mathcal{I}_{1/2}(x) &= \sqrt{\pi/2} \cdot x^{-1/2} I_{1/2}(x) = \frac{\sinh x}{x}. \end{aligned} \tag{2.4}$$

Recall that $\mathcal{J}_{-1/2}$ has the infinite product representation (1.4) and [1, p. 75]

$$\mathcal{J}_{1/2}(x) = \frac{\sin x}{x} = \prod_{n\geq 1} \left(1 - \frac{x^2}{n^2\pi^2}\right).$$

Thus using the relations (2.6) and (2.3) it is clear that $j_{-1/2,n} = (2n-1)\pi/2$ and $j_{1/2,n} = n\pi$ for all $n = 1, 2, \ldots$. Consequently for all $n = 1, 2, \ldots$ we have $\sqrt{\Delta_{1/2}(n)} = j_{1/2,1} = \pi$ and

$$\sqrt{\Delta_{-1/2}(n)} = \frac{\pi}{2}\sqrt{2n+3} \geq \frac{\pi}{2}\sqrt{5} > \frac{\pi}{2} = j_{-1/2,1},$$

which imply that $\alpha_{-1/2} = \pi/2$ and $\alpha_{1/2} = \pi$. So in view of (2.3), if we take in (2.1) $p = -1/2$ and $p = 1/2$ respectively, then we reobtain the first inequality from (1.2) and Redheffer's inequality (1.1) respectively, with the intervals of validity $[-\pi/2, \pi/2]$ and $[-\pi, \pi]$, respectively. The situation is similar to inequality (2.2), namely if we choose in (2.2) $p = -1/2$ and $p = 1/2$ respectively, then by using (2.4), we reobtain the second inequality from (1.2) and inequality (1.3), with the same intervals of validity, i.e. $[-\pi/2, \pi/2]$ and $[-\pi, \pi]$, respectively.

Proof of Theorem 2.1. First observe that to prove (2.1) it is enough to show that

$$\mathcal{J}_p(xj_{p,1}) \geq \frac{1-x^2}{1+x^2} \tag{2.5}$$

holds for all $|x| \leq \alpha_p/j_{p,1}$. It is known that for the Bessel function of the first kind J_p the following infinite product formula [6, p. 498]

$$\mathcal{J}_p(x) = 2^p\Gamma(p+1)x^{-p}J_p(x) = \prod_{n\geq 1}\left(1 - \frac{x^2}{j_{p,n}^2}\right) \tag{2.6}$$

is valid for arbitrary x and $p \neq -1, -2, \ldots$. From this we deduce

$$\mathcal{J}_p(xj_{p,1}) = \frac{1-x^2}{1+x^2}\left[(1+x^2)\lim_{n\to\infty} Q_{p,n}\right], \text{ where } Q_{p,n} := \prod_{k=2}^{n}\left(1 - \frac{x^2 j_{p,1}^2}{j_{p,k}^2}\right). \tag{2.7}$$

In what follows we want to prove by mathematical induction that

$$(1+x^2)Q_{p,n} \geq 1 + \frac{x^2 j_{p,1}}{j_{p,n}} \tag{2.8}$$

holds for all $p > -1$, $n \geq 2$ and $|x| \leq \alpha_p/j_{p,1}$. For $n = 2$ clearly by assumptions we have

$$(1+x^2)Q_{p,2} - \left(1 + \frac{x^2 j_{p,1}}{j_{p,2}}\right) = \frac{x^2}{j_{p,2}^2}\left[\Delta_p(1) - j_{p,1}^2 x^2\right] \geq 0.$$

Now suppose that (2.8) holds for some $m \geq 2$. From the definition of $Q_{p,m}$, we easily get

$$Q_{p,m+1} = Q_{p,m}\cdot\left(1 - \frac{x^2 j_{p,1}^2}{j_{p,m+1}^2}\right), \text{ for all } m = 2, 3, 4, \ldots,$$

thus

$$\begin{aligned}(1+x^2)Q_{p,m+1} - \left(1 + \frac{x^2 j_{p,1}}{j_{p,m+1}}\right) &= (1+x^2)Q_{p,m}\left(1 - \frac{x^2 j_{p,1}^2}{j_{p,m+1}^2}\right) - \left(1 + \frac{x^2 j_{p,1}}{j_{p,m+1}}\right)\\ &\geq \left(1 + \frac{x^2 j_{p,1}}{j_{p,m}}\right)\left(1 - \frac{x^2 j_{p,1}^2}{j_{p,m+1}^2}\right) - \left(1 + \frac{x^2 j_{p,1}}{j_{p,m+1}}\right)\\ &= \frac{x^2 j_{p,1}}{j_{p,m}j_{p,m+1}^2}\left[\Delta_p(m) - j_{p,1}^2 x^2\right] \geq 0,\end{aligned}$$

and hence by induction (2.8) follows. Here we used the fact that from the hypothesis we obtain $|x| \leq \sqrt{\Delta_p(m)}/j_{p,1} \leq j_{p,m+1}/j_{p,1}$. On the other hand from the MacMahon expansion [6, p. 506],

$$j_{p,n} = (n + p/2 - 1/4)\pi + \mathcal{O}(n^{-1}), \quad n \to \infty,$$

we have $j_{p,n} \to \infty$, as n tends to infinity. Finally from (2.8) we obtain

$$\lim_{n\to\infty} (1+x^2)Q_{p,n} \geq \lim_{n\to\infty} \left(1+\frac{x^2 j_{p,1}}{j_{p,n}}\right) = 1,$$

which in view of (2.7) implies (2.5). This completes the proof of (2.1).

Proceeding similarly as in the proof of (2.1) now we prove (2.2). It suffices to show that

$$\mathcal{I}_p(xj_{p,1}) \leq \frac{1+x^2}{1-x^2} \tag{2.9}$$

holds for all $|x| < 1$. Analogously, using the factorisation (2.6), it is known that for the modified Bessel function of the first kind I_p the following infinite product formula

$$\mathcal{I}_p(x) = 2^p\Gamma(p+1)x^{-p}I_p(x) = \prod_{n\geq 1}\left(1+\frac{x^2}{j_{p,n}^2}\right)$$

is also valid for arbitrary x and $p \neq -1, -2, \ldots$. From this we get that

$$\mathcal{I}_p(xj_{p,1}) = \frac{1+x^2}{1-x^2}\left[(1-x^2)\lim_{n\to\infty} R_{p,n}\right], \text{ where } R_{p,n} := \prod_{k=2}^{n}\left(1+\frac{x^2 j_{p,1}^2}{j_{p,k}^2}\right). \tag{2.10}$$

In what follows we want to show by mathematical induction that

$$(1-x^2)R_{p,n} \leq 1 - \frac{x^2 j_{p,1}}{j_{p,n}} \tag{2.11}$$

holds for all $p > -1$, $n \geq 2$ and $|x| < 1$. For $n = 2$, clearly we have

$$(1-x^2)R_{p,2} - \left(1-\frac{x^2 j_{p,1}}{j_{p,2}}\right) = \frac{x^2}{j_{p,2}^2}\left[-\Delta_p(1) - j_{p,1}^2 x^2\right] \leq 0.$$

Now suppose that (2.11) holds for some $m \geq 2$. From the definition of $R_{p,m}$, we easily get

$$R_{p,m+1} = R_{p,m}\cdot\left(1+\frac{x^2 j_{p,1}^2}{j_{p,m+1}^2}\right), \text{ for all } m = 2, 3, 4, \ldots,$$

thus

$$\begin{aligned}
(1-x^2)R_{p,m+1} - \left(1-\frac{x^2 j_{p,1}}{j_{p,m+1}}\right) &= (1-x^2)R_{p,m}\left(1+\frac{x^2 j_{p,1}^2}{j_{p,m+1}^2}\right) - \left(1-\frac{x^2 j_{p,1}}{j_{p,m+1}}\right) \\
&\leq \left(1-\frac{x^2 j_{p,1}}{j_{p,m}}\right)\left(1+\frac{x^2 j_{p,1}^2}{j_{p,m+1}^2}\right) - \left(1-\frac{x^2 j_{p,1}}{j_{p,m+1}}\right) \\
&= \frac{x^2 j_{p,1}}{j_{p,m} j_{p,m+1}^2}\left[-\Delta_p(m) - j_{p,1}^2 x^2\right] \leq 0,
\end{aligned}$$

and hence by induction, (2.11) follows. Finally using again the fact that $j_{p,n} \to \infty$, as n tends to infinity, from (2.11) we obtain

$$\lim_{n\to\infty} (1-x^2)R_{p,n} \leq \lim_{n\to\infty}\left(1-\frac{x^2 j_{p,1}}{j_{p,n}}\right) = 1,$$

which in view of (2.10) implies (2.9). Thus the proof is complete. □

3. A LOWER BOUND FOR THE Γ FUNCTION

In an effort to popularize the method that C.P. Chen, J. W. Zhao and F. Qi used in the previous section, we illustrate it below by giving a lower bound for the Γ function.

Theorem 3.1. *If* $x \in (0,1]$, *then*

$$\Gamma(x) \geq \frac{1-x}{1+x} \cdot \frac{e^{(1-\gamma)x}}{x}, \tag{3.1}$$

where $\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right) = 0.5772156649\ldots$ *is the Euler constant.*

Proof. From the well-known Euler infinite product formula [1, p. 255] for the Γ function,

$$\frac{1}{xe^{\gamma x}\Gamma(x)} = \prod_{n\geq 1}\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}},$$

we have

$$\frac{e^{(1-\gamma)x}}{x\Gamma(x)} = \frac{1+x}{1-x}\left[(1-x)\lim_{n\to\infty} S_n\right], \text{ where } S_n = \prod_{k=2}^{n}\left(1+\frac{x}{k}\right)e^{-\frac{x}{k}},\ n=2,3,\ldots. \tag{3.2}$$

Observe that to prove (3.1), it is enough to show that for all $n = 2,3,\ldots$

$$(1-x)S_n < \left(1-\frac{x}{n}\right) \tag{3.3}$$

holds, hence from this we get $\lim\limits_{n\to\infty}(1-x)S_n \leq 1$, and consequently from (3.2) the inequality (3.1) follows. To prove (3.3) we use mathematical induction again. For $n=2$ we easily get for $x \in (0,1)$ that

$$(1-x)S_2 < \left(1-\frac{x}{2}\right) \Longleftrightarrow e^{x/2} > \frac{(1-x)\left(1+\frac{x}{2}\right)}{1-\frac{x}{2}},$$

which clearly holds because

$$e^{x/2} - \frac{(1-x)\left(1+\frac{x}{2}\right)}{1-\frac{x}{2}} = \sum_{k\geq 1}\left(x+\frac{1}{k!}\right)\frac{x^k}{2^k} > 0.$$

Now suppose that (3.3) holds for some $m \geq 2$. Then from (3.2) and (3.3) we obtain that

$$\begin{aligned}(1-x)S_{m+1} - \left(1-\frac{x}{m+1}\right) &= (1-x)S_m\left(1+\frac{x}{m+1}\right)e^{-\frac{x}{m+1}} - \left(1-\frac{x}{m+1}\right)\\ &< \left(1-\frac{x}{m}\right)\left(1+\frac{x}{m+1}\right)e^{-\frac{x}{m+1}} - \left(1-\frac{x}{m+1}\right)\end{aligned}$$

and this is negative if and only if

$$e^{\frac{x}{m+1}} - \frac{\left(1-\frac{x}{m}\right)\left(1+\frac{x}{m+1}\right)}{\left(1-\frac{x}{m+1}\right)} = \sum_{k\geq 1}\left(\frac{x}{m}+\frac{1}{m}-1+\frac{1}{k!}\right)\frac{x^k}{(m+1)^k} > 0.$$

□

Remark 3.2. Numerical experiments in Maple6 show that the lower bound from (3.1) is far from being the best possible one. For example for $x = 0.5$ we have that $\Gamma(0.5) = \sqrt{\pi} = 1.772453851\ldots$, while the right hand side of (3.1) is just $0.8235978287\ldots$. Similarly for $x = 0.25$ we have $\Gamma(0.25) = 3.6256099082\ldots$, while the right hand side of (3.1) is just $2.667561665\ldots$. In fact graphics in Maple6 suggest that the function $f : (-1,\infty) \to \mathbb{R}$ defined by

$$f(x) = \Gamma(x) - \frac{1-x}{1+x}\cdot\frac{e^{(1-\gamma)x}}{x}$$

is convex and satisfies the inequality $f(x) \geq 1$, for all $x \in (-1,0]$ or $x \in [1,\infty)$. Moreover $f(x) \in (0.94,1]$, for all $x \in [0,1]$.

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