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GENERALIZED OSTROWSKI'S INEQUALITY ON TIME SCALES

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ABSTRACT. In this paper, we generalize Ostrowski's inequality and Montgomery's identity on arbitrary time scales which were given in a recent paper [*J. Inequal. Pure. Appl. Math.*, **9**(1) (2008), Art. 6] by Bohner and Matthews. Some examples for the continuous, discrete and the quantum calculus cases are given as well.

Key words and phrases: Montgomery's identity, Ostrowski's inequality, time scales.

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1. INTRODUCTION

In 1937, Ostrowski gave a very useful formula to estimate the absolute value of derivation of a differentiable function by its integral mean. In [9], the so-called Ostrowski's inequality

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(\eta)d\eta\right| \leq \left\{\sup_{\eta\in(a,b)}|f'(\eta)|\right\}\left(\frac{(t-a)^2+(b-t)^2}{2(b-a)}\right)$$

is shown by the means of the Montgomery's identity (see [6, pp. 565]).

In a very recent paper [2], the Montgomery identity and the Ostrowski inequality were generalized respectively as follows:

Lemma A (Montgomery's identity)**.** *Let* $a, b \in \mathbb{T}$ *with* $a < b$ *and* $f \in C^1_{\mathrm{rd}}([a, b]_{\mathbb{T}}, \mathbb{R})$. *Then*

$$f(t)=\frac{1}{b-a}\left(\int_a^b f^{\sigma}(\eta)\Delta\eta+\int_a^b \Psi(t,\eta)f^{\Delta}(\eta)\Delta\eta\right)$$

215-08

holds for all $t \in \mathbb{T}$*, where* $\Psi : [a,b]_{\mathbb{T}}^2 \to \mathbb{R}$ *is defined as follows:*

$$\Psi(t,s) := \begin{cases} s-a, & s \in [a,t)_{\mathbb{T}}; \\ s-b, & s \in [t,b]_{\mathbb{T}} \end{cases}$$

for $s,t \in [a,b]_{\mathbb{T}}$*.*

Theorem A (Ostrowski's inequality)**.** *Let* $a,b \in \mathbb{T}$ *with* $a<b$ *and* $f \in C_{\mathrm{rd}}^1([a,b]_{\mathbb{T}},\mathbb{R})$*. Then*

$$\left| f(t) - \frac{1}{b-a}\int_a^b f^{\sigma}(\eta)\Delta\eta \right| \leq \left\{ \sup_{\eta\in(a,b)} |f^{\Delta}(\eta)| \right\} \left(\frac{h_2(t,a)+h_2(t,b)}{b-a} \right)$$

holds for all $t \in \mathbb{T}$*. Here,* $h_2(t,s)$ *is the second-order generalized polynomial on time scales.*

In this paper, we shall apply a new method to generalize Lemma A, Theorem A, which is completely different to the method employed in [2], however following the routine steps in [2], our results may also be proved.

The paper is arranged as follows: in §2, we quote some preliminaries on time scales from [1]; §3 includes our main results which generalize Lemma A and Theorem A by the means of generalized polynomials on time scales; in §4, as applications, we consider particular time scales $\mathbb{R}, \mathbb{Z}$ and $q^{\mathbb{N}_0}$; finally, in §5, we give extensions of the results stated in §3.

2. Time Scales Essentials

Definition 2.1. A *time scale* is a nonempty closed subset of reals.

Definition 2.2. On an arbitrary time scale $\mathbb{T}$ the following are defined: the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf(t,\infty)_{\mathbb{T}}$ for $t \in \mathbb{T}$, the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup(-\infty,t)_{\mathbb{T}}$ for $t \in \mathbb{T}$, and the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}_0^+$ is defined by $\mu(t) := \sigma(t) - t$ for $t \in \mathbb{T}$. For convenience, we set $\inf\emptyset := \sup\mathbb{T}$ and $\sup\emptyset := \inf\mathbb{T}$.

Definition 2.3. Let t be a point in $\mathbb{T}$. If $\sigma(t) = t$ holds, then t is called *right-dense*, otherwise it is called *right-scattered*. Similarly, if $\rho(t) = t$ holds, then t is called *left-dense*, a point which is not left-dense is called *left-scattered*.

Definition 2.4. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided that it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points of $\mathbb{T}$. The set of rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, and $C_{\mathrm{rd}}^1(\mathbb{T},\mathbb{R})$ denotes the set of functions for which the delta derivative belongs to $C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$.

Theorem 2.1 (Existence of antiderivatives)**.** *Let* f *be a rd-continuous function. Then* f *has an antiderivative* F *such that* $F^{\Delta} = f$ *holds.*

Definition 2.5. If $f \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ and $s \in \mathbb{T}$, then we define the *integral*

$$F(t) := \int_s^t f(\eta)\Delta\eta \quad \text{for } t \in \mathbb{T}.$$

Theorem 2.2. *Let* f,g *be rd-continuous functions,* $a,b,c \in \mathbb{T}$ *and* $\alpha,\beta \in \mathbb{R}$*. Then, the following are true:*

(1) $\int_a^b \left[\alpha f(\eta) + \beta g(\eta)\right]\Delta\eta = \alpha\int_a^b f(\eta)\Delta\eta + \beta\int_a^b g(\eta)\Delta\eta$,
(2) $\int_a^b f(\eta)\Delta\eta = -\int_b^a f(\eta)\Delta\eta$,
(3) $\int_a^c f(\eta)\Delta\eta = \int_a^b f(\eta)\Delta\eta + \int_b^c f(\eta)\Delta\eta$,
(4) $\int_a^b f(\eta)g^{\Delta}(\eta)\Delta\eta = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(\eta)g(\sigma(\eta))\Delta\eta$.

Definition 2.6. Let $h_k : \mathbb{T}^2 \to \mathbb{R}$ be defined as follows:

$$h_k(t,s) := \begin{cases} 1, & k = 0 \\ \int_s^t h_{k-1}(\eta, s)\Delta\eta, & k \in \mathbb{N} \end{cases} \tag{2.1}$$

for all $s, t \in \mathbb{T}$ and $k \in \mathbb{N}_0$.

Note that the function h_k satisfies

$$h_k^{\Delta_t}(t,s) = \begin{cases} 0, & k = 0 \\ h_{k-1}(t,s), & k \in \mathbb{N} \end{cases} \tag{2.2}$$

for all $s, t \in \mathbb{T}$ and $k \in \mathbb{N}_0$.

Property 1. *Using induction it is easy to see that $h_k(t,s) \geq 0$ holds for all $k \in \mathbb{N}$ and $s, t \in \mathbb{T}$ with $t \geq s$ and $(-1)^k h_k(t,s) \geq 0$ holds for all $k \in \mathbb{N}$ and $s, t \in \mathbb{T}$ with $t \leq s$.*

3. GENERALIZATION BY GENERALIZED POLYNOMIALS

We start this section by quoting the following useful change of order formula for double(iterated) integrals which is employed in our proofs.

Lemma 3.1 ([8, Lemma 1])**.** *Assume that $a, b \in \mathbb{T}$ and $f \in C_{\mathrm{rd}}(\mathbb{T}^2, \mathbb{R})$. Then*

$$\int_a^b \int_\xi^b f(\eta, \xi)\Delta\eta\Delta\xi = \int_a^b \int_a^{\sigma(\eta)} f(\eta,\xi)\Delta\xi\Delta\eta.$$

Now, we give a generalization for Montgomery's identity as follows:

Lemma 3.2. *Assume that $a, b \in \mathbb{T}$ and $f \in C^1_{\mathrm{rd}}([a,b]_\mathbb{T}, \mathbb{R})$. Define $\Psi, \Phi \in C^1_{\mathrm{rd}}([a,b]_\mathbb{T}, \mathbb{R})$ by*

$$\Psi(t,s) := \begin{cases} h_k(s,a), & s \in [a,t)_\mathbb{T} \\ h_k(s,b), & s \in [t,b]_\mathbb{T} \end{cases} \quad \text{and} \quad \Phi(t,s) := \begin{cases} h_{k-1}(s,a), & s \in [a,t)_\mathbb{T} \\ h_{k-1}(s,b), & s \in [t,b]_\mathbb{T} \end{cases}$$

for $s, t \in [a,b]_\mathbb{T}$ and $k \in \mathbb{N}$. Then

$$f(t) = \frac{1}{h_k(t,a) - h_k(t,b)} \left(\int_a^b \Phi(t,\eta) f^\sigma(\eta)\Delta\eta + \int_a^b \Psi(t,\eta) f^\Delta(\eta)\Delta\eta \right) \tag{3.1}$$

is true for all $t \in [a,b]_\mathbb{T}$ and all $k \in \mathbb{N}$.

Proof. Note that we have $\Psi^{\Delta_s} = \Phi$. Clearly, for all $t \in [a,b]_\mathbb{T}$ and all $k \in \mathbb{N}$, from (3.1), (2.1) and (2.2) we have

$$\begin{aligned} &\int_a^t \Phi(t,\eta) f^\sigma(\eta)\Delta\eta + \int_a^t \Psi(t,\eta) f^\Delta(\eta)\Delta\eta \\ &= \int_a^t h_{k-1}(\eta,a) f^\sigma(\eta)\Delta\eta + \int_a^t h_k(\eta,a) f^\Delta(\eta)\Delta\eta \\ &= \int_a^t \int_a^{\sigma(\eta)} h_{k-1}(\eta,a) f^\Delta(\xi)\Delta\xi\Delta\eta + f(a)h_k(t,a) \\ &\qquad + \int_a^t \int_a^\eta \left[h_k(\xi,a) f^\Delta(\eta)\right]^{\Delta_\xi} \Delta\xi\Delta\eta. \end{aligned} \tag{3.2}$$

Applying Lemma 3.1 and considering (2.1), the right-hand side of (3.2) takes the form

$$\begin{aligned}
&\int_a^t \int_\xi^t h_{k-1}(\eta,a)f^\Delta(\xi)\Delta\eta\Delta\xi + f(a)h_k(t,a) + \int_a^t \int_a^\eta h_{k-1}(\xi,a)f^\Delta(\eta)\Delta\xi\Delta\eta \\
&= \int_a^t \int_a^t h_{k-1}(\eta,a)f^\Delta(\xi)\Delta\eta\Delta\xi + f(a)h_k(t,a) \\
&= f(t)h_k(t,a), \qquad (3.3)
\end{aligned}$$

and very similarly, from Lemma 3.1, (3.1), (2.1) and (2.2), we obtain

$$\begin{aligned}
&\int_t^b \Phi(t,\eta)f^\sigma(\eta)\Delta\eta + \int_t^b \Psi(t,\eta)f^\Delta(\eta)\Delta\eta \\
&= \int_t^b h_{k-1}(\eta,b)f^\sigma(\eta)\Delta\eta + \int_t^b h_k(\eta,b)f^\Delta(\eta)\Delta\eta \\
&= \int_t^b \int_t^{\sigma(\eta)} h_{k-1}(\eta,b)f^\Delta(\xi)\Delta\xi\Delta\eta - f(t)h_k(t,b) - \int_t^b \int_\eta^b \left[h_k(\xi,b)f^\Delta(\eta)\right]^{\Delta_\xi}\Delta\xi\Delta\eta, \\
&= \int_t^b \int_\xi^b h_{k-1}(\eta,b)f^\Delta(\xi)\Delta\eta\Delta\xi - f(t)h_k(t,b) - \int_t^b \int_\eta^b h_{k-1}(\xi,b)f^\Delta(\eta)\Delta\xi\Delta\eta \\
&= -f(t)h_k(t,b). \qquad (3.4)
\end{aligned}$$

By summing (3.3) and (3.4), we get the desired result. □

Now, we give the following generalization of Ostrowski's inequality.

Theorem 3.3. *Assume that* $a, b \in \mathbb{T}$ *and* $f \in C^1_{\mathrm{rd}}([a,b]_\mathbb{T}, \mathbb{R})$. *Then*

$$\left|f(t) - \frac{1}{h_k(t,a) - h_k(t,b)} \int_a^b \Phi(t,\eta)f^\sigma(\eta)\Delta\eta\right| \leq M\left(\frac{h_{k+1}(t,a) + (-1)^{k+1}h_{k+1}(t,b)}{h_k(t,a) - h_k(t,b)}\right)$$

is true for all $t \in [a,b]_\mathbb{T}$ *and all* $k \in \mathbb{N}$, *where* Φ *is as introduced in (3.1) and* $M := \sup_{\eta\in(a,b)} |f^\Delta(\eta)|$.

Proof. From Lemma 3.2 and (3.1), for all $k \in \mathbb{N}$ and $t \in [a,b]_\mathbb{T}$, we get

$$\begin{aligned}
&\left|f(t) - \frac{1}{h_k(t,a) - h_k(t,b)} \int_a^b \Phi(t,\eta)f^\sigma(\eta)\Delta\eta\right| \\
&= \left|\frac{1}{h_k(t,a) - h_k(t,b)} \int_a^b \Psi(t,\eta)f^\Delta(\eta)\Delta\eta\right| \\
&= \left|\frac{1}{h_k(t,a) - h_k(t,b)} \left(\int_a^t h_k(\eta,a)f^\Delta(\eta)\Delta\eta + \int_t^b h_k(\eta,b)f^\Delta(\eta)\Delta\eta\right)\right| \\
&\leq \frac{M}{h_k(t,a) - h_k(t,b)} \left(\left|\int_a^t h_k(\eta,a)\Delta\eta\right| + \left|\int_t^b h_k(\eta,b)\Delta\eta\right|\right), \qquad (3.5)
\end{aligned}$$

and considering Property 1 and (2.1) on the right-hand side of (3.5), we have

$$\begin{aligned}
&\frac{M}{h_k(t,a)-h_k(t,b)}\left(\int_a^t h_k(\eta,a)\Delta\eta+\int_t^b(-1)^k h_k(\eta,b)\Delta\eta\right)\\
&=\frac{M}{h_k(t,a)-h_k(t,b)}\left(\int_a^t h_k(\eta,a)\Delta\eta+(-1)^{k+1}\int_b^t h_k(\eta,b)\Delta\eta\right)\\
&=M\left(\frac{h_{k+1}(t,a)+(-1)^{k+1}h_{k+1}(t,b)}{h_k(t,a)-h_k(t,b)}\right),
\end{aligned}$$

which completes the proof. □

Remark 1. It is clear that Lemma 3.2 and Theorem 3.3 reduce to Lemma A and Theorem A respectively by letting $k=1$.

4. APPLICATIONS FOR GENERALIZED POLYNOMIALS

In this section, we give examples on particular time scales for Theorem 3.3. First, we consider the continuous case.

Example 4.1. Let $\mathbb{T}=\mathbb{R}$. Then, we have $h_k(t,s)=(t-s)^k/k!=(-1)^k(s-t)^k/k!$ for all $s,t\in\mathbb{R}$ and $k\in\mathbb{N}$. In this case, Ostrowski's inequality reads as follows:

$$\begin{aligned}
&\left|f(t)-\frac{k!}{(t-a)^k+(-1)^{k+1}(b-t)^k}\int_a^b\Phi(t,\eta)f(\eta)d\eta\right|\\
&\qquad\qquad\leq\frac{M}{k+1}\left(\frac{(t-a)^{k+1}+(b-t)^{k+1}}{(t-a)^k+(-1)^{k+1}(b-t)^k}\right),
\end{aligned}$$

where M is the maximum value of the absolute value of the derivative f' over $[a,b]_{\mathbb{R}}$, and $\Phi(t,s)=(s-a)^k/k!$ for $s\in[a,t)_{\mathbb{R}}$ and $\Phi(t,s)=(s-b)^k/k!$ for $s\in[t,b]_{\mathbb{R}}$.

Next, we consider the discrete calculus case.

Example 4.2. Let $\mathbb{T}=\mathbb{Z}$. Then, we have $h_k(t,s)=(t-s)^{(k)}/k!=(-1)^k(s-t+k)^{(k)}/k!$ for all $s,t\in\mathbb{Z}$ and $k\in\mathbb{N}$, where the usual factorial function $^{(k)}$ is defined by $n^{(k)}:=n!/k!$ for $k\in\mathbb{N}$ and $n^{(0)}:=1$ for $n\in\mathbb{Z}$. In this case, Ostrowski's inequality reduces to the following inequality:

$$\begin{aligned}
&\left|f(t)-\frac{k!}{(t-a)^{(k)}+(-1)^{k+1}(b-t+k)^{(k)}}\sum_{\eta=a}^{b-1}\Phi(t,\eta)f(\eta+1)\right|\\
&\qquad\qquad\leq\frac{M}{k+1}\left(\frac{(t-a)^{(k+1)}+(b-t+k)^{(k+1)}}{(t-a)^{(k)}+(-1)^{k+1}(b-t+k)^{(k)}}\right),
\end{aligned}$$

where M is the maximum value of the absolute value of the difference Δf over $[a,b-1]_{\mathbb{Z}}$, and $\Phi(t,s)=(s-a)^{(k)}/k!$ for $s\in[a,t-1]_{\mathbb{Z}}$ and $\Phi(t,s)=(s-b)^{(k)}/k!$ for $s\in[t,b]_{\mathbb{Z}}$.

Before giving the quantum calculus case, we need to introduce the following notations from [7]:

$$[k]_q := \frac{q^k - 1}{q - 1} \qquad \text{for } q \in \mathbb{R}/\{1\} \text{ and } k \in \mathbb{N}_0,$$

$$[k]! := \prod_{j=1}^{k} [j]_q \qquad \text{for } k \in \mathbb{N}_0,$$

$$(t - s)_q^k := \prod_{j=0}^{k-1} (t - q^j s) \qquad \text{for } s, t \in q^{\mathbb{N}_0} \text{ and } k \in \mathbb{N}_0.$$

It is shown in [1, Example 1.104] that

$$h_k(t, s) := \frac{(t - s)_q^k}{[k]!} \qquad \text{for } s, t \in q^{\mathbb{N}_0} \text{ and } k \in \mathbb{N}_0$$

holds.

And finally, we consider the quantum calculus case.

Example 4.3. Let $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$. Therefore, for the quantum calculus case, Ostrowski's inequality takes the following form:

$$\left| f(t) - \frac{[k]!(q-1)a}{(t-a)_q^k - (t-b)_q^k} \sum_{\eta=0}^{\log_q(b/(qa))} q^\eta \Phi(t, q^\eta a) f(q^{\eta+1} a) \right| \leq \frac{M}{[k+1]_q} \left(\frac{(t-a)_q^{k+1} + (-1)^{k+1}(t-b)_q^{k+1}}{(t-a)_q^k - (t-b)_q^k} \right),$$

where M is the maximum value of the absolute value of the q-difference $D_q f$ over $[a, b/q]_{q^{N_0}}$, and $\Phi(t, s) = (s - a)_q^k / [k]!$ for $s \in [a, t/q]_{q^{\mathbb{N}_0}}$ and $\Phi(t, s) = (s - b)^k / [k]!$ for $s \in [t, b]_{q^{\mathbb{N}_0}}$. Here, the q-difference operator D_q is defined by $D_q f(t) := [f(qt) - f(t)]/[(q - 1)t]$.

5. GENERALIZATION BY ARBITRARY FUNCTIONS

In this section, we replace the generalized polynomials $h_k(t, s)$ appearing in the definitions of $\Phi(t, s)$ and $\Psi(t, s)$ by arbitrary functions.

Since the proof of the following results can be done easily, we just give the statements of the results without proofs.

Lemma 5.1. *Assume that* $a, b \in \mathbb{T}$, $f \in C^1_{\mathrm{rd}}([a, b]_\mathbb{T}, \mathbb{R})$, *and that* $\psi, \phi \in C^1_{\mathrm{rd}}([a, b]_\mathbb{T}, \mathbb{R})$ *with* $\psi(b) = \phi(a) = 0$ *and* $\psi(t) - \phi(t) \neq 0$ *for all* $t \in [a, b]_\mathbb{T}$. *Set* $\Psi, \Phi \in C_{\mathrm{rd}}([a, b]_\mathbb{T}, \mathbb{R})$ *by*

$$\Psi(t, s) := \begin{cases} \phi(s), & s \in [a, t)_\mathbb{T} \\ \psi(s), & s \in [t, b]_\mathbb{T} \end{cases} \quad \text{and} \quad \Phi(t, s) := \Psi^{\Delta_s}(t, s) \tag{5.1}$$

for $s, t \in [a, b]_\mathbb{T}$. *Then*

$$f(t) = \frac{1}{\psi(t) - \phi(t)} \int_a^b \left[\Psi(t, \eta) f(\eta)\right]^{\Delta_\eta} \Delta\eta$$
$$= \frac{1}{\psi(t) - \phi(t)} \left(\int_a^b \Phi(t, \eta) f^\sigma(\eta) \Delta\eta + \int_a^b \Psi(t, \eta) f^\Delta(\eta) \Delta\eta \right)$$

is true for all $t \in [a, b]_\mathbb{T}$.

Theorem 5.2. *Assume that* $a, b \in \mathbb{T}$, $f \in C^1_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$, *and that* $\psi, \phi \in C^1_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$ *with* $\psi(b) = \phi(a) = 0$ *and* $\psi(t) - \phi(t) \neq 0$ *for all* $t \in [a,b]_{\mathbb{T}}$. *Then*

$$\left| f(t) - \frac{1}{\psi(t) - \phi(t)} \int_a^b \Phi(t,\eta) f^\sigma(\eta) \Delta\eta \right| \leq \frac{M}{|\psi(t) - \phi(t)|} \left(\int_a^b |\Psi(t,\eta)| \Delta\eta \right)$$

is true for all $t \in [a,b]_{\mathbb{T}}$, *where* Ψ, Φ *are as introduced in (5.1) and* $M := \sup_{\eta \in (a,b)} |f^\Delta(\eta)|$.

Remark 2. Letting $\phi(t) = h_k(t,a)$ and $\psi(t) = h_k(t,b)$ for some $k \in \mathbb{N}$, we obtain the results of §3, which reduce to the results in [2, § 3] by letting $k = 1$. This is for Ostrowski-polynomial type inequalities.

Remark 3. For instance, we may let $\phi(t) = \mathrm{e}_\lambda(t,a) - 1$ and $\psi(t) = \mathrm{e}_\lambda(t,b) - 1$ for some $\lambda \in \mathcal{R}^+([a,b]_{\mathbb{T}}, \mathbb{R}^+)$ to obtain new Ostrowski-exponential type inequalities.

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