ON THE BEHAVIOR OF $r-$DERIVATIVE NEAR THE ORIGIN OF SINE SERIES WITH CONVEX COEFFICIENTS

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Abstract: In this paper we will give the behavior of the $r-$derivative near origin of sine series with convex coefficients.

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Sine Series With Convex Coefficients

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1. Introduction and Preliminaries

Let us denote by

$$\sum_{n=1}^{\infty} a_n \sin nx, \tag{1.1}$$

the sine series of the function $f(x)$ with coefficients a_n such that $a_n \downarrow 0$ or $a_n \to 0$ and $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1} \geq 0$, $\Delta a_n = a_n - a_{n+1}$. It is a known fact that under these conditions, series (1.1) converges uniformly in the interval $\delta \leq x \leq 2\pi - \delta$, $\forall \delta > 0$ (see [2, p. 95]). In the following we will denote by $g(x)$ the sum of the series (1.1), i.e

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx. \tag{1.2}$$

Many authors have investigated the behaviors of the series (1.1), near the origin with convex coefficients. Young in [9] gave the estimation for $|g(x)|$ near the origin from the upper side. Later Salem (see [4], [5]) proved the following estimation for the behavior of the function $g(x)$ near the origin

$$g(x) \sim m a_m,$$

for

$$\frac{\pi}{m+1} < x \leq \frac{\pi}{m}, \quad m = 1, 2, \ldots .$$

Hartman and Winter (see [3]), proved that

$$\lim_{x \to 0} \frac{g(x)}{x} = \sum_{n=1}^{\infty} n a_n,$$

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holds for $a_n \downarrow 0$. In this context Telyakovskii (see [7]) has proved the behavior near the origin of the sine series with convex coefficients. He has compared his own results with those of Shogunbenkov (see [6]) and Aljancic et al. (see [1]).

In the sequel we will mention some results which are useful for further work. Dirichlet's kernels are denoted by

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}},$$

$$\widetilde{D}_n(t) = \sum_{k=1}^{n} \sin kt = \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}},$$

and

$$\overline{D}_n(t) = -\frac{1}{2}\cot\frac{t}{2} + \widetilde{D}_n(t) = -\frac{\cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}.$$

Let $E_n(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{ikt}$ and $E_{-n}(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{-ikt}$, then the following holds:

Lemma 1.1 ([8]). *Let r be a non-negative integer. Then for all $0 < x \leq \pi$ and all $n \geq 1$ the following estimates hold*

1. $\left|E_{-n}{}^{(r)}(x)\right| \leq \frac{4\pi n^r}{|x|}$;
2. $\left|\widetilde{D}_n^{(r)}(x)\right| \leq \frac{4\pi n^r}{|x|}$;
3. $\left|\overline{D}_n{}^{(r)}(x)\right| \leq \frac{4\pi n^r}{|x|} + O\left(\frac{1}{|x|^{r+1}}\right)$.

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2. Results

Theorem 2.1. *Let a_n be a sequence of scalars such that:*

1. $a_n \downarrow 0$;
2. $\sum_{n=1}^{\infty} n^r \Delta a_n < \infty$, *for* $r = 0, 1, 2, \dots,$

then for $\frac{\pi}{m+1} < x \leq \frac{\pi}{m}$, $m = 1, 2, \dots$ *the following estimate is valid*

$$g^{(r)}(x) = \sum_{n=1}^{m} n^r a_n \left(nx + \frac{r\pi}{2}\right) + O\left\{\sum_{n=1}^{m} a_n \left[n^r \left(\frac{n}{m} + \frac{r}{2}\right)^3 + n^3 m^{r-3}\right]\right\} + o(m).$$

Proof. Applying Abel's transform we obtain

$$g(x) = \sum_{n=1}^{\infty} \Delta a_n \widetilde{D}_n(x), \tag{2.1}$$

where $\widetilde{D}_n(x) = \sum_{k=1}^{n} \sin kx$ is Dirichlet's conjugate kernel. Let us denote by $g^{(r)}(x)$ the $r-$th derivatives for the function g. Let

$$\sum_{n=1}^{\infty} \Delta a_n \widetilde{D}_n^{(r)}(x), \tag{2.2}$$

be the r-th derivatives of the series in the relation (2.1).

From the given conditions in the theorem and Lemma 1.1(2), series (2.2) converges uniformly in $(0, \pi]$, so the following relation holds

$$g^{(r)}(x) = \sum_{n=1}^{\infty} \Delta a_n \widetilde{D}_n^{(r)}(x). \tag{2.3}$$

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From the last relation we have

$$g^{(r)}(x) = \sum_{n=1}^{m} \Delta a_n \widetilde{D}_n^{(r)}(x) + \sum_{n=m+1}^{\infty} \Delta a_n \widetilde{D}_n^{(r)}(x) = I_1(x) + I_2(x). \tag{2.4}$$

In the following we will estimate sums $I_1(x)$ and $I_2(x)$. Let us start with estimation of the second sum. From the second condition in Lemma 1.1, the second condition of the theorem and fact that $\frac{\pi}{m+1} < x \leq \frac{\pi}{m}$, we have

$$I_2(x) \leq 4\pi \cdot \sum_{n=m+1}^{\infty} \Delta a_n \frac{n^r}{x} \leq 8m \sum_{n=m+1}^{\infty} n^r \Delta a_n = o(m). \tag{2.5}$$

For the first sum we have the following estimation

$$I_1(x) = \sum_{n=1}^{m} \Delta a_n \widetilde{D}_n^{(r)}(x) = \sum_{n=1}^{m} a_n \left[\widetilde{D}_n^{(r)}(x) - \widetilde{D}_{n-1}^{(r)}(x)\right] - a_{m+1}\widetilde{D}_m^{(r)}(x),$$

where $\widetilde{D}_0^{(r)}(x) = 0$. Knowing that

$$\widetilde{D}_n^{(r)}(x) - \widetilde{D}_{n-1}^{(r)}(x) = n^r \sin\left(nx + \frac{r\pi}{2}\right),$$

taking into consideration Lemma 1.1 and the conditions in Theorem 2.1, we have

$$I_1(x) = \sum_{n=1}^{m} n^r \sin\left(nx + \frac{r\pi}{2}\right) + O(m^{r+1}a_m).$$

In the last relation we can use the known fact that $\sin x = x + O(x^3)$ for $x \to 0$. The following relation then holds

$$I_1(x) = \sum_{n=1}^{m} n^r a_n \left(nx + \frac{r\pi}{2}\right) + O\left[\sum_{n=1}^{m} n^r a_n \left(nx + \frac{r\pi}{2}\right)^3\right] + 8m^{r+1}a_m.$$

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Taking into consideration the fact that a_n is a monotone sequence we obtain

$$ma_m \leq \frac{4}{m^3}\sum_{n=1}^{m} n^3 a_n,$$

from which it follows that

$$m^{r+1}a_m \leq 4m^{r-3}\sum_{n=1}^{m} n^3 a_n.$$

From the above relations we have the following estimation for $I_1(x)$,

$$I_1(x) = \sum_{n=1}^{m} n^r a_n \left(nx + \frac{r\pi}{2}\right) + O\left\{\sum_{n=1}^{m} a_n \left[n^r \left(nx + \frac{r\pi}{2}\right)^3 + n^3 m^{r-3}\right]\right\}. \tag{2.6}$$

Now proof of Theorem 2.1 follows from (2.4), (2.5) and (2.6). ∎

Remark 1. The above result is a generalization of that given in [7].

Corollary 2.2. *Let a_n be sequence of scalars such that $a_n \downarrow 0$. Then for $\frac{\pi}{m+1} < x \leq \frac{\pi}{m}$, $m = 1, 2, \dots,$ the following relation holds*

$$g(x) = \sum_{n=1}^{m} n a_n x + O\left(\frac{1}{m^3}\sum_{n=1}^{m} n^3 a_n\right).$$

Theorem 2.3. *Let (a_n) be a sequence of scalars such that the following conditions hold:*

1. $a_n \to 0$ *and* $\Delta a_n \geq 0$
2. $\sum_{n=1}^{\infty} n^{r+1}\Delta^2 a_n < \infty$, *for* $r = 0, 1, 2, \dots$.

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Then for $\frac{\pi}{m+1} < x \leq \frac{\pi}{m}$, $m = 1, 2, \dots$ *the following estimate is valid*

$$g^{(r)}(x) \leq M(r) \left\{ m^{r+2}[a_m + \Delta a_m] + \sum_{n=1}^{m-1} n^{r+1} \left(\frac{n}{m} + \frac{r}{2} \right) \Delta a_n + o(m) \right\},$$

where $M(r)$ *is a constant which depends only on* r.

Proof. Applying Abel's transform we obtain

$$\sum_{n=1}^{\infty} n^r \Delta a_n = \sum_{n=1}^{\infty} \Delta^2 a_n \sum_{i=1}^{n} i^r \leq \sum_{n=1}^{\infty} n^{r+1} \Delta^2 a_n < \infty.$$

From the convergence of the series $\sum_{n=1}^{\infty} n^r \Delta a_n$ and Condition 2 in Lemma 1.1 we obtain that

$$\sum_{n=1}^{\infty} \Delta a_n \widetilde{D}_n^{(r)}(x)$$

converges uniformly in $(0, \pi]$, so the following relation is valid

$$g^{(r)}(x) = \sum_{n=1}^{\infty} \Delta a_n \widetilde{D}_n^{(r)}(x).$$

From the other side we have that

$$\widetilde{D}_n^{(r)}(x) = \frac{1}{2} \left(\cot \frac{x}{2} \right)^{(r)} + \overline{D}_n{}^{(r)}(x),$$

respectively,

$$g^{(r)}(x) = \frac{a_m}{2} \left(\cot \frac{x}{2} \right)^{(r)} + \sum_{n=1}^{m-1} \Delta a_n \widetilde{D}_n{}^{(r)}(x) + \sum_{n=m}^{\infty} \Delta a_n \overline{D}_n{}^{(r)}(x)$$

(2.7) $$= \frac{a_m}{2} \left(\cot \frac{x}{2} \right)^{(r)} + J_1(x) + J_2(x).$$

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For $\frac{\pi}{m+1} < x \leq \frac{\pi}{m}$, we will have the following estimation

$$\left(\cot\frac{x}{2}\right)^{(r)} \leq \frac{M}{x^{r+1}} \leq M(r)m^{r+2}. \tag{2.8}$$

On the other hand it is known that

$$\widetilde{D}_n^{(r)}(x) = \sum_{i=1}^{n} i^r \sin\left(ix + \frac{r\pi}{2}\right) \leq n^{r+1}\left(nx + \frac{r\pi}{2}\right) \leq \pi n^{r+1}\left(\frac{n}{m} + \frac{r}{2}\right).$$

From last two relations we have the following estimation for $J_1(x)$,

$$J_1(x) \leq \pi \sum_{n=1}^{m-1} n^{r+1}\left(\frac{n}{m} + \frac{r}{2}\right)\Delta a_n. \tag{2.9}$$

In the following we will estimate the second sum $J_2(x)$. Applying the Abel transform we have

$$\begin{aligned} J_2(x) &= \sum_{n=m}^{\infty} \Delta^2 a_n \sum_{i=0}^{n} \overline{D}_i^{(r)}(x) - \Delta a_m \sum_{i=0}^{m-1} \overline{D}_i^{(r)}(x) \\ &= \sum_{n=m}^{\infty} \Delta^2 a_n \left\{\sum_{i=0}^{n} \overline{D}_i^{(r)}(x) - \sum_{i=0}^{m-1} \overline{D}_i^{(r)}(x)\right\}, \end{aligned}$$

because $\sum_{n=m}^{\infty} \Delta^2 a_n = \Delta a_m$.

Taking into consideration Lemma 1.1, we have the following estimation

$$\sum_{i=0}^{n} \left|\overline{D}_i^{(r)}(x)\right| \leq 4\pi \sum_{i=0}^{n} \frac{i^r}{x} + M \sum_{i=0}^{n} \frac{1}{x^{r+1}} \leq M(r)mn^{r+1}.$$

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In a similar way we can prove that

$$\sum_{i=0}^{m-1} \left| \overline{D}_i^{(r)}(x) \right| \leq M(r) m^{r+2}.$$

Now the estimation of $J_2(x)$ can be expressed in the following way

$$|J_2(x)| \leq M(r) \left\{ m \sum_{n=m}^{\infty} n^{r+1} \Delta^2 a_n + m^{r+2} \Delta a_m \right\} = M(r)\{m^{r+2} \Delta a_m + o(m)\}. \tag{2.10}$$

The proof of the theorem follows from relations (2.7), (2.8), (2.9) and (2.10). ∎

Remark 2. The above theorem is a generalization of the result obtained in [7], from the upper side for the case $m \geq 11$.

Corollary 2.4. *Let $a_n \to 0$ be a convex sequence of scalars. If*

$$\frac{\pi}{m+1} < x \leq \frac{\pi}{m}, m \geq 11$$

then the following estimation holds

$$\frac{a_m}{2} \cot \frac{x}{2} + \frac{1}{2m} \sum_{n=1}^{m-1} n^2 \Delta a_n \leq g(x) \leq \frac{a_m}{2} \cot \frac{x}{2} + \frac{6}{m} \sum_{n=1}^{m-1} n^2 \Delta a_n.$$

Remark 3. Telyakovskii compared his own results with those given by Hartman, Winter (see [3]), then with results given by Salem (see [4], [5]). Taking into consideration Corollary 2.2 and Corollary 2.4 for the case $r = 0$, we can compare our results with the results mentioned above.

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