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CERTAIN INEQUALITIES FOR CONVEX FUNCTIONS

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ABSTRACT. Classical inequalities like Jensen and its reverse are used to obtain some elementary numerical inequalities for convex functions. Furthermore, imposing restrictions on the data points several new constrained inequalities are given.

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1. INTRODUCTION

It is well known ([1], [2]) that a continous function, f, convex in a real interval $I \subseteq \mathbb{R}$ has the property

$$f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k a_k\right) \le \frac{1}{P_n}\sum_{k=1}^{n} p_k f(a_k), \tag{1.1}$$

where $a_k \in I$, $1 \le k \le n$ are given data points and $p_1, p_2, \dots, p_n$ is a set of nonnegative real numbers constrained by $\sum_{k=1}^{j} p_k = P_j$. If f is concave the preceding inequality is reversed.

A broad consideration of inequalities for convex functions can be found, among others, in ([3], [4]). Furthermore, in [5] a reverse of Jensen's inequality is presented. It states that if $p_1, p_2, \dots, p_n$ are real numbers such that $p_1 > 0, p_k \le 0$ for $2 \le k \le n$ and $P_n > 0$, then

$$f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k a_k\right) \ge \frac{1}{P_n}\sum_{k=1}^{n} p_k f(a_k) \tag{1.2}$$

holds, where $f : I \to \mathbb{R}$ is a convex function in I and $a_k \in I$, $1 \le k \le n$ are such that $\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k \in I$. If f is concave (1.2) is reversed. Our aim in this paper is to use the preceding

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results to get new inequalities for convex functions. In addition, when the x_k's are constrained some inequalities are obtained.

2. UNCONSTRAINED INEQUALITIES

In the sequel, applying the preceding results and some numerical identities, some elementary inequalities are obtained. We begin with:

Theorem 2.1. *Let $a_0, a_1, \dots, a_n$ be nonnegative real numbers. Then, the following inequality*

$$\exp\left[\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{2^n}\right] \le \frac{1}{8^n}\left[\sum_{k=0}^{n}\binom{n}{k}\frac{e^{a_k}}{(1+a_k)^2}\right]\left[\sum_{k=0}^{n}\binom{n}{k}(1+a_k)\right]^2$$

holds.

Proof. Since $f(t) = \frac{e^t}{(1+t)^2}$ is convex in $[0,+\infty)$, then setting $p_k = \binom{n}{k}\big/2^n$, $0 \le k \le n$, into (1.1) and taking into account the well known identity $\sum_{k=0}^{n}\binom{n}{k} = 2^n$, we have

$$\exp\left(\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{2^n}\right)\left[1+\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}a_k\right]^{-2} \le \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{e^{a_k}}{(1+a_k)^2}.$$

After rearranging terms, the inequality claimed immediately follows and the proof is complete. □

Theorem 2.2. *Let $p_1, p_2, \dots, p_n$ be a set of nonnegative real numbers constrained by $\sum_{k=1}^{j} p_k = P_j$. If $a_1, a_2, \dots, a_n$ are positive real numbers, then*

$$\left[\prod_{k=1}^{n}\left(a_k+\sqrt{1+a_k^2}\right)^{p_k}\right]^{\frac{1}{P_n}} \le \frac{1}{P_n}\sum_{k=1}^{n}p_k a_k + \sqrt{1+\left(\frac{1}{P_n}\sum_{k=1}^{n}p_k a_k\right)^2}$$

holds.

Proof. Let $f:(0,+\infty)\to\mathbb{R}$ be the function defined by $f(t) = \ln(t+\sqrt{1+t^2})$. Then, we have $f'(t) = \frac{1}{\sqrt{1+t^2}} > 0$ and $f''(t) = -\frac{t}{\sqrt{(1+t^2)^3}} \le 0$. Therefore, f is concave and applying (1.1) yields

$$\ln\left[\frac{1}{P_n}\sum_{k=1}^{n}p_k a_k + \sqrt{1+\left(\frac{1}{P_n}\sum_{k=1}^{n}p_k a_k\right)^2}\right]$$
$$\ge \frac{1}{P_n}\sum_{k=1}^{n}p_k \ln\left(a_k+\sqrt{1+a_k^2}\right) = \ln\left[\prod_{k=1}^{n}\left(a_k+\sqrt{1+a_k^2}\right)^{p_k}\right]^{\frac{1}{P_n}}.$$

Taking into account that $f(t) = \log(t)$ is injective, the statement immediately follows and this completes the proof. □

Setting $p_k = \dfrac{1}{n}$, $1 \le k \le n$ into the preceding result we get

Corollary 2.3. *Let $a_1, a_2, \dots, a_n$ be a set of positive real numbers. Then*

$$\prod_{k=1}^{n}\left(a_k+\sqrt{1+a_k^2}\right)^{1/n} \le \frac{1}{n}\left(\sum_{k=1}^{n}a_k + \sqrt{n^2+\left(\sum_{k=1}^{n}a_k\right)^2}\right)$$

holds.

Let T_n be the n^{th} triangular number defined by $T_n = \frac{n(n+1)}{2}$. Then, setting $a_k = T_k$, $1 \leq k \leq n$ into the preceding result, we get

Corollary 2.4. *For all* $n \geq 1$,

$$\prod_{k=1}^{n}\left(T_k + \sqrt{1+T_k^2}\right)^{\frac{1}{n}} \leq \frac{1}{3}\left(T_{n+1} + \sqrt{9+T_{n+1}^2}\right)$$

holds.

An interesting result involving Fibonacci numbers that can be proved using convex functions is the following

Theorem 2.5. *Let n be a positive integer and ℓ be a whole number. Then,*

$$\left(F_1^{\ell} + F_2^{\ell} + ... + F_n^{\ell}\right)\left[\frac{1}{F_1^{\ell-4}} + \frac{1}{F_2^{\ell-4}} + \cdots + \frac{1}{F_n^{\ell-4}}\right] \geq F_n^2 F_{n+1}^2$$

holds, where F_n is the n^{th} Fibonacci number defined by $F_0 = 0$, $F_1 = 1$ and for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

Proof. Taking into account that $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$, as is well known, and the fact that the function $f : (0,\infty) \to \mathbb{R}$, defined by $f(t) = 1/t$ is convex, we get after setting $p_i = \frac{F_i^2}{F_n F_{n+1}}$, $1 \leq i \leq n$ and $a_i = F_n F_i^{\ell-2}$, $1 \leq i \leq n$:

$$\frac{1}{\frac{F_1^{\ell}}{F_{n+1}} + \frac{F_2^{\ell}}{F_{n+1}} + \cdots + \frac{F_n^{\ell}}{F_{n+1}}} \leq \frac{1}{F_n^2 F_{n+1}}\left[\frac{1}{F_1^{\ell-4}} + \frac{1}{F_2^{\ell-4}} + \cdots + \frac{1}{F_n^{\ell-4}}\right].$$

From the preceding expression immediately follows

$$\left(F_1^{\ell} + F_2^{\ell} + \cdots + F_n^{\ell}\right)\left[\frac{1}{F_1^{\ell-4}} + \frac{1}{F_2^{\ell-4}} + \cdots + \frac{1}{F_n^{\ell-4}}\right] \geq F_n^2 F_{n+1}^2,$$

and this completes the proof. □

Finally, using the reverse Jensen's inequality, we state and prove:

Theorem 2.6. *Let $a_0, a_1, \ldots, a_n$ be positive real numbers such that $a_0 \geq a_1 \geq \cdots \geq a_n$ and let $p_0 = n(n+1)$ and $p_k = -k, k = 1, 2, \ldots, n$. Then*

$$\left(\sum_{k=0}^{n} p_k a_k\right)\left(\sum_{k=0}^{n} \frac{p_k}{a_k}\right) \leq \binom{n+1}{2}^2. \tag{2.1}$$

Proof. Setting $f(t) = \frac{1}{t}$, that is convex in $(0,+\infty)$, and taking into account that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ from (1.2) we have

$$f\left(\frac{2}{n(n+1)}\sum_{k=0}^{n} p_k a_k\right) \geq \frac{2}{n(n+1)}\sum_{k=0}^{n} p_k f(a_k)$$

or

$$\left(\frac{2}{n(n+1)}\sum_{k=0}^{n} p_k a_k\right)^{-1} \geq \frac{2}{n(n+1)}\sum_{k=0}^{n} \frac{p_k}{a_k}$$

from which, after rearranging terms, (2.1) immediately follows and the proof is complete. □

3. CONSTRAINED INEQUALITIES

In the sequel, imposing restrictions on $x_1, x_2, \dots, x_n$, some inequalities with constraints are given. We begin with the following.

Theorem 3.1. *Let $p_1, p_2, \dots, p_n \in [0,1)$ be a set of real numbers constrained by $\sum_{k=1}^{j} p_k = P_j$. If $x_1, x_2, \dots, x_n$ are positive real numbers such that $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1$, then*

$$\left(\sum_{k=1}^{n} p_k x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k^{p_k}}\right) \geq P_n^2$$

holds.

Proof. Taking into account the weighted AM-HM inequality, we have

$$\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k \geq \frac{P_n}{\sum_{k=1}^{n}\left(\dfrac{p_k}{x_k}\right)}.$$

Since $0 \leq p_k < 1$ for $1 \leq k \leq n$, then $\frac{p_k}{x_k} \leq \frac{1}{x_k^{p_k}}$. From which, we get

$$\frac{P_n}{\sum_{k=1}^{n}\left(\frac{p_k}{x_k}\right)} \geq \frac{P_n}{\sum_{k=1}^{n} \frac{1}{x_k^{p_k}}}.$$

Then,

$$\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k \geq \frac{P_n}{\sum_{k=1}^{n} \frac{1}{x_k^{p_k}}}$$

and the statement immediately follows. □

Corollary 3.2. *If $x_1, x_2, \dots, x_n$ are positive real numbers such that $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1$, then*

$$\frac{1}{n} \leq \sum_{k=1}^{n} \frac{1}{x_k^{1/x_k}}.$$

Proof. Setting $p_k = 1/x_k$, $1 \leq k \leq n$ into Theorem 3.1 yields

$$\left(\sum_{k=1}^{n} p_k x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k^{p_k}}\right) = n\left(\sum_{k=1}^{n} \frac{1}{x_k^{1/x_k}}\right) \geq \left(\sum_{k=1}^{n} \frac{1}{x_k}\right)^2 = 1$$

completing the proof. □

Finally, we give two inequalities similar to the ones obtained in [6] for the triangle.

Theorem 3.3. *Let a, b and c be positive real numbers such that $a + b + c = 1$. Then, the following inequality*

$$a^{a(a+2b)} \cdot b^{b(b+2c)} \cdot c^{c(c+2a)} \geq \frac{1}{3}$$

holds.

Proof. Since $a + b + c = 1$, then $a^2 + b^2 + c^2 + 2(ab + bc + ca) = 1$. Therefore, choosing $p_1 = a^2$, $p_2 = b^2$, $p_3 = c^2$, $p_4 = 2ab$, $p_5 = 2bc$, $p_6 = 2ca$ and $x_1 = 1/a$, $x_2 = 1/b$, $x_3 = 1/c$,

$x_4 = 1/a$, $x_5 = 1/b$, $x_6 = 1/c$, and applying Jensen's inequality to the function $f(t) = \ln t$ that is concave for all $t \geq 0$, we obtain

$$\ln\left(a^2\frac{1}{a} + b^2\frac{1}{b} + c^2\frac{1}{c} + 2ab\frac{1}{a} + 2bc\frac{1}{b} + 2ca\frac{1}{c}\right) \geq a^2\ln\frac{1}{a} + b^2\ln\frac{1}{b} + c^2\ln\frac{1}{c} + 2ab\ln\frac{1}{a} + 2bc\ln\frac{1}{b} + 2ca\ln\frac{1}{c},$$

from which, we get

$$\ln 3 \geq \ln\left(\frac{1}{a^{a(a+2b)}\cdot b^{b(b+2c)}\cdot c^{c(c+2a)}}\right)$$

and this completes the proof. □

Theorem 3.4. *Let* a, b, c *be positive numbers such that* $ab + bc + ca = abc$. *Then,*

$$\sqrt[b]{a}\sqrt[c]{b}\sqrt[a]{c}(a+b+c) \geq abc$$

holds.

Proof. Since $ab + bc + ca = abc$, then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. So, choosing $p_1 = \frac{1}{a}$, $p_2 = \frac{1}{b}$, $p_3 = \frac{1}{c}$ and $x_1 = ab$, $x_2 = bc$, $x_3 = ca$, and applying Jensen's inequality to $f(t) = \ln t$ again, we get

$$\ln(a+b+c) \geq \frac{1}{a}\ln ab + \frac{1}{b}\ln bc + \frac{1}{c}\ln ca$$

or

$$a + b + c \geq a^{\frac{1}{a}+\frac{1}{c}}\cdot b^{\frac{1}{b}+\frac{1}{a}}\cdot c^{\frac{1}{c}+\frac{1}{b}}.$$

Now, taking into account that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, we obtain: $a + b + c \geq a^{1-\frac{1}{b}}\cdot b^{1-\frac{1}{c}}\cdot c^{1-\frac{1}{a}}$, from which the statement immediately follows and the proof is complete. □

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