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ON AN OPEN PROBLEM POSED IN THE PAPER "INEQUALITIES OF POWER-EXPONENTIAL FUNCTIONS"

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ABSTRACT. In the article, some inequalities of power-exponential functions are obtained. An answer to an open problem proposed by Feng Qi and Lokenath Debnath in the paper [F. Qi and L. Debnath, *Inequalities of power-exponential functions*, J. Inequal. Pure Appl. Math. **1** (2000), no. 2, Art. 15. http://jipam.vu.edu.au/article.php?sid=109] is given.

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1. INTRODUCTION

In the paper [6], the following inequalities for power-exponential functions were proved

$$\frac{y^{x^y}}{x^{y^x}} > \frac{y}{x} > \frac{y^x}{x^y}, \qquad \left(\frac{y}{x}\right)^{xy} > \frac{y^y}{x^x}, \tag{1.1}$$

where $0 < x < y < 1$ or $1 < x < y$. At the end of the paper, F. Qi and L. Debnath proposed the following problem.

Problem 1.1. Adopting the following notations:

$$f_1(x,y) = x, \tag{1.2}$$

$$f_{k+1}(x,y) = x^{f_k(y,x)}, \tag{1.3}$$

$$F_k(x,y) = \frac{f_k(y,x)}{f_k(x,y)} \tag{1.4}$$

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for $0 < x < y < 1$ or $1 < x < y$, and $k \geq 1$, prove or disprove the following inequalities:

$$F_{2k-1}(x,y) > F_{2k}(x,y), \tag{1.5}$$

$$F_{2k+4}(x,y) > F_{2k+1}(x,y). \tag{1.6}$$

That is,

$$F_2(x,y) < F_1(x,y) < F_4(x,y) < F_3(x,y) < F_6(x,y) < \cdots . \tag{1.7}$$

There is a rich literature on inequalities for power-exponential functions, see [1, 2, 3, 4, 5, 6]. It is well-known that, if $0 < x < y < e$, then

$$x^y < y^x. \tag{1.8}$$

If $e < x < y$, then the inequality (1.8) is reversed. If $0 < x < e$, then

$$(e+x)^{e-x} > (e-x)^{e+x}. \tag{1.9}$$

For details about these inequalities, please refer to [1, p. 82] and [4, p. 365]. In what follows we will continue to use the notations (1.2), (1.3) and (1.4).

2. MAIN RESULTS

Theorem 2.1. *Let $e < x < y$. Then the following inequalities hold:*

$$f_{2n}(y,x) < f_{2n}(x,y) < f_{2n+1}(x,y) < f_{2n+1}(y,x) < f_{2n+2}(y,x). \tag{2.1}$$

Remark 1. The inequalities (2.1) can be rewritten as follows

$$y^x < x^y < x^{y^x} < y^{x^y} < y^{x^{y^x}} < x^{y^{x^y}} < x^{y^{x^{y^x}}} < y^{x^{y^{x^y}}} < \cdots .$$

Proof. We prove (2.1) by mathematical induction. It is evident that $f_n(x,y) < f_{n+1}(x,y)$, $f_n(y,x) < f_{n+1}(y,x)$. The inequality $f_2(y,x) = y^x < x^y = f_2(x,y)$, is known and it implies $x^{y^x} < y^{x^y}$ which is $f_3(x,y) < f_3(y,x)$. Suppose that (2.1) holds for $n \leq k$. To prove (2.1) for $n = k+1$, it is sufficient to show that $f_{2k+2}(y,x) < f_{2k+2}(x,y)$. In fact, if $f_{2k+2}(y,x) < f_{2k+2}(x,y)$, then

$$f_{2k+3}(x,y) = x^{f_{2k+2}(y,x)} < y^{f_{2k+2}(x,y)} = f_{2k+3}(y,x).$$

The inequality $f_{2k+2}(y,x) < f_{2k+2}(x,y)$ is equivalent to

$$f_{2k}(x,y)\ln y - f_{2k}(y,x)\ln x > \ln\ln y - \ln\ln x. \tag{2.2}$$

We prove

$$f_{2k}(x,y)\ln y - f_{2k}(y,x)\ln x > \ln y - \ln x, \tag{2.3}$$

which gives (2.2), because

$$\ln y - \ln x > \ln\ln y - \ln\ln x.$$

The inequality (2.3) can be rewritten as

$$(f_{2k}(x,y) - 1)\ln y > (f_{2k}(y,x) - 1)\ln x$$

or as

$$\frac{\ln y}{\ln x} > \frac{\int_0^{f_{2k-1}(x,y)} y^t dt \ln y}{\int_0^{f_{2k-1}(y,x)} x^t dt \ln x},$$

which is equivalent to

$$\int_0^{f_{2k-1}(y,x)} x^t dt - \int_0^{f_{2k-1}(x,y)} y^t dt > 0. \tag{2.4}$$

Denote by

$$H(x,y)=\int_0^{f_{2k-1}(y,x)} x^t dt-\int_0^{f_{2k-1}(x,y)} y^t dt.$$

The direct computation yields

$$\frac{\partial H(x,y)}{\partial y}=f_{2k}(x,y)\frac{\partial f_{2k-1}(y,x)}{\partial y}-f_{2k}(y,x)\frac{\partial f_{2k-1}(x,y)}{\partial y}-\int_0^{f_{2k-1}(x,y)} t y^{t-1} dt.$$

Using mathematical induction, we obtain

$$\begin{aligned}(2.5)\quad \frac{\partial f_{2k-1}(y,x)}{\partial y} &= f_{2k-1}(y,x)f_{2k-2}(x,y)\cdots f_2(x,y)\ln^{k-1}x\ln^{k-1}y\\ &\quad+\sum_{j=1}^{k-1} f_{2k-1}(y,x)f_{2k-2}(x,y)\cdots f_{2k-2j}(x,y)\frac{1}{y}\ln^{j-1}x\ln^{j-1}y,\quad \text{for}\quad k>1.\end{aligned}$$

$$\begin{aligned}(2.6)\quad \frac{\partial f_{2k-1}(x,y)}{\partial y} &= f_{2k-1}(x,y)f_{2k-2}(y,x)\cdots f_2(y,x)\frac{x}{y}\ln^{k-1}x\ln^{k-2}y\\ &\quad+\sum_{j=1}^{k-2} f_{2k-1}(x,y)f_{2k-2}(y,x)\cdots f_{2k-2j-1}(x,y)\frac{1}{y}\ln^{j}x\ln^{j-1}y,\quad \text{for}\quad k>1.\end{aligned}$$

Using (2.5) and (2.6) we get

$$\begin{aligned}\frac{\partial H(x,y)}{\partial y} &= f_{2k}(x,y)\cdots f_2(x,y)\ln^{k-1}x\ln^{k-1}y\\ &\qquad+\sum_{j=1}^{k-1} f_{2k}(x,y)\cdots f_{2k-2j}(x,y)\frac{1}{y}\ln^{j-1}x\ln^{j-1}y\\ &\qquad- f_{2k}(y,x)\cdots f_2(y,x)\frac{x}{y}\ln^{k-1}x\ln^{k-2}y\\ &\qquad-\sum_{j=1}^{k-2} f_{2k}(y,x)\cdots f_{2k-2j-1}(x,y)\frac{1}{y}\ln^{j}x\ln^{j-1}y-\int_0^{f_{2k-1}(x,y)} t y^{t-1}dt\\ &= h_1(x,y)+h_2(x,y)+h_3(x,y),\end{aligned}$$

where

$$h_1(x,y)=\left(f_{2k}(x,y)\cdots f_2(x,y)\ln y-f_{2k}(y,x)\cdots f_2(y,x)\frac{x}{y}\right)\ln^{k-1}x\ln^{k-2}y,$$

$$h_2(x,y)=\frac{1}{y}\sum_{j=1}^{k-2}\left(f_{2k}(x,y)\cdots f_{2k-2j-2}(x,y)\ln y-f_{2k}(y,x)\cdots f_{2k-2j-1}(x,y)\right)\ln^{j}x\ln^{j-1}y,$$

$$h_3(x,y)=f_{2k}(x,y)f_{2k-1}(y,x)f_{2k-2}(x,y)\frac{1}{y}-\int_0^{f_{2k-1}(x,y)} t y^{t-1}dt.$$

Since (2.1) holds for $n = 1, \dots, k$ and $\ln y - \frac{x}{y} > 0$, $\ln y > 1$, we obtain that $h_1(x, y) > 0$, $h_2(x, y) > 0$. Next, we get

$$h_3(x,y) = f_{2k}(x,y) f_{2k-1}(y,x) f_{2k-2}(x,y) \frac{1}{y} - f_{2k}(y,x) f_{2k-1}(x,y) \frac{1}{y \ln y} + f_{2k}(y,x) \frac{1}{y \ln^2 y} - \frac{1}{y \ln^2 y} > 0$$

following the same arguments. So we have $\frac{\partial H(x,y)}{\partial y} > 0$. This implies that (2.4) holds because $H(x, x) = 0$. The proof is complete. □

Theorem 2.2. *Let $e < x < y$. The inequalities $F_{2n+2}(x,y) < F_{2n-1}(x,y)$ hold.*

Proof. Put $f_0(x,y) = f_0(y,x) = 1$ and $f_{-1}(x,y) = f_{-1}(y,x) = 0$. The inequality

$$F_{2n+2}(x,y) < F_{2n-1}(x,y)$$

is equivalent to

$$f_{2n+2}(y,x) f_{2n-1}(x,y) < f_{2n-1}(y,x) f_{2n+2}(x,y),$$

which can be rewritten as

$$f_{2n-2}(x,y) \ln y + f_{2n+1}(y,x) \ln x > f_{2n+1}(x,y) \ln y + f_{2n-2}(y,x) \ln x.$$

Rewriting the above inequality we have

$$\frac{f_{2n+1}(y,x) - f_{2n-2}(y,x)}{f_{2n+1}(x,y) - f_{2n-2}(x,y)} > \frac{\ln y}{\ln x},$$

which is equivalent to

$$\int_{f_{2n-3}(x,y)}^{f_{2n}(x,y)} y^t dt - \int_{f_{2n-3}(y,x)}^{f_{2n}(y,x)} x^t dt > 0. \tag{2.7}$$

The inequality (2.7) holds because

$$x < y, \quad f_{2n-1}(x,y) < f_{2n-1}(y,x), \quad f_{2n}(y,x) < f_{2n}(x,y).$$

□

Theorem 2.3. *Let $e < x < y$. The following inequalities hold:*

$$F_{2n}(x,y) < F_{2n-1}(x,y). \tag{2.8}$$

The proof of Theorem 2.3 is similar to the proof of Theorem 2.2 therefore, we omit it.

The answer to the open problem proposed by Feng Qi and Lokenath Debnath [6] is: for $e < x < y$ the inequalities $F_{2k-1}(x,y) > F_{2k}(x,y)$, $k = 1, 2, \dots$ hold and the inequalities $F_{2k+4}(x,y) > F_{2k+1}(x,y)$, $k = 0, 1, \dots$ are reversed.

REFERENCES

[1] P.S. BULLEN, *A Dictionary of Inequalities*, Pitman Monographs and Surveys in Pure and Applied Mathematics 97, Addison Wesley Longman Limited, 1998.

[2] LIU-QING HAN, BAI-NI GUO AND FENG QI, New proofs for inequalities of power-exponential functions, *Mathematics and Informatics Quarterly* **11**(3) (2001), 130–132, *RGMIA Research Report Collection*, **4**(1) (2001), Article 2, 9–13. [ONLINE: `http://www.staff.vu.edu.au/rgmia/v4n1.asp`].

[3] G.H. HARDY, J.E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, 2nd edition, Cambridge University Press, Cambridge, 1952.

[4] J.-C. KUANG, Applied Inequalities (Changyong Budengshi), 2nd edition, Hunan Education Press, Changsha, China, (Chinese), 1993.

[5] FENG QI, A method of constructing inequalities about ex, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.*,**8** (1997), 16–23.

[6] FENG QI AND L. DEBNATH, Inequalities of power-exponential functions, *J. Inequal. Pure and Appl. Math.*, **1**(2) (2000), Art. 15. [ONLINE: http://jipam.vu.edu.au/article.php?sid=109].