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HARDY-TYPE INEQUALITIES FOR HERMITE EXPANSIONS

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Abstract

Hardy-type inequalities are established for Hermite expansions for $f \in H^p(\mathbb{R})$, $0 < p \leq 1$.

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Key words: Atomic decomposition, Fourier-Hermite coefficient, Hardy spaces, Hermite functions.

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Contents

1. Introduction

Hardy's inequality for a Fourier transform $\mathcal{F}$ is stated as

$$\int_{\mathbb{R}} \frac{|\mathcal{F}f(\xi)|^p}{|\xi|^{2-p}} d\xi \leq C\|f\|^p_{\mathrm{Re}\, H^p} \qquad 0 < p \leq 1,$$

where $\mathrm{Re}\, H^p$ denotes the real Hardy space consisting of the boundary values of real parts of functions in the Hardy space H^p on the unit disc in the plane. Kanjin in [1] has proved Hardy's inequalities for Hermite and Laguerre expansions for functions in H^1. In [4] Satake has obtained Hardy's inequalities for Laguerre expansions for H^p where $0 < p \leq 1$. In connection with regularity properties of spherical means on $\mathbb{C}^n$, Thangavelu [6] has proved a Hardy's inequality for special Hermite functions. These type of inequalities for higher dimensional expansions are studied in [2], [3]. In this short note we obtain such inequalities for Hermite expansions for one dimension, namely for $f \in H^p(\mathbb{R}),\ 0 < p \leq 1$. In fact, it is to be noted from Theorem 2.1 that the resulting inequality for Hermite expansions $(0 < p \leq 1)$ is sharper than the inequalities for the classical Fourier transform as well as the Laguerre function expansion.

An H^p atom, $0 < p \leq 1$ is defined to be a function a satisfying the following conditions:

i. a is supported in an interval $[b, b+h]$

ii. $|a(x)| \leq h^{-1/p}$ almost everywhere and

iii. $\int_{\mathbb{R}} x^k a(x)dx = 0$ for all $k = 0, 1, 2, \ldots, \left[\frac{1}{p} - 1\right]$.

Making use of the atomic decomposition we define the Hardy space H^p to be the collection of all functions f satisfying $f = \sum_{k=0}^{\infty} \lambda_k a_k$, where a_j is an H^p - atom, λ_k is a sequence of complex numbers with $\sum |\lambda_k|^p < \infty$ and

$$C\|f\|_{H^p} \leq \left(\sum |\lambda_k|^p\right)^{\frac{1}{p}} \leq C'\|f\|_{H^p}.$$

For various other definitions of H^p-spaces we refer to Stein [5].

2. The Main Result

Let H_k denote the Hermite polynomials

$$H_k(x) = (-1)^k \frac{d^k}{dx^k}\left(e^{-x^2}\right)e^{x^2}, \qquad k = 0, 1, 2, \dots .$$

Then the Hermite functions $\tilde{h}_k$ are defined by

$$\tilde{h}_k(x) = H_k(x)e^{-\frac{1}{2}x^2}, \qquad k = 0, 1, 2, \dots .$$

The normalized Hermite functions h_k are defined as

$$h_k(x) = (2^k k!\sqrt{\pi})^{-\frac{1}{2}}\tilde{h}_k(x).$$

These functions $\{h_k\}$ form an orthonormal basis for $L^2(\mathbb{R})$. They are eigenfunctions for the Hermite operator $H = -\Delta + x^2$ with eigenvalues $2k + 1$. For more results concerning Hermite expansions, we refer to [7].

The following inequalities for Hermite functions are well known:

$$|h_k(x)| \leq Ck^{-\frac{1}{12}} \quad \text{and} \quad \left|\frac{d}{dx}h_k(x)\right| \leq Ck^{\frac{5}{12}}.$$

Using these inequalities and the relation

$$\frac{d}{dx}h_k(x) = \left(\frac{k}{2}\right)^{\frac{1}{2}} h_{k-1}(x) + \left(\frac{k+1}{2}\right)^{\frac{1}{2}} h_{k+1}(x)$$

we obtain the estimate

$$\left|\frac{d^m}{dx^m}h_k(x)\right| \leq Ck^{-\frac{1}{12}+\frac{m}{2}} \quad \text{for} \quad m = 0, 1, 2, \dots,$$

which can be verified easily by applying induction on m.

Theorem 2.1. *Let $\{h_k\}$ be the normalized Hermite functions on $\mathbb{R}$. Let $0 < p \leq 1$ and $m = \left[\frac{1}{p}\right]$. Then for every $f \in H^p(\mathbb{R})$, the Fourier - Hermite coefficient of f, namely,*

$$\hat{f}(k) = \int_{\mathbb{R}} f(x)h_k(x)dx, \qquad k = 0, 1, 2, 3, \dots$$

satisfies the inequality

$$\sum_{k=0}^{\infty} \frac{|\hat{f}(k)|^p}{(k+1)^\sigma} \leq C\|f\|_{H^p},$$

where C is a constant and $\sigma = \frac{2-p}{12}\left\{\frac{18m+11}{2m+1}\right\} = \left(\frac{3}{4} + \frac{1}{12m+6}\right)(2-p)$.

Proof. In order to prove the theorem, it is enough to prove that

$$\sum_{k=0}^{\infty} \frac{|\hat{f}(k)|^p}{(k+1)^\sigma} \leq C$$

for an H^p-atom f. Let f be an H^p atom. By considering the remainder term of the Taylor series expansion for $h_k(x)$, we write the Fourier-Hermite coefficient of f as

$$\hat{f}(k) = \frac{1}{m!} \int_b^{b+h} f(x) \frac{d^m}{dt^m} h_k(t)(x-b)^m dx,$$

where $t \in [b, x]$ and $m = \left[\frac{1}{p}\right]$.

Then

$$
\begin{aligned}
|\hat{f}(k)| &\leq Ch^m \int_b^{b+h} |f(x)| \left| \frac{d^m}{dt^m} h_k(t) \right| dx \\
&\leq Ch^m k^{-\frac{1}{12}+\frac{m}{2}} \int_b^{b+h} |f(x)| dx \\
&\leq Ch^m k^{-\frac{1}{12}+\frac{m}{2}} h^{-\frac{1}{p}+1}.
\end{aligned}
$$

Consider

$$
\begin{aligned}
\sum_{k=0}^{\infty} \frac{|\hat{f}(k)|^p}{(k+1)^\sigma} &= \sum_{k\leq\gamma} \frac{|\hat{f}(k)|^p}{(k+1)^\sigma} + \sum_{k>\gamma} \frac{|\hat{f}(k)|^p}{(k+1)^\sigma} \\
&= S_1 + S_2.
\end{aligned}
$$

We choose $\gamma = h^{-6\frac{(2m+1)}{6m+5}}$.

Then

$$
S_1 \leq Ch^{mp-1+p} \sum_{k\leq\gamma} \frac{k^{\frac{-p}{12}+\frac{mp}{2}}}{(k+1)^\sigma}.
$$

Since $\sigma = \frac{2-p}{12}\left\{\frac{18m+11}{2m+1}\right\}$ and $m = \left[\frac{1}{p}\right]$, we get

$$
\left(\frac{m}{2} - \frac{1}{12}\right) p - \sigma + 1 = \frac{(6m+5)\left\{(m+1)p - 1\right\}}{2m+1} > 0.
$$

Thus

$$
S_1 \leq Ch^{mp-1+p} \gamma^{\left(\frac{m}{2}-\frac{1}{12}\right)p-\sigma+1} \leq C
$$

by the choice of γ.

On the other hand, applying Hölder's inequality with $P = \frac{2}{p}$, we get,

$$\begin{aligned} S_2 &= \sum_{k>\gamma} \frac{|\hat{f}(k)|^p}{(k+1)^\sigma} \\ &\leq \left(\sum_{k>\gamma} |\hat{f}(k)|^2 \right)^{\frac{p}{2}} \left(\sum_{k>\gamma} \frac{1}{(k+1)^{\frac{2\sigma}{2-p}}} \right)^{\frac{2-p}{2}} \\ &\leq \|f\|_2^p \gamma^{\left(-\frac{2\sigma}{2-p}+1\right)^{\frac{2-p}{2}}}. \end{aligned}$$

Using property (ii) of an H^p-atom, we get $\|f\|_2^p \leq h^{-1+\frac{p}{2}}$ and thus

$$S_2 \leq h^{-1+\frac{p}{2}} \gamma^{-\sigma + \left(\frac{2-p}{2}\right)} \leq C$$

again by the choice of γ, thus proving our assertion. □

Remark 1. *In the case of higher dimensions, the result has been proved with $\sigma = \left(\frac{n}{4} + \frac{1}{2}\right)(2-p)$ (see [3]). However, here, we need an additional factor $\frac{1}{12m+6}$ which approaches 0 as $p \to 0$. But when $p = 1$, the value of $\sigma = \frac{29}{36}$, which coincides with the result of Kanjin in [1].*

References

[1] Y. KANJIN, Hardy's inequalities for Hermite and Laguerre expansions, *Bull. London Math. Soc.*, **29** (1997), 331–337.

[2] R. RADHA, Hardy type inequalities, *Taiwanese J. Math.*, **4** (2000), 447–456.

[3] R. RADHA AND S. THANGAVELU, Hardy's inequalities for Hermite and Laguerre expansions, *Proc. Amer. Math. Soc.*, **132**(12) (2004), 3525–3536.

[4] M. SATAKE, Hardy's inequalities for Laguerre expansions, *J. Math. Soc.*, **52**(1) (2000), 17–24.

[5] E.M. STEIN, *Harmonic Analysis: Real variable methods, Orthogonality and Oscillatory integrals*, Princeton Univ. Press, 1993.

[6] S. THANGAVELU, On regularity of twisted spherical means and special Hermite expansion, *Proc. Ind. Acad. Sci.*, **103** (1993), 303–320.

[7] S. THANGAVELU, *Lectures on Hermite and Laguerre expansions*, Mathematical Notes, **42**, Princeton Univ. Press, 1993.