

# Journal of Inequalities in Pure and Applied Mathematics

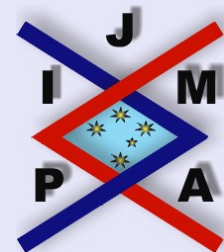
## HARDY-TYPE INEQUALITIES FOR HERMITE EXPANSIONS

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Abstract

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## Abstract

Hardy-type inequalities are established for Hermite expansions for  $f \in H^p(\mathbb{R})$ ,  $0 < p \leq 1$ .

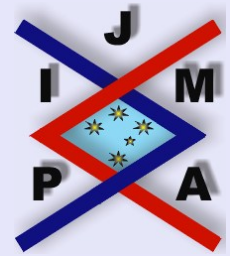
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*Key words:* Atomic decomposition, Fourier-Hermite coefficient, Hardy spaces, Hermite functions.

One of the authors (R.R.) wishes to thank Prof. S. Thangavelu for initiating her into this work.

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# 1. Introduction

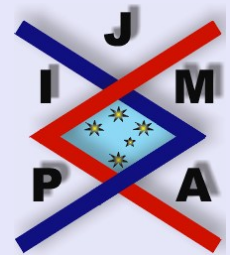
Hardy's inequality for a Fourier transform  $\mathcal{F}$  is stated as

$$\int_{\mathbb{R}} \frac{|\mathcal{F}f(\xi)|^p}{|\xi|^{2-p}} d\xi \leq C \|f\|_{\text{Re } H^p}^p \quad 0 < p \leq 1,$$

where  $\text{Re } H^p$  denotes the real Hardy space consisting of the boundary values of real parts of functions in the Hardy space  $H^p$  on the unit disc in the plane. Kanjin in [1] has proved Hardy's inequalities for Hermite and Laguerre expansions for functions in  $H^1$ . In [4] Satake has obtained Hardy's inequalities for Laguerre expansions for  $H^p$  where  $0 < p \leq 1$ . In connection with regularity properties of spherical means on  $\mathbb{C}^n$ , Thangavelu [6] has proved a Hardy's inequality for special Hermite functions. These type of inequalities for higher dimensional expansions are studied in [2], [3]. In this short note we obtain such inequalities for Hermite expansions for one dimension, namely for  $f \in H^p(\mathbb{R})$ ,  $0 < p \leq 1$ . In fact, it is to be noted from Theorem 2.1 that the resulting inequality for Hermite expansions ( $0 < p \leq 1$ ) is sharper than the inequalities for the classical Fourier transform as well as the Laguerre function expansion.

An  $H^p$  atom,  $0 < p \leq 1$  is defined to be a function  $a$  satisfying the following conditions:

- i.  $a$  is supported in an interval  $[b, b + h]$
- ii.  $|a(x)| \leq h^{-1/p}$  almost everywhere and
- iii.  $\int_{\mathbb{R}} x^k a(x) dx = 0$  for all  $k = 0, 1, 2, \dots, \left[\frac{1}{p} - 1\right]$ .



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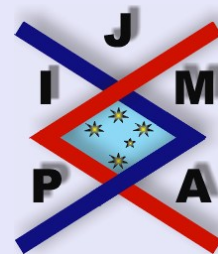
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Making use of the atomic decomposition we define the Hardy space  $H^p$  to be the collection of all functions  $f$  satisfying  $f = \sum_{k=0}^{\infty} \lambda_k a_k$ , where  $a_j$  is an  $H^p$ -atom,  $\lambda_k$  is a sequence of complex numbers with  $\sum |\lambda_k|^p < \infty$  and

$$C\|f\|_{H^p} \leq \left( \sum |\lambda_k|^p \right)^{\frac{1}{p}} \leq C'\|f\|_{H^p}.$$

For various other definitions of  $H^p$ -spaces we refer to Stein [5].




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## 2. The Main Result

Let  $H_k$  denote the Hermite polynomials

$$H_k(x) = (-1)^k \frac{d^k}{dx^k} \left( e^{-x^2} \right) e^{x^2}, \quad k = 0, 1, 2, \dots$$

Then the Hermite functions  $\tilde{h}_k$  are defined by

$$\tilde{h}_k(x) = H_k(x) e^{-\frac{1}{2}x^2}, \quad k = 0, 1, 2, \dots$$

The normalized Hermite functions  $h_k$  are defined as

$$h_k(x) = (2^k k! \sqrt{\pi})^{-\frac{1}{2}} \tilde{h}_k(x).$$

These functions  $\{h_k\}$  form an orthonormal basis for  $L^2(\mathbb{R})$ . They are eigenfunctions for the Hermite operator  $H = -\Delta + x^2$  with eigenvalues  $2k + 1$ . For more results concerning Hermite expansions, we refer to [7].

The following inequalities for Hermite functions are well known:

$$|h_k(x)| \leq C k^{-\frac{1}{12}} \quad \text{and} \quad \left| \frac{d}{dx} h_k(x) \right| \leq C k^{\frac{5}{12}}.$$

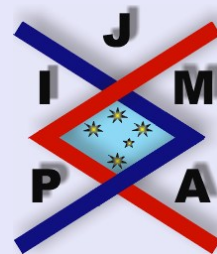
Using these inequalities and the relation

$$\frac{d}{dx} h_k(x) = \left( \frac{k}{2} \right)^{\frac{1}{2}} h_{k-1}(x) + \left( \frac{k+1}{2} \right)^{\frac{1}{2}} h_{k+1}(x)$$

we obtain the estimate

$$\left| \frac{d^m}{dx^m} h_k(x) \right| \leq C k^{-\frac{1}{12} + \frac{m}{2}} \quad \text{for } m = 0, 1, 2, \dots,$$

which can be verified easily by applying induction on  $m$ .



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**Theorem 2.1.** Let  $\{h_k\}$  be the normalized Hermite functions on  $\mathbb{R}$ . Let  $0 < p \leq 1$  and  $m = \left\lceil \frac{1}{p} \right\rceil$ . Then for every  $f \in H^p(\mathbb{R})$ , the Fourier - Hermite coefficient of  $f$ , namely,

$$\hat{f}(k) = \int_{\mathbb{R}} f(x)h_k(x)dx, \quad k = 0, 1, 2, 3, \dots$$

satisfies the inequality

$$\sum_{k=0}^{\infty} \frac{|\hat{f}(k)|^p}{(k+1)^\sigma} \leq C \|f\|_{H^p},$$

where  $C$  is a constant and  $\sigma = \frac{2-p}{12} \left\{ \frac{18m+11}{2m+1} \right\} = \left( \frac{3}{4} + \frac{1}{12m+6} \right) (2-p)$ .

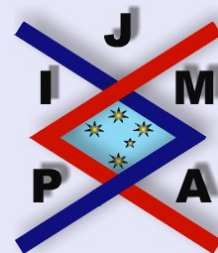
*Proof.* In order to prove the theorem, it is enough to prove that

$$\sum_{k=0}^{\infty} \frac{|\hat{f}(k)|^p}{(k+1)^\sigma} \leq C$$

for an  $H^p$ -atom  $f$ . Let  $f$  be an  $H^p$  atom. By considering the remainder term of the Taylor series expansion for  $h_k(x)$ , we write the Fourier-Hermite coefficient of  $f$  as

$$\hat{f}(k) = \frac{1}{m!} \int_b^{b+h} f(x) \frac{d^m}{dt^m} h_k(t) (x-b)^m dx,$$

where  $t \in [b, x]$  and  $m = \left\lceil \frac{1}{p} \right\rceil$ .



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Then

$$\begin{aligned}
 |\hat{f}(k)| &\leq Ch^m \int_b^{b+h} |f(x)| \left| \frac{d^m}{dt^m} h_k(t) \right| dx \\
 &\leq Ch^m k^{-\frac{1}{12} + \frac{m}{2}} \int_b^{b+h} |f(x)| dx \\
 &\leq Ch^m k^{-\frac{1}{12} + \frac{m}{2}} h^{-\frac{1}{p} + 1}.
 \end{aligned}$$

Consider

$$\begin{aligned}
 \sum_{k=0}^{\infty} \frac{|\hat{f}(k)|^p}{(k+1)^\sigma} &= \sum_{k \leq \gamma} \frac{|\hat{f}(k)|^p}{(k+1)^\sigma} + \sum_{k > \gamma} \frac{|\hat{f}(k)|^p}{(k+1)^\sigma} \\
 &= S_1 + S_2.
 \end{aligned}$$

We choose  $\gamma = h^{-6 \frac{(2m+1)}{6m+5}}$ .

Then

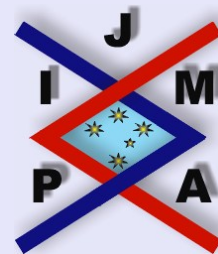
$$S_1 \leq Ch^{mp-1+p} \sum_{k \leq \gamma} k^{\frac{-p}{12} + \frac{mp}{2}} \frac{1}{(k+1)^\sigma}.$$

Since  $\sigma = \frac{2-p}{12} \left\{ \frac{18m+11}{2m+1} \right\}$  and  $m = \left[ \frac{1}{p} \right]$ , we get

$$\left( \frac{m}{2} - \frac{1}{12} \right) p - \sigma + 1 = \frac{(6m+5) \{ (m+1)p - 1 \}}{2m+1} > 0.$$

Thus

$$S_1 \leq Ch^{mp-1+p} \gamma^{\left( \frac{m}{2} - \frac{1}{12} \right) p - \sigma + 1} \leq C$$



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by the choice of  $\gamma$ .

On the other hand, applying Hölder's inequality with  $P = \frac{2}{p}$ , we get,

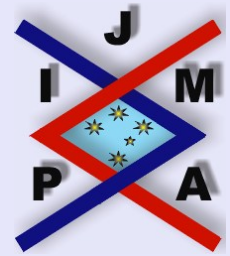
$$\begin{aligned} S_2 &= \sum_{k>\gamma} \frac{|\hat{f}(k)|^p}{(k+1)^\sigma} \\ &\leq \left( \sum_{k>\gamma} |\hat{f}(k)|^2 \right)^{\frac{p}{2}} \left( \sum_{k>\gamma} \frac{1}{(k+1)^{\frac{2\sigma}{2-p}}} \right)^{\frac{2-p}{2}} \\ &\leq \|f\|_2^p \gamma^{(-\frac{2\sigma}{2-p}+1)\frac{2-p}{2}}. \end{aligned}$$

Using property (ii) of an  $H^p$ -atom, we get  $\|f\|_2^p \leq h^{-1+\frac{p}{2}}$  and thus

$$S_2 \leq h^{-1+\frac{p}{2}} \gamma^{-\sigma+(\frac{2-p}{2})} \leq C$$

again by the choice of  $\gamma$ , thus proving our assertion.  $\square$

**Remark 1.** *In the case of higher dimensions, the result has been proved with  $\sigma = (\frac{n}{4} + \frac{1}{2})(2-p)$  (see [3]). However, here, we need an additional factor  $\frac{1}{12m+6}$  which approaches 0 as  $p \rightarrow 0$ . But when  $p = 1$ , the value of  $\sigma = \frac{29}{36}$ , which coincides with the result of Kanjin in [1].*




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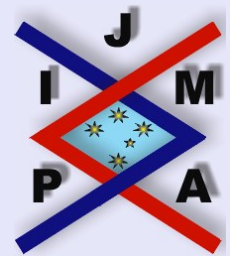
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