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CONVEX FUNCTIONS AND INEQUALITIES FOR INTEGRALS

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ABSTRACT. In this paper we present inequalities for integrals of functions that are the composition of nonnegative convex functions on an open convex set of a vector space $\mathbb{R}^m$ and vector-valued functions in a weakly compact subset of a Banach vector space generated by m L^p_μ-spaces for $1 \leq p < +\infty$ and inequalities when these vector-valued functions are in a weakly* compact subset of a Banach vector space generated by m L^∞_μ-spaces.

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1. INTRODUCTION

When studying extremum problems and integral estimates in many areas of applied mathematics, we may require the convexity of functions and the weak compactness of sets. Many properties of convex functions and weakly compact sets can be found in the literature (e.g., see [2], [3], [4], [5] and [7]). In some research fields such as the existence of solutions of differential equations (e.g., see [1] and [6]), we usually enconter some problems on the estimates of integrals of functions that are the composition of convex functions on an open convex set of a

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vector space and vector-valued functions in a weakly (or weakly*) compact subset in a Banach space. The estimates of integrals of this kind of composite function is interesting and important in many application areas. Inequalities for integrals of composite functions are necessary, therefore, for solving many problems in applied mathematics.

Let us first introduce some notations which will be used throughout this paper. $\mathbb{R}$ denotes the real number system, $\mathbb{R}^n$ is the usual vector space of real n-tuples $x = (x_1, x_2, \dots, x_n)$, μ is a nonnegative Lebesgue measure of $\mathbb{R}^n$, $L^p_\mu(\mathbb{R}^n)$ represents a Banach space where each measurable function $u(x)$ has the following norm

$$\|u\|_p = \left(\int_{\mathbb{R}^n} |u(x)|^p d\mu \right)^{\frac{1}{p}} \tag{1.1}$$

for any $p \in [1, +\infty)$, $(L^p_\mu(\mathbb{R}^n))^m$ denotes a Banach vector space where each measurable vector-valued function has m components in $L^p_\mu(\mathbb{R}^n)$, $L^\infty_\mu(\mathbb{R}^n)$ represents a Banach space where each measurable function $u(x)$ has the following norm

$$\|u\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |u(x)| \text{ (or say } \|u\|_\infty = \inf_{\substack{E \subseteq \mathbf{R}^n \\ \mu(E^C)=0}} \sup_{x \in E} |u(x)|), \tag{1.2}$$

where E^C represents the complement set of E in $\mathbb{R}^n$, and $(L^\infty_\mu(\mathbb{R}^n))^m$ denotes a Banach vector space where each measurable vector-valued function has m components in $L^\infty_\mu(\mathbb{R}^n)$.

Below, we give the definition of weak convergence of a sequence in $L^p_\mu(\mathbb{R}^n)$, where $p \in [1, +\infty)$. Assume that $q = p/(p-1)$ as $p \in (1, +\infty)$ and that $q = \infty$ as $p = 1$. If $u \in L^p_\mu(\mathbb{R}^n)$ and it is assumed that a sequence $\{u_i\}_{i=1}^{+\infty}$ in $L^p_\mu(\mathbb{R}^n)$ satisfies

$$\lim_{i \to +\infty} \int_{\mathbb{R}^n} u_i v d\mu = \int_{\mathbb{R}^n} u v d\mu \tag{1.3}$$

for all $v \in L^q_\mu(\mathbb{R}^n)$, then the sequence $\{u_i\}_{i=1}^{+\infty}$ is said to be weakly convergent in $L^p_\mu(\mathbb{R}^n)$ to u as $i \to +\infty$. Similarly, we introduce the definition of weak* convergence of a sequence in $L^\infty_\mu(\mathbb{R}^n)$. If $u \in L^\infty_\mu(\mathbb{R}^n)$ and it is assumed that a sequence $\{u_i\}_{i=1}^{+\infty}$ in $L^\infty_\mu(\mathbb{R}^n)$ satisfies the equality (1.3) for all $v \in L^1_\mu(\mathbb{R}^n)$, then the sequence $\{u_i\}_{i=1}^{+\infty}$ is said to be weakly* convergent in $L^\infty_\mu(\mathbb{R}^n)$ to u as $i \to +\infty$. Then we define the weak (or weak*) convergence of a sequence in $(L^p_\mu(\mathbb{R}^n))^m$ (or $(L^\infty_\mu(\mathbb{R}^n))^m$). If $1 \le p < +\infty$ and $\{u_{ji}\}_{i=1}^{+\infty}$ is weakly convergent in $L^p_\mu(\mathbb{R}^n)$ to $\hat{u}_j$ for all $j = 1, 2, \dots, m$ as $i \to +\infty$, then a sequence $\{u_i = (u_{1i}, u_{2i}, \dots, u_{mi})\}_{i=1}^{+\infty}$ is called weakly convergent in a Banach vector space $(L^p_\mu(\mathbb{R}^n))^m$ to $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_m)$ as $i \to +\infty$. Similarly, if $\{u_{ji}\}_{i=1}^{+\infty}$ is weakly* convergent in $L^\infty_\mu(\mathbb{R}^n)$ to $\hat{u}_j$ for all $j = 1, 2, \dots, m$ as $i \to +\infty$, then a sequence $\{u_i = (u_{1i}, u_{2i}, \dots, u_{mi})\}_{i=1}^{+\infty}$ is said to be weakly* convergent in a Banach vector space $(L^\infty_\mu(\mathbb{R}^n))^m$ to $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_m)$ as $i \to +\infty$.

We now recall the definition of a convex function. If $f(x)$ is a function with its values being real numbers or $\pm\infty$ and its domain is a subset S of an m dimensional vector space $\mathbb{R}^m$ such that $\{(x, y) | x \in S, y \in \mathbb{R}, y \ge f(x)\}$ is convex as a subset of an $m+1$ dimensional vector space $\mathbb{R}^{m+1}$, then $f(x)$ is called a convex function on S. It is known that $f(x)$ is convex from S to $(-\infty, +\infty]$ if and only if

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_k x_k) \le \lambda_1 f(x_1) + \lambda_2 f(x_2) + \dots + \lambda_k f(x_k) \tag{1.4}$$

whenever S is a convex subset of $\mathbb{R}^m$, $x_i \in S$ ($i = 1, 2, \dots$) $\lambda_1 \ge 0$, $\lambda_2 \ge 0$, $\dots$, $\lambda_k \ge 0$, $\lambda_1 + \lambda_2 + \dots + \lambda_k = 1$. This is called Jensen's inequality when $S = \mathbb{R}^m$.

There are inequalities for integrals of functions that are the composition of convex functions on a vector space $\mathbb{R}^m$ and vector-valued functions in a weakly* compact subset of $(L^\infty_\mu(\mathbb{R}^n))^m$ (see [6] or Theorem 3.2 in Section 3). If $F(x, y)$ is a special nonnegative convex function defined by $F(x,y) = (x-y)\log(\frac{x}{y})$ for $(x,y) \in (0,+\infty) \times (0,\infty)$ and $\{(a_i,b_i)\}_{i=1}^{+\infty}$ is a nonnegative sequence weakly convergent in $(L^1_\mu(\mathbb{R}^n))^2$ to (a,b), then similar inequalities for integrals of the composite functions can also be obtained (see [1]), as follows:

$$\varliminf_{i\to+\infty} \int_{\mathbb{R}^n} F(a_i,b_i)d\mu \geq \int_{\mathbb{R}^n} F(a,b)d\mu.$$

Below, we extend the two results mentioned above to a more general case. More precisely, this paper aims to show inequalities for integrals of functions that are the composition of nonnegative convex functions on an open convex set of a vector space $\mathbb{R}^m$ and vector-valued functions in a weakly compact subset of a Banach vector space generated by m L^p_μ-spaces for $1 \leq p < +\infty$. We also show inequalities for integrals of functions when these vector-valued functions are in a weakly* compact subset of a Banach vector space generated by m L^∞_μ-spaces.

2. INEQUALITIES FOR WEAKLY CONVERGENT SEQUENCES

Some basic concepts have been introduced in the previous section. In this section we show inequalities for integrals of functions which are the composition of nonnegative convex functions on an open convex set of a vector space $\mathbb{R}^m$ and vector-valued functions in a weakly compact subset of a Banach vector space generated by m L^p_μ-spaces for any given $p \in [1,+\infty)$. That is the following

Theorem 2.1. *Suppose that a sequence $\{u_i\}_{i=1}^{+\infty}$ weakly converges in $(L^p_\mu(\mathbb{R}^n))^m$ to u as $i \to +\infty$, where $p \in [1,+\infty)$ and m and n are two positive integers. Assume that all the values of u and u_i ($i = 1,2,3,\dots$) belong to an open convex set K in $\mathbb{R}^m$ and that $f(x)$ is a nonnegative convex function from K to $\mathbb{R}$. Then*

$$\varliminf_{i\to+\infty} \int_\Omega f(u_i)d\mu \geq \int_\Omega f(u)d\mu \tag{2.1}$$

for any measurable set $\Omega \subseteq \mathbb{R}^n$.

In order to prove Theorem 2.1, let us first recall the following lemma:

Lemma 2.2. *Assume $u_n \to u$ weakly in a normed linear space. Then there exists, for any $\varepsilon > 0$, a convex combination $\sum_{k=1}^n \lambda_k u_k$ ($\lambda_k \geq 0, \sum_{k=1}^n \lambda_k = 1$) of $\{u_k : k = 1,2,\dots\}$ such that $\|u - \sum_{k=1}^n \lambda_k u_k\| \leq \varepsilon$ where $\|v\|$ is a norm of v in the space.*

This is called Mazur's lemma. Its proof can be found in [5] and [7]. Using this lemma, we can give a proof of Theorem 2.1.

Proof of Theorem 2.1. Let Ω_R be a bounded set defined by $\Omega_R = \Omega \cap \{w : |w| < R, w \in \mathbb{R}^n\}$ for all $R > 0$. Put $\alpha_i = \int_{\Omega_R} f(u_i)d\mu$ ($i = 1,2,\dots$) and $\alpha = \varliminf_{i\to+\infty} \int_{\Omega_R} f(u_i)d\mu$ for all the bounded sets Ω_R. Then there exists a subsequence of $\{\alpha_i\}_{i=1}^{+\infty}$ such that this subsequence, denoted without loss of generality by $\{\alpha_i\}_{i=1}^{+\infty}$, converges to α as $i \to +\infty$.

Take $K_l = \{x : x \in K, |x| \leq l, \rho(\partial K, x) \geq 1/l\}$ and $D_l(u) = \{\omega : \omega \in \mathbb{R}^n, u(\omega) \in K_l\}$ for any fixed positive integer l, where $\rho(\partial K, x)$ is defined as a distance between the point x and the boundary ∂K of K. Then all the sets K_l are bounded, closed and convex subsets of K such that $\lim_{l\to+\infty} K_l = K$ and $K_l \subset K_{l+1}$ for any positive integer l. It can also be easily proven that $\lim_{l\to+\infty} D_l(u) = \mathbb{R}^n$ and that $D_l(u) \subseteq D_{l+1}(u)$ for any positive integer l. Since $f(x)$ is a convex function defined on an open convex set K, $f(x)$ is continuous in K (see [3]). Thus $f(x)$ is uniformly continuous in K_{l+1}, that is, for any given positive number ε, there exists a positive number δ such that

$$|f(x) - f(y)| < \varepsilon/\mu(\Omega_R) \tag{2.2}$$

as $|x - y| < \delta$ for any x and y in K_{l+1}.

Since u_i weakly converges in $(L^p_\mu(\mathbb{R}^n))^m$ to u, using Lemma 2.2, one know that for any natural number j, there exists a convex combination $\sum_{k=j}^{N(j)} \lambda_k u_k$ of $\{u_k : k = j, j+1, \dots\}$ such that $\left\| u - \sum_{k=j}^{N(j)} \lambda_k^{(j)} u_k \right\|_p \le \frac{1}{j}$, where $N(j)$ is a natural number which depends on j and $\{u_k : k = j, j+1, \dots\}$, $\|v\|_p$ represents a norm of v in $(L^p_\mu(\mathbb{R}^n))^m$, $\lambda_k^{(j)} \ge 0$ and $\sum_{k=j}^{N(j)} \lambda_k^{(j)} = 1$. Put $v_j = \sum_{k=j}^{N(j)} \lambda_k^{(j)} u_k$. Then, as j tends to infinity, v_j converges in $(L^p_\mu(\mathbb{R}^n))^m$ to u. Thus there exists a subsequence of $\{v_j\}_{j=1}^{+\infty}$ such that this subsequence, denoted without loss of generality by $\{v_j\}_{j=1}^{+\infty}$, converges almost everywhere in $\mathbb{R}^n$ to u as j goes to infinity. In particular, for any given $R > 0$, v_j converges almost everywhere in Ω_R to u as j goes to infinity. By Egorov's theorem, it is known that for any given positive number σ, there exists a measurable set E_R in Ω_R with $\mu(E_R) < \sigma$ such that v_j converges uniformly in $\Omega_R \backslash E_R$ to u. Therefore for the above number δ, there exists a natural number N such that

$$\sup_{\omega \in (\Omega_R \backslash E_R) \cap D_l(u)} |u(\omega) - v_j(\omega)| < \delta \tag{2.3}$$

for all $j > N$. Since all the values of u in $D_l(u)$ are in the closed set K_l and $K_l \subset K_{l+1}$, δ can be chosen to be sufficiently small such that all the values of v_j in $(\Omega_R \backslash E_R) \cap D_l(u)$ fall in K_{l+1} and (2.2) and (2.3) still hold for this choice of δ. Combining (2.2) and (2.3), we know that for any given positive number ε, there exists a natural number N such that

$$f(u) < f(v_j) + \varepsilon/\mu(\Omega_R) \tag{2.4}$$

in $(\Omega_R \backslash E_R) \cap D_l(u)$ for all $j > N$. Since all the values of $\{u_i\}_{i=1}^{+\infty}$ and u are in K and $f(x)$ is convex and nonnegative, integrating (2.4) gives

$$\int_{\Omega_R \cap D_l(u)} f(u) d\mu \le \sum_{k=j}^{N(j)} \lambda_k^{(j)} \int_{\Omega_R} f(u_k) d\mu + \sigma \max_{x \in K_l} f(x) + \varepsilon. \tag{2.5}$$

(2.5) can be equivalently written as

$$\int_{\Omega_R \cap D_l(u)} f(u) d\mu \le \sum_{k=j}^{N(j)} \lambda_k^{(j)} \alpha_k + \sigma \max_{x \in K_l} f(x) + \varepsilon. \tag{2.6}$$

Notice that $\alpha_k \to \alpha$ as $k \to +\infty$. By first letting $j \to +\infty$ and then $\varepsilon \to 0$, (2.6) gives

$$\int_{\Omega_R \cap D_l(u)} f(u) d\mu \le \alpha + \sigma \max_{x \in K_l} f(x). \tag{2.7}$$

The fact that $\lim_{l \to +\infty} D_l(u) = \mathbb{R}^n$ shows that the limit of $\Omega_R \cap D_l(u)$ in (2.7) is Ω_R. By first letting $\sigma \to 0$ and then $l \to +\infty$, (2.7) reads

$$\int_{\Omega_R} f(u) d\mu \le \alpha \le \varliminf_{i \to +\infty} \int_{\Omega} f(u_i) d\mu, \tag{2.8}$$

where the last inequality is obtained using the nonnegativity of $f(x)$. Finally, by letting $R \to +\infty$ and using the Lebesgue dominated convergence theorem, (2.8) leads to (2.1). This completes the proof. □

3. INEQUALITIES FOR WEAKLY* CONVERGENT SEQUENCES

In the previous section we have given inequalities for integrals of composite functions for weakly convergent sequences in $(L^p_\mu(\mathbb{R}^n))^m$ for $1 \leq p < +\infty$. In this section we present a similar result for weakly* convergent sequences in $(L^\infty_\mu(\mathbb{R}^n))^m$.

Using the process of the proof in Theorem 2.1, we can prove the following theorem.

Theorem 3.1. *Assume that a sequence $\{u_i\}_{i=1}^{+\infty}$ weakly* converges in $(L^\infty_\mu(\mathbb{R}^n))^m$ to u as $i \to +\infty$, where m and n are two positive integers. Assume that all the values of u and u_i $(i = 1, 2, 3, \dots)$ belong to an open convex set K of $\mathbb{R}^m$ and that $f(x)$ is a nonnegative convex function from K to $\mathbb{R}$. Then the inequality (2.1) holds for any measurable set $\Omega \subseteq \mathbb{R}^n$.*

Proof. Put $\Omega_R = \Omega \cap \{w : |w| < R, w \in \mathbb{R}^n\}$. Then Ω_R is a bounded set in $\mathbb{R}^n$ for all fixed positive real numbers R. Since $u_i \to u$ weakly* in $(L^\infty_\mu(\mathbb{R}^n))^m$, $u_i \to u$ weakly* in $(L^\infty_\mu(\Omega_R))^m$. Hence, by $L^\infty(\Omega_R) \subset L^1(\Omega_R)$, it can be easily shown that $u_i \to u$ weakly in $(L^1_\mu(\Omega_R))^m$. Then, using the process of the proof of Theorem 2.1, we obtain

$$\varliminf_{i\to+\infty} \int_{\Omega_R} f(u_i)d\mu \geq \int_{\Omega_R} f(u)d\mu. \tag{3.1}$$

It follows from the nonnegativity of the convex function f that

$$\varliminf_{i\to+\infty} \int_{\Omega} f(u_i)d\mu \geq \int_{\Omega_R} f(u)d\mu. \tag{3.2}$$

Finally, by the Lebesgue monotonous convergence theorem, as $R \to +\infty$, (3.2) implies (2.1). Our proof is completed. □

Furthermore, by removing the nonnegativity of $f(x)$ and assuming that the convex set K is closed, we can deduce the following result.

Theorem 3.2. *Assume that a sequence $\{u_i\}_{i=1}^{+\infty}$ weakly* converges in $(L^\infty_\mu(\mathbb{R}^n))^m$ to u as $i \to +\infty$, where m and n are two positive integers. Assume that all the values of u and u_i $(i = 1, 2, 3, \dots)$ belong to a closed convex set K in $\mathbb{R}^m$ and that $f(x)$ is a continuous convex function from K to $\mathbb{R}$. Then the inequality (2.1) holds for any bounded measurable set $\Omega \subset \mathbb{R}^n$.*

We can also obtain Theorem 3.1 from Theorem 3.2. Theorem 3.2 can be easily proved using Lemma 2.2. In fact, Theorem 3.2 is a part of the results given by Ying [6]. However, we still give its proof below.

Proof of Theorem 3.2. Put $\alpha_i = \int_\Omega f(u_i)d\mu$ $(i = 1, 2, \dots)$ and $\alpha = \varliminf_{i\to+\infty} \int_\Omega f(u_i)d\mu$ for any bounded set Ω. Then there exists a subsequence of $\{\alpha_i\}_{i=1}^{+\infty}$ such that this subsequence, denoted without loss of generality by $\{\alpha_i\}_{i=1}^{+\infty}$, converges to α as $i \to +\infty$.

Take $\widetilde{K} = K \cap \{x : x \in \mathbb{R}^m, |x| \leq \|u\|_\infty + 1\}$. Then, since K is closed and $f(x)$ is continuous in K, $\widetilde{K}$ is a bounded closed set and $f(x)$ is uniformly continuous in $\widetilde{K}$, that is, for any given positive number ε, there exists a positive number $\delta < 1$ such that

$$|f(x) - f(y)| < \varepsilon/\mu(\Omega) \tag{3.3}$$

as $|x - y| < \delta$ for any x and y in $\widetilde{K}$.

Since $u_i \to u$ weakly* in $(L^\infty_\mu(\mathbb{R}^n))^m$, $u_i \to u$ weakly* in $(L^\infty_\mu(\Omega))^m$. Hence, by $L^\infty(\Omega) \subset L^1(\Omega)$, it can be easily shown that $u_i \to u$ weakly in $(L^1_\mu(\Omega))^m$. It follows from Lemma 2.2 that, for any natural number j, there exists a convex combination $\sum_{k=j}^{N(j)} \lambda_k^{(j)} u_k$ of $\{u_k : k = j, j+1, \dots\}$ such that $\left\| u - \sum_{k=j}^{N(j)} \lambda_k^{(j)} u_k \right\|_\infty \leq \frac{1}{j}$ where $N(j)$ is a natural number which depends on j and $\{u_k : k = j, j+1, \dots\}$, $\|v\|_\infty$ represents a norm of v in $(L^\infty_\mu(\Omega))^m$, $\lambda_k^{(j)} \geq 0$ and $\sum_{k=j}^{N(j)} \lambda_k^{(j)} = 1$. Put $v_j = \sum_{k=j}^{N(j)} \lambda_k^{(j)} u_k$. Then v_j converges in $(L^\infty_\mu(\Omega))^m$ to u as j tends to ∞. In particular, for the above number δ, there exists a natural number N such that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} |u(\omega) - v_j(\omega)| < \delta \tag{3.4}$$

for all $j > N$. Since all the values of u and u_i are in K and K is convex, all the values of u and v_j in Ω are in $\widetilde{K}$ for all $j > N$. Combining (3.3) and (3.4), we know that for any given positive number ε, there exists a natural number N such that

$$f(u) < f(v_j) + \varepsilon/\mu(\Omega) \tag{3.5}$$

almost everywhere in Ω for all $j > N$. Thus, integrating (3.5) gives

$$\int_\Omega f(u)d\mu \leq \int_\Omega f(v_j)d\mu + \varepsilon \tag{3.6}$$

for all $j > N$. By the convexity of the function $f(x)$, integrating (3.6) gives

$$\int_\Omega f(u)d\mu \leq \sum_{k=j}^{N(j)} \lambda_k^{(j)} \int_\Omega f(u_k)d\mu + \varepsilon. \tag{3.7}$$

(3.7) can be equivalently written as

$$\int_\Omega f(u)d\mu \leq \sum_{k=j}^{N(j)} \lambda_k^{(j)} \alpha_k + \varepsilon. \tag{3.8}$$

By letting $j \to +\infty$ and using the convergence of α_k to α, (3.8) gives

$$\int_\Omega f(u)d\mu \leq \alpha + \varepsilon. \tag{3.9}$$

By letting $\varepsilon \to 0$, (3.9) leads to (2.1). This completes the proof. □

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