journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 8 (2007), Issue 1, Article 20, 6 pp.

SUBORDINATION AND SUPERORDINATION RESULTS FOR Φ-LIKE FUNCTIONS

T.N. SHANMUGAM, S. SIVASUBRAMANIAN, AND MASLINA DARUS

DEPARTMENT OF INFORMATION TECHNOLOGY,
SALALAH COLLEGE OF ENGINEERING
SALALAH, SULTANATE OF OMAN
drtns2001@yahoo.com

DEPARTMENT OF MATHEMATICS,
EASWARI ENGINEERING COLLEGE
RAMAPURAM, CHENNAI-600 089
INDIA
sivasaisastha@rediffmail.com

SCHOOL OF MATHEMATICAL SCIENCES,
FACULTY OF SCIENCES AND TECHNOLOGY,
UKM, MALAYSIA
maslina@pkrisc.cc.ukm.my

Received 24 June, 2006; accepted 29 December, 2006
Communicated by N.E. Cho

ABSTRACT. Let q_1 be convex univalent and q_2 be univalent in $\Delta := \{z : |z| < 1\}$ with $q_1(0) = q_2(0) = 1$. Let f be a normalized analytic function in the open unit disk Δ. Let Φ be an analytic function in a domain containing $f(\Delta)$, $\Phi(0) = 0$ and $\Phi'(0) = 1$. We give some applications of first order differential subordination and superordination to obtain sufficient conditions for the function f to satisfy

$$q_1(z) \prec \frac{z(f*g)'(z)}{\Phi(f*g)(z)} \prec q_2(z)$$

where g is a fixed function.

Key words and phrases: Differential subordination, Differential superordination, Convolution, Subordinant.

2000 *Mathematics Subject Classification.* 30C45.

1. INTRODUCTION AND MOTIVATIONS

Let $\mathcal{A}$ be the class of all normalized analytic functions $f(z)$ in the open unit disk $\Delta := \{z : |z| < 1\}$ satisfying $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{H}$ be the class of functions analytic in Δ and for any $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

We would like to thank the referee for his insightful suggestions.
175-06

$f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$. Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If p and $\phi(p(z), zp'^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'^2p''(z); z), \tag{1.1}$$

then p is a solution of the differential superordination (1.1). If f is subordinate to F, then F is called a superordinate of f. An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (1.1). A univalent subordinant $\bar{q}$ that satisfies $q \prec \bar{q}$ for all subordinants q of (1.1) is said to be the best subordinant. Recently Miller and Mocanu [5] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'^2p''(z); z) \Rightarrow q(z) \prec p(z). \tag{1.2}$$

Using the results of Miller and Mocanu [4], Bulboacă [2] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [1]. In an earlier investigation, Shanmugam et al. [8] obtained sufficient conditions for a normalized analytic function $f(z)$ to satisfy $q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$ and $q_1(z) \prec \frac{z^2f'(z)}{\{f(z)\}^2} \prec q_2(z)$ where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$. A systematic study of the subordination and superordination has been studied very recently by Shanmugam *et al.* in [9] and [10] (see also the references cited by them).

Let Φ be an analytic function in a domain containing $f(\Delta)$ with $\Phi(0) = 0$ and $\Phi'(0) = 1$. For any two analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, the Hadamard product or convolution of $f(z)$ and $g(z)$, written as $(f * g)(z)$ is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

The function $f \in \mathcal{A}$ is called Φ-like if

$$\Re\left(\frac{zf'(z)}{\Phi(f(z))}\right) > 0 \quad (z \in \Delta). \tag{1.3}$$

The concept of $\Phi-$ like functions was introduced by Brickman [3] and he established that a function $f \in \mathcal{A}$ is univalent if and only if f is Φ-like for some Φ. For $\Phi(w) = w$, the function f is starlike. In a later investigation, Ruscheweyh [7] introduced and studied the following more general class of Φ-like functions.

Definition 1.1. Let Φ be analytic in a domain containing $f(\Delta)$, $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(w) \neq 0$ for $w \in f(\Delta) \setminus \{0\}$. Let $q(z)$ be a fixed analytic function in Δ , $q(0) = 1$. The function $f \in \mathcal{A}$ is called Φ-like with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z) \quad (z \in \Delta). \tag{1.4}$$

When $\Phi(w) = w$, we denote the class of all Φ-like functions with respect to q by $S^*(q)$.

Using the definition of $\Phi-$ like functions, we introduce the following class of functions.

Definition 1.2. Let g be a fixed function in $\mathcal{A}$. Let Φ be analytic in a domain containing $f(\Delta)$, $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(w) \neq 0$ for $w \in f(\Delta) \setminus \{0\}$. Let $q(z)$ be a fixed analytic function in Δ, $q(0) = 1$. The function $f \in \mathcal{A}$ is called Φ-like with respect to $S_g^*(q)$ if

$$\frac{z(f * g)'(z)}{\Phi(f * g)(z)} \prec q(z) \quad (z \in \Delta). \tag{1.5}$$

We note that $S^*_{\frac{z}{1-z}}(q) := S^*(q)$.

In the present investigation, we obtain sufficient conditions for a normalized analytic function f to satisfy

$$q_1(z) \prec \frac{z(f*g)'(z)}{\Phi(f*g)(z)} \prec q_2(z).$$

We shall need the following definition and results to prove our main results. In this sequel, unless otherwise stated, α and γ are complex numbers.

Definition 1.3 ([4, Definition 2, p. 817])**.** Let Q be the set of all functions f that are analytic and injective on $\bar{\Delta} - E(f)$, where

$$E(f) = \left\{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Lemma 1.1 ([4, Theorem 3.4h, p. 132])**.** *Let q be univalent in the open unit disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(\omega) \neq 0$ when $\omega \in q(\Delta)$. Set $\xi(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + \xi(z)$. Suppose that*

(1) *$\xi(z)$ is starlike univalent in Δ, and*
(2) $\Re\left\{\frac{zh'(z)}{\xi(z)}\right\} > 0 \ (z \in \Delta)$.

If p is analytic in Δ with $p(\Delta) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{1.6}$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.2. [2, Corollary 3.1, p. 288] *Let q be univalent in Δ, ϑ and φ be analytic in a domain D containing $q(\Delta)$. Suppose that*

(1) $\Re\left[\frac{\vartheta'(q(z))}{\varphi(q(z))}\right] > 0$ *for $z \in \Delta$, and*
(2) *$\xi(z) = zq'(z)\varphi(q(z))$ is starlike univalent function in Δ.*

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\Delta) \subset D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in Δ, and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{1.7}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

2. MAIN RESULTS

By making use of Lemma 1.1, we prove the following result.

Theorem 2.1. *Let $q(z) \neq 0$ be analytic and univalent in Δ with $q(0) = 1$ such that $\frac{zq'(z)}{q(z)}$ is starlike univalent in Δ. Let $q(z)$ satisfy*

$$\Re\left[1 + \frac{\alpha q(z)}{\gamma} - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right] > 0. \tag{2.1}$$

Let

$$\Psi(\alpha,\gamma,g;z) := \alpha\left\{\frac{z(f*g)'(z)}{\Phi(f*g)(z)}\right\} + \gamma\left\{1 + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{z\left(\Phi(f*g)(z)\right)'}{\Phi(f*g)(z)}\right\}. \tag{2.2}$$

If q satisfies

$$\Psi(\alpha,\gamma,g;z) \prec \alpha q(z) + \frac{\gamma zq'(z)}{q(z)}, \tag{2.3}$$

then

$$\frac{z(f*g)'(z)}{\Phi(f*g)(z)} \prec q(z)$$

and q is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$p(z) := \frac{z(f*g)'(z)}{\Phi(f*g)(z)}. \tag{2.4}$$

Then the function $p(z)$ is analytic in Δ with $p(0)=1$. By a straightforward computation

$$\frac{zp'(z)}{p(z)} = \left\{1 + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{z[\Phi(f*g)(z)]'}{\Phi(f*g)(z)}\right\}$$

which, in light of hypothesis (2.3) of Theorem 2.1, yields the following subordination

$$\alpha p(z) + \frac{\gamma zp'(z)}{p(z)} \prec \alpha q(z) + \frac{\gamma zq'(z)}{q(z)}. \tag{2.5}$$

By setting

$$\theta(\omega) := \alpha\omega \quad \text{and} \quad \phi(\omega) := \frac{\gamma}{\omega},$$

it can be easily observed that $\theta(\omega)$ and $\phi(\omega)$ are analytic in $\mathbb{C}\setminus\{0\}$ and that

$$\phi(\omega) \neq 0 \quad (\omega \in \mathbb{C}\setminus\{0\}).$$

Also, by letting

$$\xi(z) = zq'(z)\phi(q(z)) = \frac{\gamma}{q(z)}zq'(z). \tag{2.6}$$

and

$$h(z) = \theta\{q(z)\} + \xi(z) = \alpha q(z) + \frac{\gamma}{q(z)}zq'(z), \tag{2.7}$$

we find that $\xi(z)$ is starlike univalent in Δ and that

$$\Re\left[1 + \frac{\alpha q(z)}{\gamma} - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right] > 0$$

by the hypothesis (2.1). The assertion of Theorem 2.1 now follows by an application of Lemma 1.1. □

When $\Phi(\omega) = \omega$ in Theorem 2.1 we get:

Corollary 2.2. *Let $q(z) \neq 0$ be univalent in Δ with $q(0) = 1$. If q satisfies*

$$(\alpha-\gamma)\frac{z(f*g)'(z)}{(f*g)(z)} + \gamma\left\{1 + \frac{z(f*g)''(z)}{(f*g)'(z)}\right\} \prec \alpha q(z) + \frac{\gamma zq'(z)}{q(z)},$$

then

$$\frac{z(f*g)'(z)}{(f*g)(z)} \prec q(z)$$

and q is the best dominant.

For $g(z) = \frac{z}{1-z}$ and $\Phi(\omega) = \omega$, we get the following corollary.

Corollary 2.3. *Let $q(z) \neq 0$ be univalent in Δ with $q(0) = 1$. If q satisfies*

$$(\alpha - \gamma)\frac{zf'(z)}{f(z)} + \gamma\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \prec \alpha q(z) + \frac{\gamma z q'(z)}{q(z)},$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and q is the best dominant.

For the choice $\alpha = \gamma = 1$ and $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Corollary 2.3, we have the following result of Ravichandran and Jayamala [6].

Corollary 2.4. *If $f \in \mathcal{A}$ and*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Theorem 2.5. *Let $\gamma \neq 0$. Let $q(z) \neq 0$ be convex univalent in Δ with $q(0) = 1$ such that $\frac{zq'(z)}{q(z)}$ is starlike univalent in Δ. Suppose that $q(z)$ satisfies*

$$\Re\left[\frac{\alpha q(z)}{\gamma}\right] > 0. \tag{2.8}$$

If $f \in \mathcal{A}$, $\frac{z(f*g)'(z)}{\Phi(f*g)(z)} \in \mathcal{H}[1,1] \cap Q$, $\Psi(\alpha, \gamma, g; z)$ as defined by (2.2) is univalent in Δ and

$$\alpha q(z) + \frac{\gamma z q'(z)}{q(z)} \prec \Psi(\alpha, \gamma, g; z), \tag{2.9}$$

then

$$q(z) \prec \frac{z(f*g)'(z)}{\Phi(f*g)(z)}$$

and q is the best subordinant.

Proof. By setting

$$\vartheta(w) := \alpha w \quad \text{and} \quad \varphi(w) := \frac{\gamma}{\omega},$$

it is easily observed that $\vartheta(w)$ is analytic in $\mathbb{C}$, $\varphi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that

$$\varphi(w) \neq 0, \quad (w \in \mathbb{C} \setminus \{0\}).$$

The assertion of Theorem 2.5 follows by an application of Lemma 1.2. □

For $\Phi(\omega) = \omega$ in Theorem 2.5, we get

Corollary 2.6. *Let $q(z) \neq 0$ be convex univalent in Δ with $q(0) = 1$. If $f \in \mathcal{A}$ and*

$$\alpha q(z) + \frac{\gamma z q'(z)}{q(z)} \prec (\alpha - \gamma)\left\{\frac{z(f*g)'(z)}{(f*g)(z)}\right\} + \gamma\left\{1 + \frac{z(f*g)''(z)}{(f*g)'(z)}\right\},$$

then

$$q(z) \prec \frac{z(f*g)'(z)}{(f*g)(z)}$$

and q is the best subordinant.

Combining Theorem 2.1 and Theorem 2.5 we get the following sandwich theorem.

Theorem 2.7. *Let* q_1 *be convex univalent and* q_2 *be univalent in* Δ *satisfying (2.8) and (2.1) respectively such that* $q_1(0) = 1$, $q_2(0) = 1$, $\frac{zq_1'(z)}{q_1(z)}$ *and* $\frac{zq_2'(z)}{q_2(z)}$ *are starlike univalent in* Δ *with*

$$q_1(z) \neq 0 \quad \textit{and} \quad q_2(z) \neq 0.$$

Let $f \in \mathcal{A}$, $\frac{z(f*g)'(z)}{\Phi(f*g)(z)} \in \mathcal{H}[1,1] \cap Q$, *and* $\Psi(\alpha, \gamma, g; z)$ *as defined by (2.2) be univalent in* Δ. *Further, if*

$$\alpha q_1(z) + \frac{\gamma z q_1'(z)}{q_1(z)} \prec \Psi(\alpha, \gamma, g; z) \prec \alpha q_2(z) + \frac{\gamma z q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \frac{z(f*g)'(z)}{\Phi(f*g)(z)} \prec q_2(z)$$

and q_1 *and* q_2 *are respectively the best subordinant and best dominant.*

REFERENCES

[1] T. BULBOACĂ, A class of superordination-preserving integral operators, *Indag. Math*. (N.S.), **13**(3) (2002), 301–311.

[2] T. BULBOACĂ, Classes of first-order differential superordinations, *Demonstr. Math.*, **35**(2) (2002), 287–292.

[3] L. BRICKMAN, Φ-like analytic functions. I, *Bull. Amer. Math. Soc.*, **79** (1973), 555–558.

[4] S.S. MILLER AND P.T. MOCANU, *Differential Subordinations*, Dekker, New York, 2000.

[5] S.S. MILLER AND P.T. MOCANU, Subordinants of differential superordinations, *Complex Var. Theory Appl.*, **48**(10) (2003), 815–826.

[6] V. RAVICHANDRAN AND M. JAYAMALA, On sufficient conditions for Caratheodory functions, *Far East J. Math. Sci.*, **12**(2) (2004), 191–201.

[7] St. RUSCHEWEYH, A subordination theorem for F-like functions, *J. London Math. Soc.*, **13**(2) (1976), 275–280.

[8] T.N. SHANMUGAM, V. RAVICHANDRAN AND S. SIVASUBRAMANIAN, Differential sandwich theorems for some subclasses of analytic Functions, *Aust. J. Math. Anal. Appl.*, **3**(1) (2006), Art. 8, 11 pp. (electronic).

[9] T.N. SHANMUGAM, S. SIVASUBRAMANIAN AND H.M. SRIVASTAVA, Differential sandwich theorems for certain subclasses of analytic functions involving multiplier transformations, *Int. Transforms Spec. Functions*, **17**(12) (2006), 889–899.

[10] T.N. SHANMUGAM, S. SIVASUBRAMANIAN AND M. DARUS, On certain subclasses of functions involving a linear Operator, *Far East J. Math. Sci.*, **23**(3), (2006), 329–339.

[11] T.N. SHANMUGAM, S. SIVASUBRAMANIAN AND S. OWA, On sandwich theorems for some subclasses of analytic functions involving a linear operator, to appear in *Integral Transforms and Special functions*.