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MARKOFF-TYPE INEQUALITIES IN WEIGHTED L^2-NORMS

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ABSTRACT. We give exact estimations of certain weighted L^2-norms of the k-th derivative of polynomials which have a curved majorant. They are all obtained as applications of special quadrature formulae.

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1. INTRODUCTION

The following problem was raised by P.Turán:

Let $\varphi(x) \geq 0$ for $-1 \leq x \leq 1$ and consider the class $P_{n,\varphi}$ of all polynomials of degree n such that $|p_n(x)| \leq \varphi(x)$ for $-1 \leq x \leq 1$. How large can $\max_{[-1,1]} \left|p_n^{(k)}(x)\right|$ be if p_n is an arbitrary polynomial in $P_{n,\varphi}$?

The aim of this paper is to consider the solution in the weighted L^2-norm for the majorant $\varphi(x) = \frac{\alpha+\beta-2\alpha x^2}{\sqrt{1-x^2}}$, $0 \leq \alpha \leq \beta$.

Let us denote by

$$x_i = \cos\frac{(2i-1)\pi}{2n}, i = 1, 2, \ldots, n, \text{ the zeros of } T_n(x) = \cos n\theta, x = \cos\theta, \tag{1.1}$$

the Chebyshev polynomial of the first kind,

$$y_i^{(k)} \text{ the zeros of } U_{n-1}^{(k)}(x), \quad U_{n-1}(x) = \frac{\sin n\theta}{\sin\theta}, \quad x = \cos\theta, \tag{1.2}$$

the Chebyshev polynomial of the second kind and

$$G_{n-1}(x) = \beta U_{n-1}(x) - \alpha U_{n-3}(x), \quad 0 \leq \alpha \leq \beta. \tag{1.3}$$

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Let $H_{\alpha,\beta}$ be the class of all real polynomials p_{n-1}, of degree $\leq n-1$ such that

$$|p_{n-1}(x_i)| \leq \frac{\alpha+\beta-2\alpha x_i^2}{\sqrt{1-x_i^2}}, \quad i=1,2,\dots,n, \tag{1.4}$$

where the x_i's are given by (1.1) and $0 \leq \alpha \leq \beta$.

2. RESULTS

Theorem 2.1. *If $p_{n-1} \in H_{\alpha,\beta}$ then we have*

$$\begin{aligned} &\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[p'_{n-1}(x)\right]^2 dx \\ &\qquad \leq \frac{2\pi(n-1)}{15} \left[(\alpha-\beta)^2 n(n+1)(n-2)(n-3)\right. \\ &\qquad\qquad \left. + 5(n-1)\left(\beta^2 n(n+1) - \alpha^2(n-2)(n-3)\right)\right] \end{aligned} \tag{2.1}$$

with equality for $p_{n-1} = G_{n-1}$.

Two cases are of special interest:

I. Case $\alpha=\beta=\frac{1}{2}$, $\varphi(x)=\sqrt{1-x^2}$ ***(circular majorant),*** $G_{n-1}=T_{n-1}$.
Note that $P_{n-1,\varphi} \subset H_{\frac{1}{2},\frac{1}{2}}$, $T_{n-1} \notin P_{n-1,\varphi}$, $T_{n-1} \in H_{\frac{1}{2},\frac{1}{2}}$.

Corollary 2.2. *If $p_{n-1} \in H_{\frac{1}{2},\frac{1}{2}}$ then we have*

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[p'_{n-1}(x)\right]^2 dx \leq \pi(n-1)^3, \tag{2.2}$$

with equality for $p_{n-1} = T_{n-1}$.

II. Case $\alpha=0$, $\beta=1$, $\varphi(x)=\frac{1}{\sqrt{1-x^2}}$, $G_{n-1}=U_{n-1}$.
Note that $P_{n-1,\varphi} \subset H_{0,1}$, $U_{n-1} \in P_{n-1,\varphi}$, $U_{n-1} \in H_{0,1}$.

Corollary 2.3. *If $p_{n-1} \in H_{0,1}$ then we have*

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[p'_{n-1}(x)\right]^2 dx \leq \frac{2\pi n(n^4-1)}{15}, \tag{2.3}$$

with equality for $p_{n-1} = U_{n-1}$.

In this second case we have a more general result:

Theorem 2.4. *If $p_{n-1} \in H_{0,1}$ and*

$$r(x) = b(b-2a)x^2 + 2c(b-a)x + a^2 + c^2$$

with $0<a<b, |c|<b-a, b \neq 2a$ then we have

$$\begin{aligned} &\int_{-1}^{1} r(x)\left(1-x^2\right)^{k-1/2} \left[p_{n-1}^{(k+1)}(x)\right]^2 dx \\ &\qquad \leq \frac{\pi(n+k+1)!}{(n-k-2)!} \left[\frac{\left[2\left(n^2-k^2\right)-3(2k+1)\right]\left[(a-b)^2+c^2\right]}{(2k+1)(2k+3)(2k+5)} + \frac{a^2+c^2}{2k+3}\right], \end{aligned} \tag{2.4}$$

where $k=0,\dots,n-2$, with equality for $p_{n-1} = U_{n-1}$.

Setting $a=1, b=c=0$ one obtains the following

Corollary 2.5. *If* $p_{n-1} \in H_{0,1}$ *then we have*

(2.5) $$\int_{-1}^{1} \left(1-x^2\right)^{k-1/2} \left[p_{n-1}^{(k+1)}(x)\right]^2 dx \le \frac{2\pi (n+k+1)!}{(n-k-2)!} \frac{n^2+k^2+3k+1}{(2k+1)(2k+3)(2k+5)},$$

$k = 0, \dots, n-2$ *, with equality for* $p_{n-1} = U_{n-1}$.

3. LEMMAS

Here we state and prove some lemmas which help us in proving the above theorems.

Lemma 3.1. *Let* p_{n-1} *be such that* $|p_{n-1}(x_i)| \le \frac{\alpha+\beta-2\alpha x_i^2}{\sqrt{1-x_i^2}}, i = 1, 2, \dots, n$, *where the* x_i*'s are given by (1.1). Then we have*

(3.1) $$\left|p'_{n-1}(y_j)\right| \le \left|G'_{n-1}(y_j)\right|, \quad k = 0, 1, \dots, n-1,$$

and

(3.2) $$\left|p'_{n-1}(1)\right| \le \left|G'_{n-1}(1)\right|, \quad \left|p'_{n-1}(-1)\right| \le \left|G'_{n-1}(-1)\right|.$$

Proof. By the Lagrange interpolation formula based on the zeros of T_n and using $T'_n(x_i) = \frac{(-1)^{i+1} n}{\left(1-x_i^2\right)^{1/2}}$, we can represent any algebraic polynomial p_{n-1} by

$$p_{n-1}(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{T_n(x)}{x-x_i} (-1)^{i+1} \left(1-x_i^2\right)^{1/2} p_{n-1}(x_i).$$

From

$$G_{n-1}(x_i) = (-1)^{i+1} \frac{\alpha+\beta-2\alpha x_i^2}{\sqrt{1-x_i^2}}$$

we have

$$G_{n-1}(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{T_n(x)}{x-x_i} \left(\alpha+\beta-2\alpha x_i^2\right).$$

Differentiating with respect to x we obtain

$$p'_{n-1}(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{T'_n(x)(x-x_i) - T_n(x)}{(x-x_i)^2} (-1)^{i+1} \left(1-x_i^2\right)^{1/2} p_{n-1}(x_i).$$

On the roots of $T'_n(x) = nU_{n-1}(x)$ and using (1.4) we find

$$\begin{aligned} \left|p'_{n-1}(y_j)\right| &\le \frac{1}{n} \sum_{i=1}^{n} \frac{|T_n(y_j)|}{(y_j-x_i)^2} \left(\alpha+\beta-2\alpha x_i^2\right) \\ &= \frac{|T_n(y_j)|}{n} \sum_{i=1}^{n} \frac{\alpha+\beta-2\alpha x_i^2}{(y_j-x_i)^2} = \left|G'_{n-1}(y_j)\right|. \end{aligned}$$

For $l_i(x) = \frac{T_n(x)}{x-x_i}$ taking into account that $l'_i(1) > 0$ (see [6]) it follows that

$$\left|p'_{n-1}(1)\right| \le \frac{1}{n} \sum_{i=1}^{n} l'_i(1) \left(\alpha+\beta-2\alpha x_i^2\right) = \left|G'_{n-1}(1)\right|.$$

Similarly $\left|p'_{n-1}(-1)\right| \le \left|G'_{n-1}(-1)\right|$. □

We shall need the result of Duffin and Schaeffer [2]:

Lemma 3.2 (Duffin – Schaeffer). *If* $q(x) = c\prod_{i=1}^{n}(x - x_i)$ *is a polynomial of degree* n *with* n *distinct real zeros and if* $p \in P_n$ *is such that*

$$|p'(x_i)| \le |q'(x_i)| \quad (i = 1, 2, \dots, n),$$

then for $k = 1, 2, \dots, n-1$,

$$\left|p^{(k+1)}(x)\right| \le \left|q^{(k+1)}(x)\right|$$

whenever $q^{(k)}(x) = 0$.

Lemma 3.3. *Let* p_{n-1} *be such that* $|p_{n-1}(x_i)| \le \frac{1}{\sqrt{1-x_i^2}}$, $i = 1, 2, \dots, n$, *where the* x_i*'s are given by (1.1). Then we have*

$$\left|p_{n-1}^{(k+1)}(y_j^{(k)})\right| \le \left|U_{n-1}^{(k+1)}(y_j^{(k)})\right|, \quad \textit{whenever } U_{n-1}^{(k)}(y_j^{(k)}) = 0, \tag{3.3}$$

for $k = 0, 1, \dots, n-1$, *and*

$$\left|p_{n-1}^{(k+1)}(1)\right| \le \left|U_{n-1}^{(k+1)}(1)\right|, \left|p_{n-1}^{(k+1)}(-1)\right| \le \left|U_{n-1}^{(k+1)}(-1)\right|. \tag{3.4}$$

Proof. For $\alpha = 0$, $\beta = 1$, $G_{n-1} = U_{n-1}$ and (3.1) give $\left|p'_{n-1}(y_j)\right| \le \left|U'_{n-1}(y_j)\right|$ and (3.2)

$$\left|p'_{n-1}(1)\right| \le \left|U'_{n-1}(1)\right|, \quad \left|p'_{n-1}(-1)\right| \le \left|U'_{n-1}(-1)\right|.$$

Now the proof is concluded by applying the Duffin-Schaeffer lemma. □

The following proposition was proved in [3].

Lemma 3.4. *A real polynomial* r *of exact degree 2 satisfies* $r(x) > 0$ *for* $-1 \le x \le 1$ *if and only if*

$$r(x) = b(b-2a)x^2 + 2c(b-a)x + a^2 + c^2$$

with $0 < a < b,\ |c| < b - a,\ b \ne 2a$.

We need the following quadrature formulae:

Lemma 3.5. *For any given* n *and* k, $0 \le k \le n-1$, *let* $y_i^{(k)}$, $i = 1, \dots, n-k-1$, *be the zeros of* $U_{n-1}^{(k)}$. *Then the quadrature formulae*

$$\int_{-1}^{1}\left(1-x^2\right)^{k-1/2} f(x)\,dx = A_0\left[f(-1) + f(1)\right] + \sum_{i=1}^{n-k-1} s_i f\left(y_i^{(k)}\right), \tag{3.5}$$

where

$$A_0 = \frac{2^{2k-1}(2k+1)\Gamma(k+1/2)^2(n-k-1)!}{(n+k)!}, \qquad s_i > 0$$

and

$$\int_{-1}^{1}\left(1-x^2\right)^{k-1/2} f(x)\,dx \tag{3.6}$$

$$= B_0\left[f(-1) + f(1)\right] + C_0\left[f'(-1) - f'(1)\right] + \sum_{i=1}^{n-k-2} v_i f\left(y_i^{(k+1)}\right),$$

where

$$C_0 = \frac{2^{2k}(2k+3)\Gamma(k+3/2)^2(n-k-2)!}{(n+k+1)!},$$

$$B_0 = C_0 \frac{2\left(n^2-(k+2)^2\right)(2k+3)+4(k+1)(2k+5)}{(2k+1)(2k+5)}$$

have algebraic degree of precision $2n-2k-1$.

The quadrature formulae

$$\int_{-1}^{1} r(x)\left(1-x^2\right)^{k-1/2} f(x)\,dx = A_1 f(-1) + B_1 f(1) + \sum_{i=1}^{n-k-1} s_i r\left(y_i^{(k)}\right) f\left(y_i^{(k)}\right), \tag{3.7}$$

where

$$A_1 = \frac{2^{2k-1}(2k+1)\Gamma(k+1/2)^2(n-k-1)!(a-b+c)^2}{(n+k)!},$$

$$B_1 = \frac{2^{2k-1}(2k+1)\Gamma(k+1/2)^2(n-k-1)!(a-b-c)^2}{(n+k)!}$$

and

$$\int_{-1}^{1} r(x)\left(1-x^2\right)^{k-1/2} f(x)\,dx = C_1 f(-1) + D_1 f(1) + C_2 f'(-1) - D_2 f'(1) + \sum_{i=1}^{n-k-2} v_i r\left(y_i^{(k+1)}\right) f\left(y_i^{(k+1)}\right), \tag{3.8}$$

$$C_1 = B_0(a-b+c)^2 + 2C_0 d, D_1 = B_0(a-b-c)^2 - 2C_0 e,$$

$$C_2 = C_0(a-b+c)^2, D_2 = C_0(a-b-c)^2,$$

$$d = 2ab+bc-ac-b^2, e = b^2-2ab+bc-ac.$$

have algebraic degree of precision $2n-2k-3$.

Proof. In order to compute the coefficients we need the following formulae

$$\begin{aligned}
&\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\lambda} P_m^{(\alpha,\beta)}(x)\,dx \\
&\qquad = \frac{(-1)^m 2^{\alpha+\lambda+1}\Gamma(\lambda+1)\Gamma(m+\alpha+1)\Gamma(\beta-\lambda+m)}{\Gamma(m+1)\Gamma(\beta-\lambda)\Gamma(m+\alpha+\lambda+2)}, \quad \lambda<\beta. \\
&\int_{-1}^{1}(1-x)^{\lambda}(1+x)^{\beta} P_m^{(\alpha,\beta)}(x)\,dx \\
&\qquad = \frac{2^{\beta+\lambda+1}\Gamma(\lambda+1)\Gamma(m+\beta+1)\Gamma(\alpha-\lambda+m)}{\Gamma(m+1)\Gamma(\alpha-\lambda)\Gamma(m+\beta+\lambda+2)}, \quad \lambda<\alpha
\end{aligned} \tag{3.9}$$

The first quadrature formula (3.5) is the Bouzitat formula of the second kind [4, *formula* (4.8.1)], for the zeros of $U_{n-1}^{(k)} = cP_{n-k-1}^{\left(k+\frac{1}{2},k+\frac{1}{2}\right)}$. Setting $\alpha=\beta=\frac{1}{2}$, $m=n-k-1$ in [4, *formula* (4.8.5)] we find A_0 and $s_i>0$ (cf. [4, *formula* (4.8.4)]).

If in the above quadrature formula (3.6), taking into account (3.9), we put

$$f(x) = (1-x)(1+x)^2 P_{n-k-2}^{\left(k+\frac{3}{2},k+\frac{3}{2}\right)}(x),$$

$$U_{n-1}^{(k+1)}(x) = cP_{n-k-2}^{\left(k+\frac{3}{2},k+\frac{3}{2}\right)}(x),$$

we obtain C_0, and for

$$f(x) = (1+x)^2 P_{n-k-2}^{\left(k+\frac{3}{2},k+\frac{3}{2}\right)}(x)$$

we find B_0.

If in formula (3.5) we replace $f(x)$ with $r(x)f(x)$ we get (3.7) and if in formula (3.6) we replace $f(x)$ with $r(x)f(x)$ we get (3.8). □

4. PROOF OF THE THEOREMS

Proof of Theorem 2.1. Setting $k=0$ in (3.5) we find the formula

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = \frac{\pi}{2n}\left[f(-1)+f(1)\right] + \frac{\pi}{n}\sum_{i=1}^{n-1} f(y_i).$$

According to this quadrature formula and using (3.1) and (3.2) we have

$$\begin{aligned}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\left[p'_{n-1}(x)\right]^2 dx &= \frac{\pi}{2n}\left(p'_{n-1}(-1)\right)^2 + \frac{\pi}{2n}\left(p'_{n-1}(1)\right)^2 + \frac{\pi}{n}\sum_{i=1}^{n-1}\left(p'_{n-1}(y_i)\right)^2 \\ &\le \frac{\pi}{2n}\left(G'_{n-1}(-1)\right)^2 + \frac{\pi}{2n}\left(G'_{n-1}(1)\right)^2 + \frac{\pi}{n}\sum_{i=1}^{n-1}\left(G'_{n-1}(y_i)\right)^2 \\ &= \int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\left[G'_{n-1}(x)\right]^2 dx.\end{aligned}$$

Now

$$\begin{aligned}\int_{-1}^{1}\frac{\left[G'_{n-1}(x)\right]^2}{\sqrt{1-x^2}}dx &= \beta^2\int_{-1}^{1}\frac{\left[U'_{n-1}(x)\right]^2}{\sqrt{1-x^2}}dx \\ &\quad - 2\alpha\beta\int_{-1}^{1}\frac{U'_{n-1}(x)\,U'_{n-3}(x)}{\sqrt{1-x^2}}dx + \alpha^2\int_{-1}^{1}\frac{\left[U'_{n-3}(x)\right]^2}{\sqrt{1-x^2}}dx.\end{aligned}$$

Using the following formula ($k=0$ in (3.6))

$$\begin{aligned}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx &= \frac{3\pi\left(3n^2-2\right)}{10n\left(n^2-1\right)}\left[f(-1)+f(1)\right] \\ &\quad + \frac{3\pi}{4n\left(n^2-1\right)}\left[f'(-1)-f'(1)\right] + \sum_{i=1}^{n-2} c_i f(y'_i)\end{aligned}$$

we find

$$\int_{-1}^{1}\frac{\left[U'_{n-1}(x)\right]^2}{\sqrt{1-x^2}} = \frac{2\pi n\left(n^4-1\right)}{15},$$

$$\int_{-1}^{1}\frac{U'_{n-1}(x)\,U'_{n-3}(x)}{\sqrt{1-x^2}} = \frac{2\pi n\left(n^2-1\right)(n-2)(n-3)}{15},$$

$$\int_{-1}^{1}\frac{\left[U'_{n-3}(x)\right]^2}{\sqrt{1-x^2}} = \frac{2\pi(n-1)\left(n^2-4n+5\right)(n-2)(n-3)}{15}$$

and

$$\int_{-1}^{1}\frac{\left[G_{n-1}'(x)\right]^2}{\sqrt{1-x^2}}dx=\frac{2\pi(n-1)}{15}\left[(\alpha-\beta)^2 n(n+1)(n-2)(n-3)\right.$$
$$\left.+5(n-1)\left(\beta^2 n(n+1)-\alpha^2(n-2)(n-3)\right)\right].$$

□

Proof of Theorem 2.4. According to the quadrature formula (3.7), positivity of s_i's, and using (3.3) and (3.4) we have

$$\begin{aligned}\int_{-1}^{1} r(x)\left(1-x^2\right)^{k-1/2}\left[p_{n-1}^{(k+1)}(x)\right]^2 dx\\
&= A_1\left[p_{n-1}^{(k+1)}(-1)\right]^2+B_1\left[p_{n-1}^{(k+1)}(1)\right]^2+\sum_{i=1}^{n-k-1}s_i r\left(y_i^{(k)}\right)\left[p_{n-1}^{(k+1)}\left(y_i^{(k)}\right)\right]^2\\
&\le A_1\left[U_{n-1}^{(k+1)}(-1)\right]^2+B_1\left[U_{n-1}^{(k+1)}(1)\right]^2+\sum_{i=1}^{n-k-1}s_i r\left(y_i^{(k)}\right)\left[U_{n-1}^{(k+1)}\left(y_i^{(k)}\right)\right]^2\\
&= \int_{-1}^{1} r(x)\left(1-x^2\right)^{k-1/2}\left[U_{n-1}^{(k+1)}(x)\right]^2 dx.\end{aligned}$$

In order to complete the proof we apply formula (3.8) to $f=\left[U_{n-1}^{(k+1)}(x)\right]^2$.

Having in mind $U_{n-1}^{(k+1)}\left(y_i^{(k+1)}\right)=0$ and the following relations deduced from [1]

$$U_{n-1}^{(k+1)}(1)=\frac{n\left(n^2-1^2\right)\cdots\left(n^2-(k+1)^2\right)}{1\cdot 3\cdots(2k+3)},$$

$$U_{n-1}^{(k+2)}(1)=\frac{n^2-(k+2)^2}{2k+5}U_{n-1}^{(k+1)}(1),$$

$$U_{n-1}^{(k+1)}(-1)\,U_{n-1}^{(k+2)}(-1)=-U_{n-1}^{(k+1)}(1)\,U_{n-1}^{(k+2)}(1),$$

we find

$$\begin{aligned}\int_{-1}^{1} r(x)\left(1-x^2\right)^{k-1/2}\left[p_{n-1}^{(k+1)}(x)\right]^2 dx\\
&= C_1\left[U_{n-1}^{(k+1)}(-1)\right]^2+D_1\left[U_{n-1}^{(k+1)}(1)\right]^2\\
&\quad+2C_2U_{n-1}^{(k+1)}(-1)\,U_{n-1}^{(k+2)}(-1)-2D_2U_{n-1}^{(k+1)}(1)\,U_{n-1}^{(k+2)}(1)\\
&= \frac{\pi(n+k+1)!}{(n-k-2)!}\left[\frac{\left[2\left(n^2-k^2\right)-3(2k+1)\right]\left[(a-b)^2+c^2\right]}{(2k+1)(2k+3)(2k+5)}+\frac{a^2+c^2}{2k+3}\right].\end{aligned}$$

□

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