Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 3, Article 62, 2005

ABSOLUTE NÖRLUND SUMMABILITY FACTORS

HÜSEYİN BOR

DEPARTMENT OF MATHEMATICS
ERCIYES UNIVERSITY
38039 KAYSERI, TURKEY
bor@erciyes.edu.tr
URL: http://fef.erciyes.edu.tr/math/hbor.htm

Received 24 May, 2005; accepted 31 May, 2005
Communicated by L. Leindler

ABSTRACT. In this paper a theorem on the absolute Nörlund summability factors has been proved under more weaker conditions by using a quasi β-power increasing sequence instead of an almost increasing sequence.

Key words and phrases: Nörlund summability, summability factors, power increasing sequences.

2000 *Mathematics Subject Classification.* 40D15, 40F05, 40G05.

1. INTRODUCTION

A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [2]).

Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n) and $w_n = na_n$. By u_n^α and t_n^α we denote the n-th Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (w_n), respectively. The series $\sum a_n$ is said to be summable $|C, \alpha|$, if (see [5], [7])

$$\sum_{n=1}^{\infty} \left| u_n^\alpha - u_{n-1}^\alpha \right| = \sum_{n=1}^{\infty} \frac{1}{n} \left| t_n^\alpha \right| < \infty. \tag{1.1}$$

Let (p_n) be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n \neq 0, \quad (n \geq 0). \tag{1.2}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \tag{1.3}$$

ISSN (electronic): 1443-5756

163-05

defines the sequence (σ_n) of the Nörlund mean of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $|N, p_n|$, if (see [9])

$$\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty. \tag{1.4}$$

In the special case when

$$p_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \quad \alpha \geq 0 \tag{1.5}$$

the Nörlund mean reduces to the (C, α) mean and $|N, p_n|$ summability becomes $|C, \alpha|$ summability. For $p_n = 1$ and $P_n = n$, we get the $(C, 1)$ mean and then $|N, p_n|$ summability becomes $|C, 1|$ summability. For any sequence (λ_n), we write $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ and $\Delta^2\lambda_n = \Delta(\Delta\lambda_n) = \Delta\lambda_n - \Delta\lambda_{n+1}$.

In [6] Kishore has proved the following theorem concerning $|C, 1|$ and $|N, p_n|$ summability methods.

Theorem 1.1. *Let $p_0 > 0$, $p_n \geq 0$ and (p_n) be a non-increasing sequence. If $\sum a_n$ is summable $|C, 1|$, then the series $\sum a_n P_n (n+1)^{-1}$ is summable $|N, p_n|$.*

Ahmad [1] proved the following theorem for absolute Nörlund summability factors.

Theorem 1.2. *Let (p_n) be as in Theorem 1.1. If*

$$\sum_{v=1}^{n} \frac{1}{v} |t_v| = O(X_n) \quad \textit{as } n \to \infty, \tag{1.6}$$

where (X_n) is a positive non-decreasing sequence and (λ_n) is a sequence such that

$$X_n \lambda_n = O(1), \tag{1.7}$$

$$n\Delta X_n = O(X_n), \tag{1.8}$$

$$\sum nX_n \left|\Delta^2\lambda_n\right| < \infty, \tag{1.9}$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|$.

Later on Bor [3] proved Theorem 1.2 under weaker conditions in the following form.

Theorem 1.3. *Let (p_n) be as in Theorem 1.1 and let (X_n) be a positive non-decreasing sequence. If the conditions (1.6) and (1.7) of Theorem 1.2 are satisfied and the sequences (λ_n) and (β_n) are such that*

$$|\Delta\lambda_n| \leq \beta_n, \tag{1.10}$$

$$\beta_n \to 0, \tag{1.11}$$

$$\sum nX_n |\Delta\beta_n| < \infty, \tag{1.12}$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|$.

Also Bor [4] has proved Theorem 1.3 under the weaker conditions in the following form.

Theorem 1.4. *Let (p_n) be as in Theorem 1.1 and let (X_n) be an almost increasing sequence. If the conditions (1.6), (1.7), (1.10) and (1.12) of Theorem 1.2 and Theorem 1.3 are satisfied, then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|$.*

2. MAIN RESULT

The aim of this paper is to prove Theorem 1.4 under more weaker conditions. For this we need the concept of a quasi β-power increasing sequence. A positive sequence (γ_n) is said to be a quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$Kn^\beta \gamma_n \geq m^\beta \gamma_m \tag{2.1}$$

holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is a quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking an example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. So we are weakening the hypotheses of the theorem replacing an almost increasing sequence by a quasi β-power increasing sequence.

Now we shall prove the following theorem.

Theorem 2.1. *Let (p_n) be as in Theorem 1.1 and let (X_n) be a quasi β-power increasing sequence. If the conditions (1.6), (1.7), (1.10) and (1.12) of Theorem 1.2 and Theorem 1.3 are satisfied, then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|$.*

We need the following lemma for the proof of our theorem.

Lemma 2.2 ([8])**.** *Under the conditions on (X_n), (λ_n) and (β_n), as taken in the statement of the theorem, the following conditions hold, when (1.12) is satisfied:*

$$n\beta_n X_n = O(1) \quad \text{as } n \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{2.3}$$

Proof of Theorem 2.1. In order to prove the theorem, we need consider only the special case in which (N, p_n) is $(C, 1)$, that is, we shall prove that $\sum a_n \lambda_n$ is summable $|C, 1|$. Our theorem will then follow by means of Theorem 1.1. Let T_n be the n-th $(C, 1)$ mean of the sequence $(na_n\lambda_n)$, that is,

$$T_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v \lambda_v. \tag{2.4}$$

Using Abel's transformation, we have

$$\begin{aligned} T_n &= \frac{1}{n+1} \sum_{v=1}^{n} v a_v \lambda_v \\ &= \frac{1}{n+1} \sum_{v=1}^{n-1} \Delta\lambda_v (v+1) t_v + \lambda_n t_n \\ &= T_{n,1} + T_{n,2}, \quad \text{say.} \end{aligned}$$

To complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} |T_{n,r}| < \infty \quad \text{for } r = 1, 2, \text{ by (1.1).} \tag{2.5}$$

Now, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \frac{1}{n} |T_{n,1}| &\leq \sum_{n=2}^{m+1} \frac{1}{n(n+1)} \left\{ \sum_{v=1}^{n-1} \frac{v+1}{v} v |\Delta\lambda_v| |t_v| \right\} \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^2} \left\{ \sum_{v=1}^{n-1} v\beta_v |t_v| \right\} \\
&= O(1) \sum_{v=1}^{m} v\beta_v |t_v| \sum_{n=v+1}^{m+1} \frac{1}{n^2} \\
&= O(1) \sum_{v=1}^{m} v\beta_v \frac{|t_v|}{v} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} \frac{|t_r|}{r} + O(1) m\beta_m \sum_{v=1}^{m} \frac{|t_v|}{v} \\
&= O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)| X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} |(v+1)\Delta\beta_v - \beta_v| X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v |\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} |\beta_v| X_v + O(1) m\beta_m X_m \\
&= O(1) \quad \text{as } m \to \infty,
\end{aligned}$$

by (1.6), (1.10), (1.12), (2.2) and (2.3).

Again

$$\begin{aligned}
\sum_{n=1}^{m} \frac{1}{n} |T_{n,2}| &= \sum_{n=1}^{m} |\lambda_n| \frac{|t_n|}{n} \\
&= \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} \frac{|t_v|}{v} + |\lambda_m| \sum_{n=1}^{m} \frac{|t_n|}{n} \\
&= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1) |\lambda_m| X_m \\
&= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) |\lambda_m| X_m = O(1) \quad \text{as } m \to \infty,
\end{aligned}$$

by (1.6), (1.7), (1.10) and (2.3). This completes the proof of the theorem. □

REFERENCES

[1] Z.U. AHMAD, Absolute Nörlund summability factors of power series and Fourier series, *Ann. Polon. Math.*, **27** (1972), 9–20.

[2] S. ALJANČIĆ AND D. ARANDELOVIĆ, *O*-regularly varying functions, *Publ. Inst. Math.*, **22** (1977), 5–22.

[3] H. BOR, Absolute Nörlund summability factors of power series and Fourier series, *Ann. Polon. Math.*, **56** (1991), 11–17.

[4] H. BOR, On the Absolute Nörlund summability factors, *Math. Commun.*, **5** (2000), 143–147.

[5] M. FETEKE, Zur Theorie der divergenten Reihen, *Math. es Termes Ertesitö* (Budapest), **29** (1911), 719–726.

[6] N. KISHORE, On the absolute Nörlund summability factors, *Riv. Math. Univ. Parma*, **6** (1965), 129–134.

[7] E. KOGBENTLIANTZ, Sur lés series absolument sommables par la méthode des moyennes arithmétiques, *Bull. Sci. Math.*, **49** (1925), 234–256.

[8] L. LEINDLER, A new application of quasi power increasing sequences, *Publ. Math. Debrecen*, **58** (2001), 791–796.

[9] F.M. MEARS, Some multiplication theorems for the Nörlund mean, *Bull. Amer. Math. Soc.*, **41** (1935), 875–880.