

Journal of Inequalities in Pure and Applied Mathematics

A REFINEMENT OF JENSEN'S INEQUALITY

J. ROOIN

Department of Mathematics
Institute for Advanced Studies in Basic Sciences
Zanjan, Iran

EMail: rooin@iasbs.ac.ir

©2000 Victoria University
ISSN (electronic): 1443-5756
160-04



volume 6, issue 2, article 38,
2005.

*Received 27 August, 2004;
accepted 16 March, 2005.*

Communicated by: *C.E.M. Pearce*

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

We refine Jensen's inequality as

$$\varphi\left(\int_X f d\mu\right) \leq \int_Y \varphi\left(\int_X f(x)\omega(x,y)d\mu(x)\right) d\lambda(y) \leq \int_X (\varphi \circ f) d\mu,$$

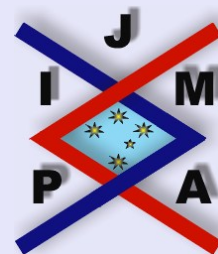
where (X, \mathcal{A}, μ) and $(Y, \mathcal{B}, \lambda)$ are two probability measure spaces, $\omega : X \times Y \rightarrow [0, \infty)$ is a weight function on $X \times Y$, I is an interval of the real line, $f \in L^1(\mu)$, $f(x) \in I$ for all $x \in X$ and φ is a real-valued convex function on I .

2000 Mathematics Subject Classification: Primary: 26D15, 28A35.

Key words: Product measure, Fubini's Theorem, Jensen's inequality.

Contents

1	Introduction	3
2	Refinement	4
	References	



A Refinement of Jensen's Inequality

J. Rooin

Title Page

Contents



Go Back

Close

Quit

Page 2 of 9

1. Introduction

The classical integral form of Jensen's inequality states that

$$(1.1) \quad \varphi \left(\int_X f d\mu \right) \leq \int_X (\varphi \circ f) d\mu,$$

where (X, \mathcal{A}, μ) is a probability measure space, I is an interval of the real line, $f \in L^1(\mu)$, $f(x) \in I$ for all $x \in X$ and φ is a real-valued convex function on I ; see e.g. [2, p. 202] or [4, p. 62]. Now suppose that (X, \mathcal{A}, μ) and $(Y, \mathcal{B}, \lambda)$ are two probability measure spaces. By a (separately) weight function on $X \times Y$ we mean a product-measurable mapping $\omega : X \times Y \rightarrow [0, \infty)$, see e.g. [4, p. 160], such that

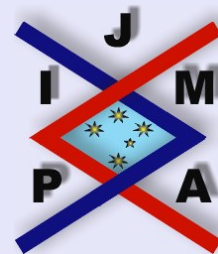
$$(1.2) \quad \int_X \omega(x, y) d\mu(x) = 1 \quad (\text{for each } y \text{ in } Y),$$

and

$$(1.3) \quad \int_Y \omega(x, y) d\lambda(y) = 1 \quad (\text{for each } x \text{ in } X).$$

For example, if we take X and Y as the unit interval $[0, 1]$ with Lebesgue measure, then $\omega(x, y) = 1 + (\sin 2\pi x)(\sin 2\pi y)$ is a weight function on $[0, 1] \times [0, 1]$.

In this paper, using a weight function ω , we refine Jensen's inequality (1.1) as in the following section. For some applications in the discrete case, see e.g. [3].



A Refinement of Jensen's
Inequality

J. Roojin

Title Page

Contents



Go Back

Close

Quit

Page 3 of 9

2. Refinement

In this section, using the terminologies of the introduction, we refine the integral form of Jensen's inequality (1.1) via a weight function ω .

Theorem 2.1. *Let (X, \mathcal{A}, μ) and $(Y, \mathcal{B}, \lambda)$ be two probability measure spaces and $\omega : X \times Y \rightarrow [0, \infty)$ be a weight function on $X \times Y$. If I is an interval of the real line, $f \in L^1(\mu)$, $f(x) \in I$ for all $x \in X$, and φ is a real convex function on I , then*

$$\int_Y \varphi \left(\int_X f(x) \omega(x, y) d\mu(x) \right) d\lambda(y)$$

has meaning and we have

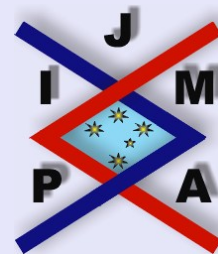
$$(2.1) \quad \varphi \left(\int_X f d\mu \right) \leq \int_Y \varphi \left(\int_X f(x) \omega(x, y) d\mu(x) \right) d\lambda(y) \leq \int_X (\varphi \circ f) d\mu.$$

Proof. The functions ω and $(x, y) \rightarrow f(x)$, and so

$$(x, y) \rightarrow f(x) \omega(x, y)$$

is product-measurable on $X \times Y$. Now since

$$(2.2) \quad \begin{aligned} \int_X \int_Y |f(x)| \omega(x, y) d\lambda(y) d\mu(x) \\ &= \int_X |f(x)| \left(\int_Y \omega(x, y) d\lambda(y) \right) d\mu(x) \\ &= \int_X |f(x)| d\mu(x) = \|f\|_{L^1(\mu)} < \infty, \end{aligned}$$



A Refinement of Jensen's Inequality

J. Rooin

Title Page

Contents



Go Back

Close

Quit

Page 4 of 9

by Fubini's theorem, the real-valued function $(x, y) \rightarrow f(x)\omega(x, y)$ on $X \times Y$ belongs to $L^1(\mu \times \lambda)$. Therefore for λ -almost all $y \in Y$, the function $x \rightarrow f(x)\omega(x, y)$ belongs to $L^1(\mu)$. Fix an arbitrary $\alpha \in I$. Define $F : Y \rightarrow \mathbb{R}$, by

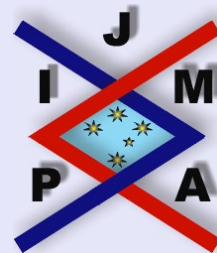
$$F(y) = \int_X f(x)\omega(x, y)d\mu(x)$$

if the integral exists, and $F(y) = \alpha$ otherwise. By Fubini's theorem, we have $F \in L^1(\lambda)$. It is easy to show that $F(y) \in I$ ($y \in Y$). So,

$$\int_Y \varphi \left(\int_X f(x)\omega(x, y)d\mu(x) \right) d\lambda(y) := \int_Y (\varphi \circ F)(y)d\lambda(y)$$

has meaning and is an extended real number belonging to $(-\infty, +\infty]$; see e.g. [4, p. 62]. Now, since $(x, y) \rightarrow f(x)\omega(x, y)$ belongs to $L^1(\mu \times \lambda)$, by (1.1) and Fubini's theorem, we have

$$\begin{aligned} \int_Y \varphi \left(\int_X f(x)\omega(x, y)d\mu(x) \right) d\lambda(y) &= \int_Y (\varphi \circ F)(y)d\lambda(y) \\ &\geq \varphi \left(\int_Y F(y)d\lambda(y) \right) \\ &= \varphi \left(\int_Y \int_X f(x)\omega(x, y)d\mu(x)d\lambda(y) \right) \\ &= \varphi \left(\int_X f(x) \left(\int_Y \omega(x, y)d\lambda(y) \right) d\mu(x) \right) \\ &= \varphi \left(\int_X f d\mu \right), \end{aligned}$$



A Refinement of Jensen's Inequality

J. Rooin

Title Page

Contents



Go Back

Close

Quit

Page 5 of 9

and the left-hand side inequality (2.1) is obtained.

For the right-hand side inequality in (2.1), we consider two cases: If $\int_X (\varphi \circ f) d\mu = +\infty$, the assertion is trivial. Suppose then, $\varphi \circ f \in L^1(\mu)$. Take an arbitrary $y \in Y$ such that $x \rightarrow f(x)\omega(x, y)$ belongs to $L^1(\mu)$, and put

$$d\nu^y = \omega^y d\mu,$$

where

$$\omega^y(x) = \omega(x, y) \quad (x \in X).$$

Trivially, (X, \mathcal{A}, ν^y) is a probability measure space, $f \in L^1(\nu^y)$ and

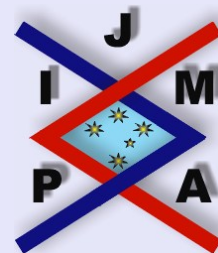
$$F(y) = \int_X f(x)\omega(x, y)d\mu(x) = \int_X f(x)d\nu^y(x).$$

Thus, by Jensen's inequality (1.1), we have

$$(2.3) \quad (\varphi \circ F)(y) = \varphi \left(\int_X f(x)d\nu^y(x) \right) \leq \int_X (\varphi \circ f)d\nu^y.$$

Since $\varphi \circ f \in L^1(\mu)$,

$$(2.4) \quad \begin{aligned} & \int_X \int_Y |(\varphi \circ f)(x)|\omega(x, y)d\lambda(y)d\mu(x) \\ &= \int_X |(\varphi \circ f)(x)|d\mu(x) \int_Y \omega(x, y)d\lambda(y) \\ &= \int_X |(\varphi \circ f)(x)|d\mu(x) < \infty, \end{aligned}$$



A Refinement of Jensen's Inequality

J. Rooin

Title Page

Contents



Go Back

Close

Quit

Page 6 of 9

and so for λ -almost all $y \in Y$, the function $x \rightarrow (\varphi \circ f)(x)\omega(x, y)$ belongs to $L^1(\mu)$ and for these y 's, we have

$$(2.5) \quad \int_X (\varphi \circ f)(x)\omega(x, y)d\mu(x) = \int_X (\varphi \circ f)(x)d\nu^y(x).$$

Thus, by (2.3) and (2.5), for λ -almost all $y \in Y$

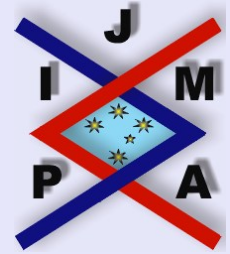
$$(2.6) \quad (\varphi \circ F)(y) \leq \int_X (\varphi \circ f)(x)\omega(x, y)d\mu(x).$$

Denote temporarily the right-hand side of (2.6) by $\psi(y)$ (put $\psi(y) = 0$, if the integral does not exist). Since by (2.4), $\psi \in L^1(\lambda)$, from $(\varphi \circ F)^+ \leq \psi^+$ (λ -a.e.), we conclude that $\int_Y (\varphi \circ F)^+ d\lambda \leq \int_Y \psi^+ d\lambda < \infty$.

On the other hand, we know that $\int_Y (\varphi \circ F)^- d\lambda < \infty$. Thus $\varphi \circ F \in L^1(\lambda)$, and so by (2.6), (2.4) and Fubini's theorem,

$$\begin{aligned} \int_Y \varphi \left(\int_X f(x)\omega(x, y)d\mu(x) \right) d\lambda(y) &= \int_Y (\varphi \circ F)(y)d\lambda(y) \\ &\leq \int_Y \psi(y)d\lambda(y) \\ &= \int_Y \int_X (\varphi \circ f)(x)\omega(x, y)d\mu(x)d\lambda(y) \\ &= \int_X (\varphi \circ f)(x)d\mu(x) \int_Y \omega(x, y)d\lambda(y) \\ &= \int_X (\varphi \circ f)d\mu. \end{aligned}$$

This completes the proof. \square



A Refinement of Jensen's Inequality

J. Rooin

Title Page

Contents



Go Back

Close

Quit

Page 7 of 9

Corollary 2.2. *If φ is a real convex function on a closed interval $[a, b]$, then we have Hermite-Hadamard inequalities [1]:*

$$(2.7) \quad \varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \varphi(t) dt \leq \frac{\varphi(a) + \varphi(b)}{2}.$$

Proof. Put $X = \{0, 1\}$ with $\mathcal{A} = 2^X$ and $\mu\{0\} = \mu\{1\} = \frac{1}{2}$, and $Y = [0, 1]$ with Lebesgue measure λ . Now, (2.7) follows from (2.1) by taking $\omega(0, y) = 2(1-y)$, $\omega(1, y) = 2y$ ($0 \leq y \leq 1$), $I = [a, b]$, $f(0) = a$, $f(1) = b$, and considering the change of variables $t = (1-y)a + yb$. \square

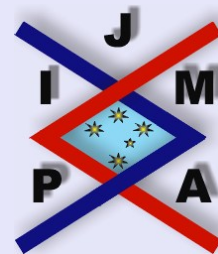
We conclude this paper by the following open problem:

Open problem. Characterize all weight functions. Actually, if $\omega(x, y)$ is a weight function, then $\theta(x, y) = \omega(x, y) - 1$ satisfies the following relations:

$$(2.8) \quad \int_X \theta(x, y) d\mu(x) = 0 \quad (\text{for each } y \text{ in } Y),$$

$$(2.9) \quad \int_Y \theta(x, y) d\lambda(y) = 0 \quad (\text{for each } x \text{ in } X).$$

So precisely, the weight functions are of the form $1 + \theta(x, y)$ with nonnegative values such that $\theta(x, y)$ is product-measurable and satisfies (2.8) and (2.9). Therefore, it is sufficient only to characterize these θ 's.



A Refinement of Jensen's Inequality

J. Roojin

Title Page

Contents



Go Back

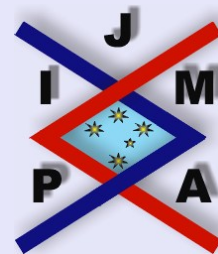
Close

Quit

Page 8 of 9

References

- [1] S.S. DRAGOMIR AND C.E.M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. [ONLINE: <http://rgmia.vu.edu.au/monographs/>]
- [2] E. HEWITT AND K. STROMBERG, *Real and Abstract Analysis*, Springer-Verlag, New York, 1965.
- [3] J. ROOIN, Some aspects of convex functions and their applications, *J. Ineq. Pure and Appl. Math.*, **2**(1) (2001), Art. 4. [ONLINE <http://jipam.vu.edu.au/article.php?sid=120>]
- [4] W. RUDIN, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1974.



A Refinement of Jensen's Inequality

J. Rooin

Title Page

Contents



Go Back

Close

Quit

Page 9 of 9