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INEQUALITIES APPLICABLE TO CERTAIN PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper explicit bounds on certain retarded integral inequalities involving functions of two independent variables are established. Some applications are also given to illustrate the usefulness of one of our results.

Key words and phrases: Explicit bounds, retarded integral inequalities, two independent variables, non-self-adjoint, Hyperbolic partial differential equations, partial derivatives.

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1. INTRODUCTION

The integral inequalities which furnish explicit bounds on unknown functions has become a rich source of inspiration in the development of the theory of differential and integral equations. Over the years a great deal of attention has been given to such inequalities and their applications. A detailed account related to such inequalities can be found in [1] – [6] and the references given therein. However, in certain situations the bounds provided by such inequalities available in the literature are inadequate and we need bounds on some new integral inequalities in order to achieve a diversity of desired goals. In this paper, we offer some basic integral inequalities in two independent variables which can be used more conveniently in specific applications. Some applications are also given to study the behavior of solutions of non-self-adjoint hyperbolic partial differential equations with several retarded arguments.

2. STATEMENT OF RESULTS

In what follows $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, \infty)$, $I_1 = [x_0, X)$, $I_2 = [y_0, Y)$ are the subsets of $\mathbb{R}$ and $\Delta = I_1 \times I_2$. The partial derivatives of a function $z(x, y)$, $x, y \in \mathbb{R}$ with respect to x, y and xy are denoted by $D_1 z(x, y)$, $D_2 z(x, y)$ and $D_1 D_2 z(x, y)$ (or z_{xy}) respectively.

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Our main results are established in the following theorems.

Theorem 2.1. *Let* $u, a, b_i \in C(\Delta, \mathbb{R}_+)$ *and* $\alpha_i \in C^1(I_1, I_1), \beta_i \in C^1(I_2, I_2)$ *be nondecreasing with* $\alpha_i(x) \leq x$ *on* $I_1, \beta_i(y) \leq y$ *on* I_2 *for* $i = 1, ..., n$ *and* $k \geq 0$ *be a constant.*

(A_1) *If*

$$u(x,y) \leq k + \int_{x_0}^{x} a(s,y)\, u(s,y)\, ds + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, u(s,t)\, dtds, \tag{2.1}$$

for $x \in I_1, y \in I_2$ *, then*

$$u(x,y) \leq kq(x,y) \exp\left(\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, q(s,t)\, dtds\right), \tag{2.2}$$

for $x \in I_1, y \in I_2$ *, where*

$$q(x,y) = \exp\left(\int_{x_0}^{x} a(\xi, y)\, d\xi\right), \tag{2.3}$$

for $x \in I_1, y \in I_2$ *.*

(A_2) *If*

$$u(x,y) \leq k + \int_{y_0}^{y} a(x,t)\, u(x,t)\, dt + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, u(s,t)\, dtds, \tag{2.4}$$

for $x \in I_1, y \in I_2$ *, then*

$$u(x,y) \leq kr(x,y) \exp\left(\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, r(s,t)\, dtds\right), \tag{2.5}$$

for $x \in I_1, y \in I_2$ *, where*

$$r(x,y) = \exp\left(\int_{y_0}^{y} a(x, \eta)\, d\eta\right), \tag{2.6}$$

for $x \in I_1, y \in I_2$*.*

Theorem 2.2. *Let* $u, a, b_i, \alpha_i, \beta_i, k$ *be as in Theorem 2.1. Let* $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing and submultiplicative function with* $g(u) > 0$ *for* $u > 0$*.*

(B_1) *If*

$$u(x,y) \leq k + \int_{x_0}^{x} a(s,y)\, u(s,y)\, ds + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(u(s,t))\, dtds, \tag{2.7}$$

for $x \in I_1, y \in I_2$ *; then for* $x_0 \leq x \leq x_1, y_0 \leq y \leq y_1; x, x_1 \in I_1, y, y_1 \in I_2$*,*

$$u(x,y) \leq q(x,y)\, G^{-1}\left[G(k) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(q(s,t))\, dtds\right], \tag{2.8}$$

where $q(x,y)$ *is given by (2.3) and* G^{-1} *is the inverse function of*

$$G(r) = \int_{r_0}^{r} \frac{ds}{g(s)}, r > 0, \tag{2.9}$$

$r_0 > 0$ *is arbitrary and* $x_1 \in I_1, y_1 \in I_2$ *are chosen so that*

$$G(k) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(q(s,t))\, dtds \in \mathrm{Dom}(G^{-1}),$$

for all x and y lying in $[x_0, x_1]$ *and* $[y_0, y_1]$ *respectively.*

(B_2) *If*

$$u(x,y) \le k + \int_{y_0}^{y} a(x,t)\, u(x,t)\, dt + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(u(s,t))\, dt ds, \tag{2.10}$$

for $x \in I_1, y \in I_2$ *; then for* $x_0 \le x \le x_2, y_0 \le y \le y_2; x, x_2 \in I_1, y, y_2 \in I_2$,

$$u(x,y) \le r(x,y)\, G^{-1}\left[G(k) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(r(s,t))\, dt ds\right], \tag{2.11}$$

where G, G^{-1} *are as in part* (B_1), $r(x,y)$ *is given by (2.6) and* $x_2 \in I_1, y_2 \in I_2$ *are chosen so that*

$$G(k) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(r(s,t))\, dt ds \in \mathrm{Dom}(G^{-1}),$$

for all x *and* y *lying in* $[x_0, x_2]$ *and* $[y_0, y_2]$ *respectively.*

The inequalities in the following theorems can be used in the qualitative analysis of certain partial integrodifferential equations involving several retarded arguments.

Theorem 2.3. *Let* $u, a, b_i, \alpha_i, \beta_i, k$ *be as in Theorem 2.1.*

(C_1) *If* $c \in C(\Delta, \mathbb{R}_+)$ *and*

$$u(x,y) \le k + \int_{x_0}^{x} a(s,y)\left(u(s,y) + \int_{x_0}^{s} c(\sigma,y)\, u(\sigma,y)\, d\sigma\right) ds + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, u(s,t)\, dt ds, \tag{2.12}$$

for $x \in I_1, y \in I_2$ *, then*

$$u(x,y) \le k p(x,y) \exp\left(\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, p(s,t)\, dt ds\right), \tag{2.13}$$

for $x \in I_1, y \in I_2$ *, where*

$$p(x,y) = 1 + \int_{x_0}^{x} a(\xi,y) \exp\left(\int_{x_0}^{\xi} [a(\sigma,y) + c(\sigma,y)]\, d\sigma\right) d\xi, \tag{2.14}$$

for $x \in I_1, y \in I_2$.

(C_2) *If* $c \in C(\Delta, \mathbb{R}_+)$ *and*

$$u(x,y) \le k + \int_{y_0}^{y} a(x,t)\left(u(x,t) + \int_{y_0}^{t} c(x,\tau)\, u(x,\tau)\, d\tau\right) dt + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, u(s,t)\, dt ds, \tag{2.15}$$

for $x \in I_1, y \in I_2$ *, then*

$$u(x,y) \le k w(x,y) \exp\left(\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, w(s,t)\, dt ds\right), \tag{2.16}$$

for $x \in I_1, y \in I_2$ *, where*

$$w(x,y) = 1 + \int_{y_0}^{y} a(x,\eta) \exp\left(\int_{y_0}^{\eta} [a(x,\tau) + c(x,\tau)]\, d\tau\right) d\eta, \tag{2.17}$$

for $x \in I_1, y \in I_2$.

Theorem 2.4. *Let* $u, a, b_i, \alpha_i, \beta_i, k$ *be as in Theorem 2.1 and* g *be as in Theorem 2.2.*

(D_1) *If* $c \in C(\Delta, \mathbb{R}_+)$ *and*

$$u(x,y) \le k + \int_{x_0}^{x} a(s,y)\left(u(s,y) + \int_{x_0}^{s} c(\sigma,y)\, u(\sigma,y)\, d\sigma\right) ds + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(u(s,t))\, dt ds, \tag{2.18}$$

for $x \in I_1, y \in I_2$*; then for* $x_0 \le x \le x_3, y_0 \le y \le y_3; x, x_3 \in I_1, y, y_3 \in I_2$ *,*

$$u(x,y) \le p(x,y)\, G^{-1}\left[G(k) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(p(s,t))\, dt ds\right], \tag{2.19}$$

where $p(x,y)$ *is given by (2.14),* G, G^{-1} *are as in part* (B_1) *in Theorem 2.2 and* $x_3 \in I_1, y_3 \in I_2$ *are chosen so that*

$$G(k) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(p(s,t))\, dt ds \in Dom(G^{-1}),$$

for all x and y lying in $[x_0, x_3]$ *and* $[y_0, y_3]$ *respectively.*

(D_2) *If* $c \in C(\Delta, \mathbb{R}_+)$ *and*

$$u(x,y) \le k + \int_{y_0}^{y} a(x,t)\left(u(x,t) + \int_{y_0}^{t} c(x,\tau)\, u(x,\tau)\, d\tau\right) dt + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(u(s,t))\, dt ds, \tag{2.20}$$

for $x \in I_1, y \in I_2$*; then for* $x_0 \le x \le x_4, y_0 \le y \le y_4; x, x_4 \in I_1, y, y_4 \in I_2$ *,*

$$u(x,y) \le w(x,y)\, G^{-1}\left[G(k) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(w(s,t))\, dt ds\right], \tag{2.21}$$

where $w(x,y)$ *is given by (2.17),* G, G^{-1} *are as in part* (B_1) *in Theorem 2.2 and* $x_4 \in I_1, y_4 \in I_2$ *are chosen so that*

$$G(k) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(w(s,t))\, dt ds \in Dom(G^{-1}),$$

for all x *and* y *lying in* $[x_0, x_4]$ *and* $[y_0, y_4]$ *respectively.*

3. PROOFS OF THEOREMS 2.1 – 2.4

We give the details of the proofs of (A_1), (B_1) and (C_1) only. The proofs of the remaining inequalities can be completed by closely looking at the proofs of the above mentioned inequalities with suitable modifications.

(A_1) Define a function $z(x,y)$ by

$$z(x,y) = k + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, u(s,t)\, dtds. \tag{3.1}$$

Then (2.1) can be restated as

$$u(x,y) \le z(x,y) + \int_{x_0}^{x} a(s,y)\, u(s,y)\, ds. \tag{3.2}$$

It is easy to observe that $z(x,y)$ is a nonnegative, continuous and nondecreasing function for $x \in I_1, y \in I_2$. Treating $y, y \in I_2$ fixed in (3.2) and using Lemma 2.1 in [4] (see also [3, Theorem 1.3.1]) to (3.2), we get

$$u(x,y) \le q(x,y)\, z(x,y), \tag{3.3}$$

for $x \in I_1, y \in I_2$, where $q(x,y)$ is defined by (2.3). From (3.1) and (3.3) we have

$$z(x,y) \le k + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, q(s,t)\, z(s,t)\, dtds. \tag{3.4}$$

Let $k > 0$ and define a function $v(x,y)$ by the right hand side of (3.4). Then it is easy to observe that

$$v(x,y) > 0, \quad v(x_0,y) = v(x,y_0) = k, \quad z(x,y) \le v(x,y)$$

and

$$\begin{aligned} D_1 v(x,y) &= \sum_{i=1}^{n} \left(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(\alpha_i(x),t)\, q(\alpha_i(x),t)\, z(\alpha_i(x),t)\, dt \right) \alpha_i'(x) \\ &\le \sum_{i=1}^{n} \left(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(\alpha_i(x),t)\, q(\alpha_i(x),t)\, v(\alpha_i(x),t)\, dt \right) \alpha_i'(x) \\ &\le v(x,y) \sum_{i=1}^{n} \left(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(\alpha_i(x),t)\, q(\alpha_i(x),t)\, dt \right) \alpha_i'(x) \end{aligned}$$

i.e.

$$\frac{D_1 v(x,y)}{v(x,y)} \le \sum_{i=1}^{n} \left(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(\alpha_i(x),t)\, q(\alpha_i(x),t)\, dt \right) \alpha_i'(x). \tag{3.5}$$

Keeping y fixed in (3.5) , setting $x = \sigma$ and integrating it with respect to σ from x_0 to $x,\ x \in I_1$, and making the change of variables we get

$$v(x,y) \le k \exp\left(\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, q(s,t)\, dtds \right), \tag{3.6}$$

for $x \in I_1, y \in I_2$. Using (3.6) in $z(x,y) \le v(x,y)$ we get

$$z(x,y) \le k \exp\left(\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, q(s,t)\, dtds \right). \tag{3.7}$$

Using (3.7) in (3.3) we get the required inequality in (2.5).

If $k \ge 0$ we carry out the above procedure with $k + \varepsilon$ instead of k, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass the limit $\varepsilon \to 0$ to obtain (2.5).

(B_1) Define a function $z(x,y)$ by

$$z(x,y) = k + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(u(s,t))\, dtds. \tag{3.8}$$

Then (2.7) can be stated as

$$u(x,y) \le z(x,y) + \int_{x_0}^{x} a(s,y)\, u(s,y)\, ds. \tag{3.9}$$

As in the proof of part (A_1), using Lemma 2.1 in [4] to (3.9) we have

$$u(x,y) \le q(x,y)\, z(x,y), \tag{3.10}$$

for $x \in I_1, y \in I_2$, where $q(x,y)$ and $z(x,y)$ are defined by (2.3) and (3.8). From (3.8) and (3.10) and the hypotheses on g we have

$$\begin{aligned} z(x,y) &\le k + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(q(s,t)\, z(s,t))\, dtds \\ &\le k + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(q(s,t))\, g(z(s,t))\, dtds. \end{aligned} \tag{3.11}$$

Let $k > 0$ and define a function $v(x,y)$ by the right hand side of (3.11). Then, it is easy to observe that $v(x,y) > 0$, $v(x_0,y) = v(x,y_0) = k$, $z(x,y) \le v(x,y)$ and

$$\begin{aligned} D_1 v(x,y) &= \sum_{i=1}^{n} \left(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(\alpha_i(x),t)\, g(q(\alpha_i(x),t))\, g(z(\alpha_i(x),t))\, dt \right) \alpha_i'(x) \\ &\le \sum_{i=1}^{n} \left(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(\alpha_i(x),t)\, g(q(\alpha_i(x),t))\, g(v(\alpha_i(x),t))\, dt \right) \alpha_i'(x) \\ &\le g(v(x,y)) \sum_{i=1}^{n} \left(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(\alpha_i(x),t)\, g(q(\alpha_i(x),t))\, dt \right) \alpha_i'(x). \end{aligned} \tag{3.12}$$

From (2.9) and (3.12) we have

$$\begin{aligned} D_1 G(v(x,y)) &= \frac{D_1 v(x,y)}{g(v(x,y))} \\ &\le \sum_{i=1}^{n} \left(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(\alpha_i(x),t)\, g(q(\alpha_i(x),t))\, dt \right) \alpha_i'(x). \end{aligned} \tag{3.13}$$

Keeping y fixed in (3.13), setting $x = \sigma$ and integrating it with respect to σ from x_0 to $x,\ x \in I_1$ and making the change of variables we get

$$G(v(x,y)) \le G(k) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(q(s,t))\, dtds. \tag{3.14}$$

Since $G^{-1}(v)$ is increasing, from (3.14) we have

$$v(x,y) \le G^{-1}\left[G(k) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(q(s,t))\, dtds \right]. \tag{3.15}$$

Using (3.15) in $z(x,y) \le v(x,y)$ and then the bound on $z(x,y)$ in (3.10) we get the required inequality in (2.8). The case $k \ge 0$ can be completed as mentioned in the proof of (A_1).

(C_1) Define a function $z(x,y)$ by (3.1). Then (2.12) can be stated as

$$u(x,y) \leq z(x,y) + \int_{x_0}^{x} a(s,y) \left(u(s,y) + \int_{x_0}^{s} c(\sigma,y) u(\sigma,y) d\sigma \right) ds. \tag{3.16}$$

Clearly, $z(x,y)$ is nonnegative, continuous and nondecreasing function for $x \in I_1, y \in I_2$. Treating $y, y \in I_2$ fixed in (3.16) and applying Theorem 1.7.4 given in [3, p. 39] to (3.16) yields

$$u(x,y) \leq p(x,y) z(x,y),$$

where $p(x,y)$ and $z(x,y)$ are defined by (2.14) and (3.1). Now by following the proof of (A_1) with suitable changes we get the desired inequality in (2.13).

4. SOME APPLICATIONS

In this section, we present applications of the inequality (A_1) given in Theorem 2.1 which display the importance of our results to the literature. Consider the following retarded non-self-adjoint hyperbolic partial differential equation

$$\begin{aligned} z_{xy}(x,y) &= D_2(a(x,y) z(x,y)) \\ &\quad + f(x,y,z(x-h_1(x), y-g_1(y)), \ldots, z(x-h_n(x), y-g_n(y))), \end{aligned} \tag{4.1}$$

with the given initial boundary conditions

$$z(x,y_0) = a_1(x), z(x_0,y) = a_2(y), a_1(x_0) = a_2(y_0) = 0, \tag{4.2}$$

where $f \in C(\Delta \times \mathbb{R}^n, \mathbb{R})$, $a_1 \in C^1(I_1,\mathbb{R})$, $a_2 \in C^1(I_2,\mathbb{R})$, and $a \in C(\Delta,\mathbb{R})$ is differentiable with respect to y; $h_i \in C(I_1,\mathbb{R}_+)$, $g_i \in C(I_2,\mathbb{R}_+)$ are nonincreasing, and such that $x - h_i(x) \geq 0$, $x - h_i(x) \in C^1(I_1,I_1)$, $y - g_i(y) \geq 0$, $y - g_i(y) \in C^1(I_2,I_2)$, $h_i'(x) < 1$, $g_i'(y) < 1$, $h_i(x_0) = g_i(y_0) = 0$ for $i = 1, \ldots, n$; $x \in I_1, y \in I_2$ and

$$M_i = \max_{x \in I_1} \frac{1}{1 - h_i'(x)}, \quad N_i = \max_{y \in I_2} \frac{1}{1 - g_i'(y)}. \tag{4.3}$$

Our first result gives the bound on the solution of the problem (4.1) – (4.2).

Theorem 4.1. *Suppose that*

$$|f(x,y,u_1,\ldots,u_n)| \leq \sum_{i=1}^{n} b_i(x,y) |u_i|, \tag{4.4}$$

$$|e(x,y)| \leq k, \tag{4.5}$$

where $b_i(x,y)$, k are as in Theorem 2.1 and

$$e(x,y) = a_1(x) + a_2(y) - \int_{x_0}^{x} a(s,y_0) a_1(s) ds. \tag{4.6}$$

If $z(x,y)$ is any solution of (4.1) – (4.2), then

$$|z(x,y)| \leq k \bar{q}(x,y) \exp\left(\sum_{i=1}^{n} \int_{\phi_i(x_0)}^{\phi_i(x)} \int_{\psi_i(y_0)}^{\psi(y)} \bar{b}_i(\sigma,\tau) \bar{q}(\sigma,\tau) d\tau d\sigma \right), \tag{4.7}$$

for $x \in I_1, y \in I_2$, where $\phi_i(x) = x - h_i(x)$, $x \in I_1$, $\psi_i(y) = y - g_i(y)$, $y \in I_2$, $\bar{b}_i(\sigma,\tau) = M_i N_i b_i(\sigma + h_i(s), \tau + g_i(t))$ for $\sigma, s \in I_1, \tau, t \in I_2$ and

$$\bar{q}(x,y) = \exp\left(\int_{x_0}^{x} |a(\xi,y)| d\xi \right), \tag{4.8}$$

for $x \in I_1, y \in I_2$.

Proof. It is easy to see that, the solution $z(x,y)$ of the problem (4.1) – (4.2) satisfies the equivalent integral equation

$$z(x,y) = e(x,y) + \int_{x_0}^{x} a(s,y)\, z(s,y)\, ds + \int_{x_0}^{x}\int_{y_0}^{y} f\left(s,t,z\left(s-h_1(s),t-g_1(t)\right),\dots,z\left(s-h_n(s),t-g_n(t)\right)\right) dtds, \tag{4.9}$$

where $e(x,y)$ is given by (4.6). From (4.9), (4.4), (4.5), (4.3) and making the change of variables we have

$$\begin{aligned} |z(x,y)| &\le k + \int_{x_0}^{x} |a(s,y)|\,|z(s,y)|\,ds \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} \sum_{i=1}^{n} b_i(s,t)\,|z(s-h_i(s),t-g_i(t))|\,dtds \\ &\le k + \int_{x_0}^{x} |a(s,y)|\,|z(s,y)|\,ds \\ &\quad + \sum_{i=1}^{n} \int_{\phi_i(x_0)}^{\phi_i(x)} \int_{\psi_i(y_0)}^{\psi_i(y)} \bar{b}_i(\sigma,\tau)\,|z(\sigma,\tau)|\,d\tau d\sigma. \end{aligned} \tag{4.10}$$

Now a suitable application of the inequality (A_1) given in Theorem 2.1 to (4.10) yields (4.7). □

The next theorem deals with the uniqueness of solutions of (4.1) – (4.2).

Theorem 4.2. . *Suppose that the function f in (4.1) satisfies the condition*

$$|f(x,y,u_1,\dots,u_n) - f(x,y,v_1,\dots,v_n)| \le \sum_{i=1}^{n} b_i(x,y)\,|u_i - v_i|, \tag{4.11}$$

where $b_i(x,y)$ are as in Theorem 2.1. Let $M_i, N_i, \phi_i, \psi_i, \bar{b}_i$ be as in Theorem 4.1. Then the problem (4.1) – (4.2) has at most one solution on Δ .

Proof. Let $u(x,y)$ and $v(x,y)$ be two solutions of (4.1) – (4.2) on Δ , then

$$\begin{aligned} u(x,y) - v(x,y) &= \int_{x_0}^{x} a(s,y)\,\{u(s,y) - v(s,y)\}\,ds \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} \{f(s,t,u(s-h_1(s),t-g_1(t)),\dots,u(s-h_n(s),t-g_n(t))) \\ &\qquad - f(s,t,v(s-h_1(s),t-g_1(t)),\dots,v(s-h_n(s),t-g_n(t)))\}\,dtds\,. \end{aligned} \tag{4.12}$$

From (4.12), (4.11), making the change of variables and in view of (4.3) we have

$$\begin{aligned} |u(x,y) - v(x,y)| &\le \int_{x_0}^{x} |a(s,y)|\,|u(s,y) - v(s,y)|\,ds \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} \sum_{i=1}^{n} b_i(s,t)\,|u(s-h_i(s),t-g_i(t)) - v(s-h_i(s),t-g_i(t))|\,dtds \end{aligned} \tag{4.13}$$

$$\leq \int_{x_0}^{x} |a(s,y)| |u(s,y) - v(s,y)| \, ds$$
$$+ \sum_{i=1}^{n} \int_{\phi_i(x_0)}^{\phi_i(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{b}_i(\sigma,\tau) |u(\sigma,\tau) - v(\sigma,\tau)| \, d\tau d\sigma.$$

A suitable application of the inequality (A_1) in Theorem 2.1 to (4.13) yields

$$|u(x,y) - v(x,y)| \leq 0.$$

Therefore $u(x,y) = v(x,y)$ i.e. there is at most one solution of the problem (4.1) – (4.2). $\square$

The following theorem shows the dependency of solutions of equation (4.1) on given initial boundary data.

Theorem 4.3. *Let $u(x,y)$ and $v(x,y)$ be the solutions of (4.1) with the given initial boundary data*

$$u(x,y_0) = c_1(x), u(x_0,y) = c_2(y), c_1(x_0) = c_2(y_0) = 0, \tag{4.14}$$

and

$$v(x,y_0) = d_1(x), v(x_0,y) = d_2(y), d_1(x_0) = d_2(y_0) = 0, \tag{4.15}$$

respectively, where $c_1, d_1 \in C^1(I_1,\mathbb{R}), c_2, d_2 \in C^1(I_2,\mathbb{R})$. Suppose that the function f satisfies the condition (4.11) in Theorem 4.2. Let

$$e_1(x,y) = c_1(x) + c_2(y) - \int_{x_0}^{x} a(s,y_0) c_1(s) \, ds, \tag{4.16}$$

$$e_2(x,y) = d_1(x) + d_2(y) - \int_{x_0}^{x} a(s,y_0) d_1(s) \, ds, \tag{4.17}$$

for $x \in I_1, y \in I_2$ and

$$|e_1(x,y) - e_2(x,y)| \leq k, \tag{4.18}$$

where k is as in Theorem 2.1. Let $M_i, N_i, \phi_i, \psi_i, \bar{b}_i, \bar{q}$ be as in Theorem 4.1. Then

$$|u(x,y) - v(x,y)| \leq k\bar{q}(x,y) \exp\left(\sum_{i=1}^{n} \int_{\phi_i(x_0)}^{\phi_i(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{b}_i(\sigma,\tau) \bar{q}(\sigma,\tau) \, d\tau d\sigma \right), \tag{4.19}$$

for $x \in I_1, y \in I_2$.

Proof. Since $u(x,y)$ and $v(x,y)$ are the solutions of (4.1) – (4.14) and (4.1) – (4.15) respectively, we have

$$u(x,y) - v(x,y) = e_1(x,y) - e_2(x,y) + \int_{x_0}^{x} a(s,y) \{u(s,y) - v(s,y)\} \, ds \tag{4.20}$$
$$+ \int_{x_0}^{x} \int_{y_0}^{y} \{ f(s,t,u(s-h_1(s),t-g_1(t)),\ldots,u(s-h_n(s),t-g_n(t)))$$
$$- f(s,t,v(s-h_1(s),t-g_1(t)),\ldots,v(s-h_n(s),t-g_n(t)))\} \, dtds,$$

for $x \in I_1, y \in I_2$. From (4.20), (4.18), (4.11), making the change of variables and in view of (4.3) we have

$$\text{(4.21)} \quad |u(x,y) - v(x,y)| \leq k + \int_{x_0}^{x} |a(s,y)| |u(s,y) - v(s,y)| ds + \sum_{i=1}^{n} \int_{\phi_i(x_0)}^{\phi_i(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{b}_i(\sigma,\tau) |u(\sigma,\tau) - v(\sigma,\tau)| d\tau d\sigma,$$

for $x \in I_1, y \in I_2$. Now a suitable application of the inequality (A_1) in Theorem 2.1 to (4.21) yields the required estimate in (4.19), which shows the dependency of solutions of (4.1) on given initial boundary data. □

We next consider the following retarded non-self-adjoint hyperbolic partial differential equations

$$\text{(4.22)} \quad z_{xy}(x,y) = D_2(a(x,y) z(x,y)) + f(x,y,z(x-h_1(x), y-g_1(y)), \ldots, z(x-h_n(x), y-g_n(y)), \mu),$$

$$\text{(4.23)} \quad z_{xy}(x,y) = D_2(a(x,y) z(x,y)) + f(x,y,z(x-h_1(x), y-g_1(y)), \ldots, z(x-h_n(x), y-g_n(y)), \mu_0),$$

with the given initial boundary conditions (4.2), where $f \in C(\Delta \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R})$, h_i, g_i are as in (4.1) and $\mu,\ \mu_0$ are real parameters.

The following theorem shows the dependency of solutions of problems (4.22) – (4.2) and (4.23) – (4.2) on parameters.

Theorem 4.4. *Suppose that*

$$\text{(4.24)} \quad |f(x,y,u_1,\ldots,u_n,\mu) - f(x,y,v_1,\ldots,v_n,\mu)| \leq \sum_{i=1}^{n} b_i(x,y) |u_i - v_i|,$$

$$\text{(4.25)} \quad |f(x,y,u_1,\ldots,u_n,\mu) - f(x,y,u_1,\ldots,u_n,\mu)| \leq m(x,y) |\mu - \mu_0|,$$

where $b_i(x,y)$ are as in Theorem 2.1 and $m : \Delta \to \mathbb{R}$ is a continuous function such that

$$\text{(4.26)} \quad \int_{x_0}^{x} \int_{y_0}^{y} m(s,t) dt ds \leq M,$$

where $M \geq 0$ is a real constant . Let $M_i, N_i, \phi_i, \psi_i, \bar{b}_i$ be as in Theorem 4.1. If $z_1(x,y)$ and $z_2(x,y)$ are the solutions of (4.22) – (4.2) and (4.23) =- (4.2), then

$$\text{(4.27)} \quad |z_1(x,y) - z_2(x,y)| \leq \bar{k}\bar{q}(x,y) \exp\left(\sum_{i=1}^{n} \int_{\phi_i(x_0)}^{\phi_i(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{b}_i(\sigma,\tau) \bar{q}(\sigma,\tau) d\tau d\sigma \right),$$

for $x \in I_1, y \in I_2$,where $\bar{k} = |\mu - \mu_0| M$ and $\bar{q}(x,y)$ is defined by (4.8).

Proof. Let $z(x,y) = z_1(x,y) - z_2(x,y)$ for $x \in I_1, y \in I_2$. As in the proof of Theorem 4.2, from the hypotheses we have

$$\begin{aligned}(4.28)\quad z(x,y) &= \int_{x_0}^{x} a(s,y)\, z(s,y)\, ds \\ &+ \int_{x_0}^{x}\int_{y_0}^{y} \{f(s,t,z_1(s-h_1(s),t-g_1(t)),\dots,z_1(s-h_n(s),t-g_n(t)),\mu) \\ &\quad - f(s,t,z_2(s-h_1(s),t-g_1(t)),\dots,z_2(s-h_n(s),t-g_n(t)),\mu) \\ &\quad + f(s,t,z_2(s-h_1(s),t-g_1(t)),\dots,z_2(s-h_n(s),t-g_n(t)),\mu) \\ &\quad - f(s,t,z_2(s-h_1(s),t-g_1(t)),\dots,z_2(s-h_n(s),t-g_n(t)),\mu_0)\}\, dtds.\end{aligned}$$

From (4.28), (4.24) – (4.26), making the change of variables and in view of (4.3) we have

$$\begin{aligned}(4.29)\ |z(x,y)| &\le \int_{x_0}^{x} |a(s,y)|\,|z(s,y)|\, ds \\ &+ \int_{x_0}^{x}\int_{y_0}^{y} \sum_{i=1}^{n} b_i(s,t)\,|z_1(s-h_i(s),t-g_i(t)) - z_2(s-h_i(s),t-g_i(t))|dtds \\ &+ \int_{x_0}^{x}\int_{y_0}^{y} m(s,t)\,|\mu-\mu_0|\, dtds \\ &\le \bar{k} + \int_{x_0}^{x} |a(s,y)|\,|z(s,y)|\, ds \\ &+ \sum_{i=1}^{n} \int_{\phi_i(x_0)}^{\phi_i(x)} \int_{\psi_i(y_0)}^{\psi(y)} \bar{b}_i(\sigma,\tau)\,|z(\sigma,\tau)|\, d\tau d\sigma.\end{aligned}$$

A suitable application of the inequality (A_1) in Theorem 2.1 to (4.29) yields (4.27), which shows the dependency of solutions of problems (4.22) – (4.2) and (4.23) – (4.2) on parameters μ and μ_0. □

We note that the inequality given in Theorem 2.1 part (A_2) can be used to study the similar properties as in Theorems 4.1 – 4.4 by replacing $D_2(a(x,y)\,z(x,y))$ by $D_1(a(x,y)\,z(x,y))$ in the equations (4.1), (4.22), (4.23) with the corresponding given initial-boundary conditions, under some suitable conditions on the functions involved therein. We also note that the inequalities given in Theorem 2.3 can be used to establish similar results as in Theorems 4.1 – 4.4 by replacing $D_2(a(x,y)\,z(x,y))$ by

$$D_2\left(Q_1\left(x,y,z(x,y), \int_{x_0}^{x} k_1(\sigma,y,z(\sigma,y))\, d\sigma\right)\right)$$

or

$$D_1\left(Q_2\left(x,y,z(x,y), \int_{y_0}^{y} k_2(x,\tau,z(x,\tau))\, d\tau\right)\right)$$

in the equations (4.1), (4.22), (4.23) with the corresponding given initial-boundary conditions and under some suitable conditions on the functions involved therein.

Further it is to be noted that the inequalities and their applications given here can be extended very easily to functions involving many independent variables.

REFERENCES

[1] D. BAINOV AND P. SIMENOV, *Integral Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, 1992 .

[2] O. LIPOVAN, A retarded Gronwall-like inequality and its applications, *J. Math. Anal. Appl.*, **252** (2000), 389–401.

[3] B.G. PACHPATTE, *Inequalities for Differential and Integral Equations*, Academic Press, New York, 1998.

[4] B.G. PACHPATTE, On some fundamental integral inequalities and their discrete analogues, *J. Inequal. Pure and Appl. Math.*, **2**(2) (2001), Art.15. [ONLINE `http://jipam.vu.edu.au`]

[5] B.G. PACHPATTE, Explicit bounds on certain integral inequalities, *J. Math. Anal. Appl.*, **267** (2002), 48–61.

[6] B.G. PACHPATTE, On some retarded integral inequalities and applications, *J. Inequal. Pure and Appl. Math.*, **3**(2) (2002), Art. 18. [ONLINE `http://jipam.vu.edu.au`]