ON SOME WEIGHTED MIXED NORM HARDY-TYPE INTEGRAL INEQUALITIES

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Abstract: In this paper, we establish a weighted mixed norm integral inequality of Hardy's type. This inequality features a free constant term and extends earlier results on weighted norm Hardy-type inequalities. It contains, as special cases, some earlier inequalities established by the authors and also provides an improvement over them.

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1. Introduction

In a recent paper [2], the authors proved the following result.

Theorem 1.1. *Let* g *be continuous and non-decreasing on* $[a,b]$*,* $0 \leq a \leq b \leq \infty$ *with* $g(x) > 0, x > 0, r \neq 1$ *and let* $f(x)$ *be non-negative and Lebesgue-Stieltjes integrable with respect to* $g(x)$ *on* $[a,b]$*. Suppose* $F_a(x) = \int_a^x f(t)dg(t)$, $F_b(x) = \int_x^b f(t)dg(t)$ *and* $\delta = \frac{1-r}{p}, r \neq 1$*. Then*

$$\int_a^b g(x)^{\delta-1}\left[g(x)^{-\delta} - g(a)^{-\delta}\right]^{1-p} F_a(x)^p dg(x) + K_1(p,\delta,a,b) \leq \left[\frac{p}{r-1}\right]^p \int_a^b g(x)^{\delta p-1}\left[g(x)f(x)\right]^p dg(x), \qquad r > 1, \tag{1.1}$$

$$\int_a^b g(x)^{\delta-1}\left[g(x)^{-\delta} - g(b)^{-\delta}\right]^{1-p} F_b(x)^p dg(x) + K_2(p,\delta,a,b) \leq \left[\frac{p}{1-r}\right]^p \int_a^b g(x)^{\delta p-1}[g(x)f(x)]^p dg(x), \qquad r < 1, \tag{1.2}$$

where

$$K_1(p,\delta,a,b) = \frac{p}{r-1} g(b)^{\delta}\left[g(b)^{-\delta} - g(a)^{-\delta}\right]^{1-p} F_a(b)^p, \quad \delta < 0, \ \textit{i.e. } r > 1$$

and

$$K_2(p,\delta,a,b) = \frac{p}{1-r} g(a)^{\delta}\left[g(a)^{-\delta} - g(b)^{-\delta}\right]^{1-p} F_b(a)^p, \quad \delta > 0, \ \textit{i.e. } r < 1.$$

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The above result generalizes Imoru [1] and therefore Shum [3]. The purpose of the present work is to obtain a weighted norm Hardy-type inequality involving mixed norms which contains the above result as a special case and also provides an improvement over it.

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2. Main Result

The main result of this paper is the following theorem:

Theorem 2.1. *Let g be a continuous function which is non-decreasing on $[a,b]$, $0 \le a \le b < \infty$, with $g(x) > 0$ for $x > 0$. Suppose that $q \ge p \ge 1$ and $f(x)$ is non-negative and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a,b]$. Let*

$$F_a(x) = \int_a^x f(t)dg(t), \theta_a(x) = \int_a^x g(t)^{(p-1)(1+\delta)} f(t)^p dg(t), \tag{2.1}$$

$$F_b(x) = \int_x^b f(t)dg(t), \theta_b(x) = \int_x^b g(t)^{(p-1)(1+\delta)} f(t)^p dg(t) \tag{2.2}$$

and $\delta = \frac{1-r}{p}, r \ne 1$. Then if $r > 1$, i.e. $\delta < 0$,

$$\left[\int_a^b g(x)^{\frac{\delta q}{p}-1} \left[g(x)^{-\delta} - g(a)^{-\delta}\right]^{\frac{q}{p}(p-1)} F_a^q(x)dg(x) + A_1(p,q,a,b,\delta)\right]^{\frac{1}{q}} \le C_1(p,q,\delta)\left[\int_a^b g(x)^{\delta p-1}\left[g(x)f(x)\right]^p dg(x)\right]^{\frac{1}{p}}, \tag{2.3}$$

and for $r < 1$, i.e. $\delta > 0$,

$$\left[\int_a^b g(x)^{\frac{\delta q}{p}-1} \left[g(x)^{-\delta} - g(b)^{-\delta}\right]^{\frac{q}{p}(p-1)} F_b^q(x)dg(x) + A_2(p,q,a,b,\delta)\right]^{\frac{1}{q}} \le C_2(p,q,\delta)\left[\int_a^b g(x)^{\delta p-1}\left[g(x)f(x)\right]^p dg(x)\right]^{\frac{1}{p}}, \tag{2.4}$$

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where

$$A_1(p,q,a,b,\delta) = \frac{p}{q}(-\delta)^{\frac{q}{p}(1-p)-1} g(b)^{\frac{\delta q}{p}} \theta_a(b)^{\frac{q}{p}}, \qquad \delta < 0,$$

$$C_1(p,q,\delta) = \left[\frac{p}{q}(-\delta)^{\frac{q}{p}(1-p)-1}\right]^{\frac{1}{q}},$$

$$A_2(p,q,a,b,\delta) = \frac{p}{q}(\delta)^{\frac{q}{p}(1-p)-1} g(a)^{\frac{\delta q}{p}} \theta_b(a)^{\frac{q}{p}}, \qquad \delta > 0$$

$$C_2(p,q,\delta) = \left[\frac{p}{q}\delta^{\frac{q}{p}(1-p)-1}\right]^{\frac{1}{q}}.$$

Proof. For the proof of Theorem 2.1 we will use the following adaptations of Jensen's inequality for convex functions,

$$\int_a^x h(x,t)^{\frac{1}{pq}} d\lambda(t) \le \left[\int_a^x d\lambda(t)\right]^{1-\frac{1}{p}} \left[\int_a^x h(x,t)^{\frac{1}{q}} d\lambda(t)\right]^{\frac{1}{p}} \tag{2.5}$$

and

$$\int_x^b h(x,t)^{\frac{1}{pq}} d\lambda(t) \le \left[\int_x^b d\lambda(t)\right]^{1-\frac{1}{p}} \left[\int_x^b h(x,t)^{\frac{1}{q}} d\lambda(t)\right]^{\frac{1}{p}}, \tag{2.6}$$

where $h(x,t) \ge 0$ for $x \ge 0$, $t \ge 0$, λ is non-decreasing and $q \ge p \ge 1$.

Let

$$h(x,t) = g(x)^{\delta q} g(t)^{pq(1+\delta)} f(t)^{pq}, \qquad d\lambda(t) = g(t)^{-(1+\delta)} dg(t), \tag{2.7}$$

$\Delta_1^q = (-\delta)^{\frac{q}{p}(1-p)}$, if $\delta < 0$ and $\Delta_2^q = (\delta)^{\frac{q}{p}(1-p)}$, if $\delta > 0$.

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Using (2.7) in (2.5), we get

$$g(x)^{\frac{\delta}{p}} \int_a^x f(t)dg(t)$$
$$\leq (-\delta)^{\frac{1}{p}(1-p)} \left[g(x)^{-\delta} - g(a)^{-\delta}\right]^{\frac{1}{p}(p-1)} g(x)^{\frac{\delta}{p}} \left[\int_a^x g(t)^{(p-1)(1+\delta)} f(t)^p dg(t)\right]^{\frac{1}{p}}.$$

Raising both sides of the above inequalities to power q and using (2.1), we obtain

$$g(x)^{\frac{\delta q}{p}} F_a(x)^q \leq \Delta_1^q g_a(x)^{\frac{q}{p}(p-1)} g(x)^{\frac{\delta q}{p}} \theta_a(x)^{\frac{q}{p}},$$

where $g_a(x) = \left[g(x)^{-\delta} - g(a)^{-\delta}\right]$.

Integrating over (a, b) with respect to $g(x)^{-1}dg(x)$ gives

$$\text{(2.8)} \quad \int_a^b g(x)^{\frac{\delta q}{p}-1} g_a(x)^{\frac{q}{p}(1-p)} F_a(x)^q dg(x) \leq \Delta_1^q \int_a^b g(x)^{\frac{\delta q}{p}-1} \theta_a(x)^{\frac{q}{p}} dg(x) = J.$$

Now integrate the right side of (2.8) by parts to obtain

$$J = \Delta_1^q \int_a^b g(x)^{\frac{\delta q}{p}-1} \theta_a(x)^{\frac{q}{p}} dg(x)$$
$$= \frac{\Delta_1^q}{(\delta q/p)} g(x)^{\frac{\delta q}{p}} \theta_a(x)^{\frac{q}{p}} |_a^b + (-\delta^{-1})\Delta_1^q$$
$$\times \int_a^b g(x)^{\frac{\delta q}{p}} g(x)^{(p-1)(1+\delta)} f(x)^p \theta_a(x)^{\frac{q}{p}-1} dg(x).$$

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However,

$$I = \int_a^b g(x)^{\frac{\delta q}{p}} g(x)^{(p-1)(1+\delta)} f(x)^p \theta_a^{\frac{q}{p}-1}(x)dg(x)$$

$$= \int_a^b g(x)^{\frac{\delta q}{p}} g(x)^{(p-1)(1+\delta)} f(x)^p \left[\int_a^x g(t)^{\delta p+p-1-\delta} f(t)^p dg(t)\right]^{\frac{q}{p}-1} dg(x)$$

$$= \int_a^b g(x)^{\delta p+p-1} f(x)^p \left[g(x)^{\delta} \int_a^x g(t)^{\delta p+p-1-\delta} f(t)^p dg(t)\right]^{\frac{q}{p}-1} dg(x).$$

Since $\delta < 0$, we have $g(x)^{-\delta} \geq g(t)^{-\delta} \ \forall t \in [a, x]$.

Consequently

$$I \leq \int_a^b g(x)^{\delta p+p-1} f(t)^p \left[\int_a^x g(t)^{\delta p+p-1} f(t)^p dg(t)\right]^{\frac{q}{p}-1} dg(x)$$

$$= \int_a^b \left[\int_a^x g(t)^{\delta p+p-1} f(t)^p dg(t)\right]^{\frac{q}{p}-1} g(x)^{\delta p+p-1} f(t)^p dg(x)$$

$$= \frac{p}{q}\left[\int_a^b g(x)^{\delta p-1}\left[f(x)g(x)\right]^p dg(x)\right]^{\frac{q}{p}}.$$

Thus (2.8) becomes

$$\text{(2.9)} \quad \int_a^b g(x)^{\frac{\delta q}{p}-1}\left[g(x)^{-\delta} - g(a)^{-\delta}\right]^{\frac{q}{p}(1-p)} F_a(x)^q dg(x) + \frac{p}{q}\Delta_1^q(-\delta^{-1}) g(b)^{\frac{\delta q}{p}} \theta_a(b)^{\frac{q}{p}}$$

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$$\leq \frac{p}{q}(-\delta^{-1})\Delta_1^q \left[\int_a^b g(x)^{\delta p-1}\left[f(x)g(x)\right]^p dg(x)\right]^{\frac{q}{p}}.$$

Taking the q^{th} root of both sides yields assertion (2.3) of the theorem.

To prove (2.4), we start with inequality (2.6) and use (2.7) with (2.2) to obtain

$$\begin{aligned} g(x)^{\frac{\delta}{p}}F_b(x) &\leq (-\delta)^{\frac{1}{p}(1-p)}g_b(x)^{\frac{1}{p}(p-1)}g(x)^{\frac{\delta}{p}}\theta_b(x)^{\frac{1}{p}} \\ &= (\delta^{-1})^{\frac{1}{p}(p-1)}(-g_b(x))^{\frac{1}{p}(p-1)}g(x)^{\frac{\delta}{p}}\theta_b(x)^{\frac{1}{p}}, \end{aligned}$$

where $g_b(x) = \left[g(b)^{-\delta} - g(x)^{-\delta}\right]$.

On rearranging and raising to power q and then integrating both sides over $[a,b]$ with respect to $g(x)^{-1}dg(x)$, we obtain

$$(2.10)\qquad \int_a^b g(x)^{\frac{\delta q}{p}-1}\left[g(x)^{-\delta} - g(b)^{-\delta}\right]^{\frac{q}{p}(1-p)} F_b(x)^q dg(x) \leq \Delta_2^q \int_a^b g(x)^{\frac{\delta q}{p}-1}\theta_b(x)dg(x).$$

We denote the right side of (2.10) by H, integrate it by parts and use the fact that for $\delta > 0$, $g(x)^\delta \leq g(t)^\delta \quad \forall t \in [x,b]$ to obtain

$$H \leq \frac{p}{\delta q}\Delta_2^q g(x)^{\frac{\delta p}{q}}\theta_b(x)^{\frac{q}{p}}\Big|_a^b + (\delta q/p)^{-1}\Delta_2^q \int_a^b g(x)^{\delta p-1}\left[f(x)g(x)\right]^p dg(x).$$

Using this in (2.10) we obtain

$$(2.11)\qquad \int_a^b g(x)^{\frac{\delta q}{p}-1}\left[g(x)^{-\delta} - g(b)^{-\delta}\right]^{\frac{q}{p}(1-p)} F_b(x)^q dg(x) + \frac{p}{q}\delta^{-1}\Delta_2^q g(a)^{\frac{q}{p}}\theta_b(a)^{\frac{q}{p}} \leq \frac{p}{q}\delta^{-1}\Delta_2^q \left[\int_a^b g(x)^{\delta p-1}\left[f(x)g(x)\right]^p dg(x)\right]^{\frac{q}{p}}.$$

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We take the q^{th} root of both sides to obtain assertion (2.4) of the theorem. □

Remark 1. Let $p = q$, and $\delta = \frac{1-r}{p} < 0$, i.e., $r > 1$, then (2.3) reduces to

$$\int_a^b g(x)^{\delta-1}\left[g(x)^{-\delta} - g(a)^{-\delta}\right]^{p-1} F_a(x)^p dg(x) + A_1(p,p,a,b,\delta) \leq C_1(p,p,\delta)\left[\int_a^b g(x)^{\delta p-1}\left[f(x)g(x)\right]^p dg(x)\right], \tag{2.12}$$

where

$$A_1(p,p,a,b,\delta) = (-\delta)^{-p} g(b)^{\delta}\theta_a(b), \qquad \delta < 0 \tag{2.13}$$

and

$$C_1(p,p,\delta) = (-\delta)^{-p} = \left[\frac{p}{r-1}\right]^p. \tag{2.14}$$

Now from (2.18) in [2] we have that, for $\delta < 0$

$$g(b)^{\delta}\theta_a(b) \geq (-\delta^{-1})^{1-p} g(b)^{\delta}\left[g(b)^{-\delta} - g(a)^{-\delta}\right]^{1-p} F_a(b)^p. \tag{2.15}$$

Thus, from (2.13) and (2.15), using notations in (1.1), we have

$$\begin{aligned} A_1(p,p,a,b,\delta) &= (-\delta)^{-p} g(b)^{\delta}\theta_a(b) \\ &\geq (-\delta)^{-p}(-\delta^{-1})^{1-p} g(b)^{\delta}\left[g(b)^{-\delta} - g(a)^{-\delta}\right]^{1-p} F_a(b)^p \\ &= (-\delta)^{-1} g(b)^{\delta}\left[g(b)^{-\delta} - g(a)^{-\delta}\right]^{1-p} F_a(b)^p \\ &= \frac{p}{r-1} g(b)^{\delta}\left[g(b)^{-\delta} - g(a)^{-\delta}\right]^{1-p} F_a(b)^p \\ &= K_1(p,\delta,a,b), \end{aligned} \tag{2.16}$$

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i.e., $A_1(p,p,a,b,\delta) = K_1(p,\delta,a,b) + B_1$ for some $B_1 \geq 0$.
Thus we can write (2.12), using (2.14), as

$$\int_a^b g(x)^{\delta-1}\left[g(x)^{-\delta} - g(a)^{-\delta}\right]^{p-1} F_a(x)^p dg(x) + K_1(p,\delta,a,b) + B_1 \leq \left[\frac{p}{r-1}\right]^p \left[\int_a^b g(x)^{\delta p-1}\left[f(x)g(x)\right]^p dg(x)\right]. \tag{2.17}$$

So, when $B_1 = 0$, (2.17) reduces to (1.1). When $B_1 \neq 0$, i.e., $B_1 > 0$, (2.17) is an improvement of (1.1). Similarly with notations in (1.2) and (2.4) in this paper we use (2.19) in [2] to prove that

$$A_2(p,p,a,b,\delta) = K_2(p,\delta,a,b) + B_2$$

for some $B_2 \geq 0$.
Thus, when $p = q$, (2.4) reduces to (1.2) if $B_2 = 0$ and is an improvement of (1.2) when $B_2 \neq 0$, i. e., when $B_2 > 0$.

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