



# ON SOME WEIGHTED MIXED NORM HARDY-TYPE INTEGRAL INEQUALITIES

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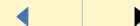
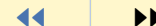
*Abstract:* In this paper, we establish a weighted mixed norm integral inequality of Hardy's type. This inequality features a free constant term and extends earlier results on weighted norm Hardy-type inequalities. It contains, as special cases, some earlier inequalities established by the authors and also provides an improvement over them.

Weighted Mixed Norm Hardy-Type  
Integral Inequalities

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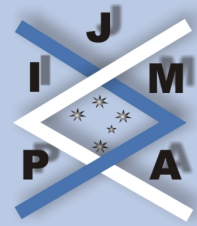
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# 1. Introduction

In a recent paper [2], the authors proved the following result.

**Theorem 1.1.** *Let  $g$  be continuous and non-decreasing on  $[a, b]$ ,  $0 \leq a \leq b \leq \infty$  with  $g(x) > 0, x > 0, r \neq 1$  and let  $f(x)$  be non-negative and Lebesgue-Stieltjes integrable with respect to  $g(x)$  on  $[a, b]$ . Suppose  $F_a(x) = \int_a^x f(t)dg(t)$ ,  $F_b(x) = \int_x^b f(t)dg(t)$  and  $\delta = \frac{1-r}{p}, r \neq 1$ . Then*

$$(1.1) \quad \int_a^b g(x)^{\delta-1} [g(x)^{-\delta} - g(a)^{-\delta}]^{1-p} F_a(x)^p dg(x) + K_1(p, \delta, a, b) \leq \left[ \frac{p}{r-1} \right]^p \int_a^b g(x)^{\delta p-1} [g(x)f(x)]^p dg(x), \quad r > 1,$$

$$(1.2) \quad \int_a^b g(x)^{\delta-1} [g(x)^{-\delta} - g(b)^{-\delta}]^{1-p} F_b(x)^p dg(x) + K_2(p, \delta, a, b) \leq \left[ \frac{p}{1-r} \right]^p \int_a^b g(x)^{\delta p-1} [g(x)f(x)]^p dg(x), \quad r < 1,$$

where

$$K_1(p, \delta, a, b) = \frac{p}{r-1} g(b)^\delta [g(b)^{-\delta} - g(a)^{-\delta}]^{1-p} F_a(b)^p, \quad \delta < 0, \text{ i.e. } r > 1$$

and

$$K_2(p, \delta, a, b) = \frac{p}{1-r} g(a)^\delta [g(a)^{-\delta} - g(b)^{-\delta}]^{1-p} F_b(a)^p, \quad \delta > 0, \text{ i.e. } r < 1.$$

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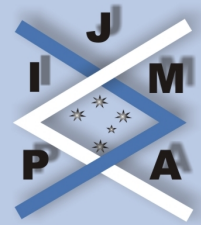
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The above result generalizes Imoru [1] and therefore Shum [3]. The purpose of the present work is to obtain a weighted norm Hardy-type inequality involving mixed norms which contains the above result as a special case and also provides an improvement over it.



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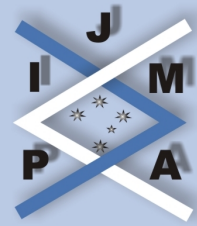
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## 2. Main Result

The main result of this paper is the following theorem:

**Theorem 2.1.** *Let  $g$  be a continuous function which is non-decreasing on  $[a, b]$ ,  $0 \leq a \leq b < \infty$ , with  $g(x) > 0$  for  $x > 0$ . Suppose that  $q \geq p \geq 1$  and  $f(x)$  is non-negative and Lebesgue-Stieltjes integrable with respect to  $g(x)$  on  $[a, b]$ . Let*

$$(2.1) \quad F_a(x) = \int_a^x f(t)dg(t), \theta_a(x) = \int_a^x g(t)^{(p-1)(1+\delta)} f(t)^p dg(t),$$

$$(2.2) \quad F_b(x) = \int_x^b f(t)dg(t), \theta_b(x) = \int_x^b g(t)^{(p-1)(1+\delta)} f(t)^p dg(t)$$

and  $\delta = \frac{1-r}{p}$ ,  $r \neq 1$ . Then if  $r > 1$ , i.e.  $\delta < 0$ ,

$$(2.3) \quad \left[ \int_a^b g(x)^{\frac{\delta q}{p}-1} [g(x)^{-\delta} - g(a)^{-\delta}]^{\frac{q}{p}(p-1)} F_a^q(x) dg(x) + A_1(p, q, a, b, \delta) \right]^{\frac{1}{q}} \\ \leq C_1(p, q, \delta) \left[ \int_a^b g(x)^{\delta p-1} [g(x)f(x)]^p dg(x) \right]^{\frac{1}{p}},$$

and for  $r < 1$ , i.e.  $\delta > 0$ ,

$$(2.4) \quad \left[ \int_a^b g(x)^{\frac{\delta q}{p}-1} [g(x)^{-\delta} - g(b)^{-\delta}]^{\frac{q}{p}(p-1)} F_b^q(x) dg(x) + A_2(p, q, a, b, \delta) \right]^{\frac{1}{q}} \\ \leq C_2(p, q, \delta) \left[ \int_a^b g(x)^{\delta p-1} [g(x)f(x)]^p dg(x) \right]^{\frac{1}{p}},$$

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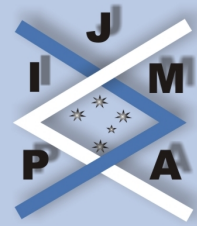
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where

$$A_1(p, q, a, b, \delta) = \frac{p}{q} (-\delta)^{\frac{q}{p}(1-p)-1} g(b)^{\frac{\delta q}{p}} \theta_a(b)^{\frac{q}{p}}, \quad \delta < 0,$$

$$C_1(p, q, \delta) = \left[ \frac{p}{q} (-\delta)^{\frac{q}{p}(1-p)-1} \right]^{\frac{1}{q}},$$

$$A_2(p, q, a, b, \delta) = \frac{p}{q} (\delta)^{\frac{q}{p}(1-p)-1} g(a)^{\frac{\delta q}{p}} \theta_b(a)^{\frac{q}{p}}, \quad \delta > 0$$

$$C_2(p, q, \delta) = \left[ \frac{p}{q} \delta^{\frac{q}{p}(1-p)-1} \right]^{\frac{1}{q}}.$$

*Proof.* For the proof of Theorem 2.1 we will use the following adaptations of Jensen's inequality for convex functions,

$$(2.5) \quad \int_a^x h(x, t)^{\frac{1}{pq}} d\lambda(t) \leq \left[ \int_a^x d\lambda(t) \right]^{1-\frac{1}{p}} \left[ \int_a^x h(x, t)^{\frac{1}{q}} d\lambda(t) \right]^{\frac{1}{p}}$$

and

$$(2.6) \quad \int_x^b h(x, t)^{\frac{1}{pq}} d\lambda(t) \leq \left[ \int_x^b d\lambda(t) \right]^{1-\frac{1}{p}} \left[ \int_x^b h(x, t)^{\frac{1}{q}} d\lambda(t) \right]^{\frac{1}{p}},$$

where  $h(x, t) \geq 0$  for  $x \geq 0, t \geq 0, \lambda$  is non-decreasing and  $q \geq p \geq 1$ .

Let

$$(2.7) \quad h(x, t) = g(x)^{\delta q} g(t)^{pq(1+\delta)} f(t)^{pq}, \quad d\lambda(t) = g(t)^{-(1+\delta)} dg(t),$$

$$\Delta_1^q = (-\delta)^{\frac{q}{p}(1-p)}, \text{ if } \delta < 0 \text{ and } \Delta_2^q = (\delta)^{\frac{q}{p}(1-p)}, \text{ if } \delta > 0.$$



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Using (2.7) in (2.5), we get

$$g(x)^{\frac{\delta}{p}} \int_a^x f(t) dg(t) \leq (-\delta)^{\frac{1}{p}(1-p)} [g(x)^{-\delta} - g(a)^{-\delta}]^{\frac{1}{p}(p-1)} g(x)^{\frac{\delta}{p}} \left[ \int_a^x g(t)^{(p-1)(1+\delta)} f(t)^p dg(t) \right]^{\frac{1}{p}}.$$

Raising both sides of the above inequalities to power  $q$  and using (2.1), we obtain

$$g(x)^{\frac{\delta q}{p}} F_a(x)^q \leq \Delta_1^q g_a(x)^{\frac{q}{p}(p-1)} g(x)^{\frac{\delta q}{p}} \theta_a(x)^{\frac{q}{p}},$$

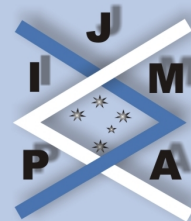
where  $g_a(x) = [g(x)^{-\delta} - g(a)^{-\delta}]$ .

Integrating over  $(a, b)$  with respect to  $g(x)^{-1} dg(x)$  gives

$$(2.8) \quad \int_a^b g(x)^{\frac{\delta q}{p}-1} g_a(x)^{\frac{q}{p}(1-p)} F_a(x)^q dg(x) \leq \Delta_1^q \int_a^b g(x)^{\frac{\delta q}{p}-1} \theta_a(x)^{\frac{q}{p}} dg(x) = J.$$

Now integrate the right side of (2.8) by parts to obtain

$$\begin{aligned} J &= \Delta_1^q \int_a^b g(x)^{\frac{\delta q}{p}-1} \theta_a(x)^{\frac{q}{p}} dg(x) \\ &= \frac{\Delta_1^q}{(\delta q/p)} g(x)^{\frac{\delta q}{p}} \theta_a(x)^{\frac{q}{p}} \Big|_a^b + (-\delta^{-1}) \Delta_1^q \\ &\quad \times \int_a^b g(x)^{\frac{\delta q}{p}} g(x)^{(p-1)(1+\delta)} f(x)^p \theta_a(x)^{\frac{q}{p}-1} dg(x). \end{aligned}$$



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However,

$$\begin{aligned}
 I &= \int_a^b g(x)^{\frac{\delta q}{p}} g(x)^{(p-1)(1+\delta)} f(x)^p \theta_a^{\frac{q}{p}-1}(x) dg(x) \\
 &= \int_a^b g(x)^{\frac{\delta q}{p}} g(x)^{(p-1)(1+\delta)} f(x)^p \left[ \int_a^x g(t)^{\delta p+p-1-\delta} f(t)^p dg(t) \right]^{\frac{q}{p}-1} dg(x) \\
 &= \int_a^b g(x)^{\delta p+p-1} f(x)^p \left[ g(x)^\delta \int_a^x g(t)^{\delta p+p-1-\delta} f(t)^p dg(t) \right]^{\frac{q}{p}-1} dg(x).
 \end{aligned}$$

Since  $\delta < 0$ , we have  $g(x)^{-\delta} \geq g(t)^{-\delta} \forall t \in [a, x]$ .

Consequently

$$\begin{aligned}
 I &\leq \int_a^b g(x)^{\delta p+p-1} f(x)^p \left[ \int_a^x g(t)^{\delta p+p-1} f(t)^p dg(t) \right]^{\frac{q}{p}-1} dg(x) \\
 &= \int_a^b \left[ \int_a^x g(t)^{\delta p+p-1} f(t)^p dg(t) \right]^{\frac{q}{p}-1} g(x)^{\delta p+p-1} f(x)^p dg(x) \\
 &= \frac{p}{q} \left[ \int_a^b g(x)^{\delta p-1} [f(x)g(x)]^p dg(x) \right]^{\frac{q}{p}}.
 \end{aligned}$$

Thus (2.8) becomes

$$\begin{aligned}
 (2.9) \quad \int_a^b g(x)^{\frac{\delta q}{p}-1} [g(x)^{-\delta} - g(a)^{-\delta}]^{\frac{q}{p}(1-p)} F_a(x)^q dg(x) \\
 + \frac{p}{q} \Delta_1^q(-\delta^{-1}) g(b)^{\frac{\delta q}{p}} \theta_a(b)^{\frac{q}{p}}
 \end{aligned}$$



$$\leq \frac{p}{q}(-\delta^{-1})\Delta_1^q \left[ \int_a^b g(x)^{\delta p-1} [f(x)g(x)]^p dg(x) \right]^{\frac{q}{p}}.$$

Taking the  $q^{th}$  root of both sides yields assertion (2.3) of the theorem.

To prove (2.4), we start with inequality (2.6) and use (2.7) with (2.2) to obtain

$$\begin{aligned} g(x)^{\frac{\delta}{p}} F_b(x) &\leq (-\delta)^{\frac{1}{p}(1-p)} g_b(x)^{\frac{1}{p}(p-1)} g(x)^{\frac{\delta}{p}} \theta_b(x)^{\frac{1}{p}} \\ &= (\delta^{-1})^{\frac{1}{p}(p-1)} (-g_b(x))^{\frac{1}{p}(p-1)} g(x)^{\frac{\delta}{p}} \theta_b(x)^{\frac{1}{p}}, \end{aligned}$$

where  $g_b(x) = [g(b)^{-\delta} - g(x)^{-\delta}]$ .

On rearranging and raising to power  $q$  and then integrating both sides over  $[a, b]$  with respect to  $g(x)^{-1}dg(x)$ , we obtain

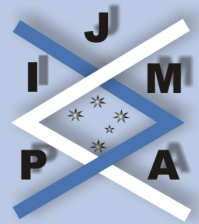
$$(2.10) \quad \int_a^b g(x)^{\frac{\delta q}{p}-1} [g(x)^{-\delta} - g(b)^{-\delta}]^{\frac{q}{p}(1-p)} F_b(x)^q dg(x) \leq \Delta_2^q \int_a^b g(x)^{\frac{\delta q}{p}-1} \theta_b(x) dg(x).$$

We denote the right side of (2.10) by  $H$ , integrate it by parts and use the fact that for  $\delta > 0$ ,  $g(x)^\delta \leq g(t)^\delta \quad \forall t \in [x, b]$  to obtain

$$H \leq \frac{p}{\delta q} \Delta_2^q g(x)^{\frac{\delta p}{q}} \theta_b(x)^{\frac{q}{p}} \Big|_a^b + (\delta q/p)^{-1} \Delta_2^q \int_a^b g(x)^{\delta p-1} [f(x)g(x)]^p dg(x).$$

Using this in (2.10) we obtain

$$(2.11) \quad \int_a^b g(x)^{\frac{\delta q}{p}-1} [g(x)^{-\delta} - g(b)^{-\delta}]^{\frac{q}{p}(1-p)} F_b(x)^q dg(x) + \frac{p}{q} \delta^{-1} \Delta_2^q g(a)^{\frac{q}{p}} \theta_b(a)^{\frac{q}{p}} \leq \frac{p}{q} \delta^{-1} \Delta_2^q \left[ \int_a^b g(x)^{\delta p-1} [f(x)g(x)]^p dg(x) \right]^{\frac{q}{p}}.$$



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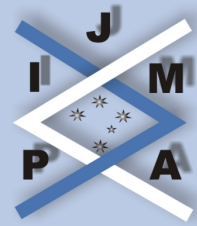
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We take the  $q^{th}$  root of both sides to obtain assertion (2.4) of the theorem. □

*Remark 1.* Let  $p = q$ , and  $\delta = \frac{1-r}{p} < 0$ , i.e.,  $r > 1$ , then (2.3) reduces to

$$(2.12) \quad \int_a^b g(x)^{\delta-1} [g(x)^{-\delta} - g(a)^{-\delta}]^{p-1} F_a(x)^p dg(x) + A_1(p, p, a, b, \delta) \\ \leq C_1(p, p, \delta) \left[ \int_a^b g(x)^{\delta p-1} [f(x)g(x)]^p dg(x) \right],$$

where

$$(2.13) \quad A_1(p, p, a, b, \delta) = (-\delta)^{-p} g(b)^\delta \theta_a(b), \quad \delta < 0$$

and

$$(2.14) \quad C_1(p, p, \delta) = (-\delta)^{-p} = \left[ \frac{p}{r-1} \right]^p.$$

Now from (2.18) in [2] we have that, for  $\delta < 0$

$$(2.15) \quad g(b)^\delta \theta_a(b) \geq (-\delta^{-1})^{1-p} g(b)^\delta [g(b)^{-\delta} - g(a)^{-\delta}]^{1-p} F_a(b)^p.$$

Thus, from (2.13) and (2.15), using notations in (1.1), we have

$$(2.16) \quad A_1(p, p, a, b, \delta) = (-\delta)^{-p} g(b)^\delta \theta_a(b) \\ \geq (-\delta)^{-p} (-\delta^{-1})^{1-p} g(b)^\delta [g(b)^{-\delta} - g(a)^{-\delta}]^{1-p} F_a(b)^p \\ = (-\delta)^{-1} g(b)^\delta [g(b)^{-\delta} - g(a)^{-\delta}]^{1-p} F_a(b)^p \\ = \frac{p}{r-1} g(b)^\delta [g(b)^{-\delta} - g(a)^{-\delta}]^{1-p} F_a(b)^p \\ = K_1(p, \delta, a, b),$$

i.e.,  $A_1(p, p, a, b, \delta) = K_1(p, \delta, a, b) + B_1$  for some  $B_1 \geq 0$ .

Thus we can write (2.12), using (2.14), as

$$(2.17) \quad \int_a^b g(x)^{\delta-1} [g(x)^{-\delta} - g(a)^{-\delta}]^{p-1} F_a(x)^p dg(x) + K_1(p, \delta, a, b) + B_1 \\ \leq \left[ \frac{p}{r-1} \right]^p \left[ \int_a^b g(x)^{\delta p-1} [f(x)g(x)]^p dg(x) \right].$$

So, when  $B_1 = 0$ , (2.17) reduces to (1.1). When  $B_1 \neq 0$ , i.e.,  $B_1 > 0$ , (2.17) is an improvement of (1.1). Similarly with notations in (1.2) and (2.4) in this paper we use (2.19) in [2] to prove that

$$A_2(p, p, a, b, \delta) = K_2(p, \delta, a, b) + B_2$$

for some  $B_2 \geq 0$ .

Thus, when  $p = q$ , (2.4) reduces to (1.2) if  $B_2 = 0$  and is an improvement of (1.2) when  $B_2 \neq 0$ , i. e., when  $B_2 > 0$ .



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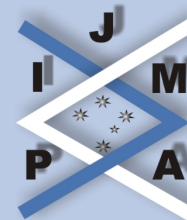
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