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ON AN INTEGRAL INEQUALITY

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ABSTRACT. In this article we give different sufficient conditions for inequality $\left(\int_a^b f(x)^\alpha \mathrm{d}x\right)^\beta \geq \int_a^b f(x)^\gamma \mathrm{d}x$ to hold.

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1. INTRODUCTION

In this paper we wish to investigate some sufficient conditions for the following inequality:

$$\left(\int_a^b f(x)^\alpha \mathrm{d}x\right)^\beta \geq \int_a^b f(x)^\gamma \mathrm{d}x. \tag{1.1}$$

This is a generalization of the inequalities that appear in the papers [4, 5, 6, 7, 8, 9].

F. Qi in [7] considered inequality (1.1) for $\alpha = n+2$, $\beta = 1/(n+1)$, $\gamma = 1$, $n \in \mathbb{N}$. He proved that under conditions

$$f \in \mathrm{C}^n([a,b]); \qquad f^{(i)}(a) \geq 0,\ 0 \leq i \leq n-1; \qquad f^{(n)}(x) \geq n!,\ x \in [a,b]$$

the inequality is valid.

Later, S. Mazouzi and F. Qi gave what appeared to be a simpler proof of the inequality under the same conditions (Corollary 3.6 in [1]). Unfortunately their proof was incorrect. Namely, they made a false substitution and arrived at the condition $f(x) \geq (n+1)(x-a)^n$ which is not true, e.g. for function $f(x) = x - a$, whereas this function obviously satisfies the conditions of the theorem if $n = 1$.

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K.-W. Yu and F. Qi ([9]) and N. Towghi ([8]) gave other conditions for the inequality (1.1) to hold under this special choice of constants α, β, γ.

T.K. Pogány in [6], by avoiding the assumption of differentiability used in [7, 8, 9], and instead using the inequalities due to Hölder, Nehari (Lemma 2.4) and Barnes, Godunova and Levin (Lemma 2.5) established some inequalities which are a special case of (1.1) when $\alpha = 1$ or $\gamma = 1$.

To obtain some conditions for the inequality (1.1) we will first proceed similarly to T.K. Pogány ([6]) and in the second part of this article we will be using a method from the paper [4].

2. Conditions Based on Inequalities Between Means

We want to transform inequality (1.1) to a form more suitable for us. It can easily be seen that if $f(x) \geq 0$, for all $x \in [a,b]$ and $\gamma > 0$, inequality (1.1) is equivalent to

$$\left[\left(\frac{\int_a^b f(x)^\alpha \mathrm{d}x}{b-a}\right)^{\frac{1}{\alpha}}\right]^{\frac{\alpha\beta}{\gamma}} (b-a)^{\frac{\beta-1}{\gamma}} \geq \left(\frac{\int_a^b f(x)^\gamma \mathrm{d}x}{b-a}\right)^{\frac{1}{\gamma}}. \tag{2.1}$$

Definition 2.1. Let f be a nonnegative and integrable function on the segment $[a,b]$. The r-mean (or the r-th power mean) of f is defined as

$$\mathrm{M}^{[r]}(f) := \begin{cases} \left(\frac{\int_a^b f(x)^r \mathrm{d}x}{b-a}\right)^{\frac{1}{r}} & (r \neq 0, +\infty, -\infty), \\ \exp\left(\frac{\int_a^b \ln f(x) \mathrm{d}x}{b-a}\right) & (r = 0), \\ m & (r = -\infty), \\ M & (r = +\infty). \end{cases}$$

where $m = \inf f(x)$ and $M = \sup f(x)$ for $x \in [a,b]$.

According to the previous definition inequality (2.1) can be written as

$$\left(\mathrm{M}^{[\alpha]}(f)\right)^{\frac{\alpha\beta}{\gamma}} (b-a)^{\frac{\beta-1}{\gamma}} \geq \mathrm{M}^{[\gamma]}(f) \tag{2.2}$$

We will be using the following inequalities:

Lemma 2.1 (power mean inequality, [2]). *If f is a nonnegative function on $[a,b]$ and $-\infty \leq r < s \leq +\infty$, then*

$$\mathrm{M}^{[r]}(f) \leq \mathrm{M}^{[s]}(f).$$

Lemma 2.2 (Berwald inequality, [3, 5]). *If f is a nonnegative concave function on $[a,b]$, then for $0 < r < s$ we have*

$$\mathrm{M}^{[s]}(f) \leq \frac{(r+1)^{1/r}}{(s+1)^{1/s}} \mathrm{M}^{[r]}(f).$$

Lemma 2.3 (Thunsdorff inequality, [3]). *If f is a nonnegative convex function on $[a,b]$ with $f(a) = 0$, then for $0 < r < s$ we have*

$$\mathrm{M}^{[s]}(f) \geq \frac{(r+1)^{1/r}}{(s+1)^{1/s}} \mathrm{M}^{[r]}(f).$$

Lemma 2.4 (Nehari inequality, [2]). *Let f, g be nonnegative concave functions on $[a,b]$. Then, for $p,q > 0$ such that $p^{-1} + q^{-1} = 1$, we have*

$$\left(\int_a^b f(x)^p \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_a^b g(x)^q \mathrm{d}x\right)^{\frac{1}{q}} \leq \mathrm{N}(p,q) \int_a^b f(x)g(x)\mathrm{d}x,$$

where $\mathrm{N}(p,q) = \frac{6}{(1+p)^{1/p}(1+q)^{1/q}}$.

Lemma 2.5 (Barnes-Godunova-Levin inequality, [3, 2]). *Let f, g be nonnegative concave functions on $[a,b]$. Then, for $p,q>1$ we have*

$$\left(\int_a^b f(x)^p \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_a^b g(x)^q \mathrm{d}x\right)^{\frac{1}{q}} \le \mathrm{B}(p,q)\int_a^b f(x)g(x)\mathrm{d}x,$$

where $\mathrm{B}(p,q) = \frac{6(b-a)^{1/p+1/q-1}}{(1+p)^{1/p}(1+q)^{1/q}}$.

Let us first state our results in a clear table. Each result is an independent set of conditions that guarantee the inequality (1.1) is valid.

Result	Conditions on constants α, β, γ, a, b	Conditions on function f (holds for all $x \in [a,b]$)	Lemma for the proof
1.	$\alpha \ge \gamma > 0$, $\alpha\beta > \gamma$	$f(x) \ge (b-a)^{\frac{-\beta+1}{\alpha\beta-\gamma}}$	Lemma 2.1
2.	$\alpha \le \gamma > 0$, $\alpha\beta < 0$	$0 \le f(x) \le (b-a)^{\frac{-\beta+1}{\alpha\beta-\gamma}}$	Lemma 2.1
3 (i).	$\alpha \ge \gamma > 0$, $\alpha\beta \ge \gamma$, $(b-a)^{\frac{\beta-1}{\gamma}} \ge 1$	$f(x) \ge 1$	Lemma 2.1
3 (ii).	$\alpha \ge \gamma > 0$, $\alpha\beta \le \gamma$, $(b-a)^{\frac{\beta-1}{\gamma}} \ge 1$	$0 \le f(x) \le 1$	Lemma 2.1
4 (i).	$0 < \alpha \le \gamma$, $\alpha\beta \ge \gamma$ $(b-a)^{\frac{\beta-1}{\gamma}} \ge \frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}}$	f concave $f(x) \ge 1$	Lemma 2.2
4 (ii).	$0 < \alpha \le \gamma$, $\alpha\beta \le \gamma$ $(b-a)^{\frac{\beta-1}{\gamma}} \ge \frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}}$	f concave $0 \le f(x) \le 1$	Lemma 2.2
5.	$0 < \alpha \le \gamma$, $\alpha\beta > \gamma$	f concave, $f(x) \ge$ $(b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}} \left(\frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}}\right)^{\frac{\gamma}{\alpha\beta-\gamma}}$	Lemma 2.2
6.	$0 < \gamma \le \alpha$, $\beta < 0$	f concave, $0 \le f(x) \le$ $(b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}} \left(\frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}}\right)^{\frac{\alpha\beta}{\alpha\beta-\gamma}}$	Lemma 2.2
7.	$0 < \gamma \le \alpha$, $\alpha\beta > \gamma$	f convex, $f(a)=0$, $f(x) \ge$ $\frac{(b-a)^{\frac{\alpha-\gamma}{\alpha(\alpha\beta-\gamma)}}}{(b^{\alpha+1}-a^{\alpha+1})^{1/\alpha}} \cdot \frac{(\alpha+1)^{\frac{\beta}{\alpha\beta-\gamma}}}{(\gamma+1)^{\frac{1}{\alpha\beta-\gamma}}} x$	Lemma 2.3
8.	$0 < \alpha \le \gamma$, $\beta < 0$	f convex, $f(a)=0$, $0 \le f(x) \le$ $(b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}} \left(\frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}}\right)^{\frac{\alpha\beta}{\alpha\beta-\gamma}}$	Lemma 2.3
9.	$0 < \gamma < \alpha$, $\beta < 0$	f concave, $0 \le f(x) \le (b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}}$ $\times \frac{6^{\frac{\alpha\beta}{\gamma(\gamma-\alpha\beta)}}}{\left(\frac{2\alpha-\gamma}{\alpha-\gamma}\right)^{\frac{\beta(\alpha-\gamma)}{\gamma(\gamma-\alpha\beta)}} \left(\frac{\alpha+\gamma}{\gamma}\right)^{\frac{\beta}{\gamma-\alpha\beta}}}$	Lemma 2.4

Remark 2.6. Observe that in the results $4(i)$. and $5.$ it is enough for the condition on f to hold in endpoints of segment $[a,b]$ (ie., for $f(a)$ and $f(b)$).

Remark 2.7. There is only one result in the table obtained with the help of Lemma 2.4 and none with the Lemma 2.5 because the constants in the conditions are quite complicated.

Remark 2.8. If we make the substitution $\gamma \mapsto 1$, $\beta \mapsto \frac{1}{\beta}$ in Result 1, Theorem 2.1 in [6] is acquired.

We will prove only a few results after which the method of proving the others will become clear.

Proof of Result 1. Lemma 2.1 implies that

$$\mathrm{M}^{[\alpha]}(f) \geq \mathrm{M}^{[\gamma]}(f). \tag{2.3}$$

Also

$$\mathrm{M}^{[\alpha]}(f) \geq \mathrm{M}^{[-\infty]}(f) \geq (b-a)^{\frac{-\beta+1}{\alpha\beta-\gamma}},$$

so by raising this inequality to a power, we get

$$\left(\mathrm{M}^{[\alpha]}(f)\right)^{\frac{\alpha\beta-\gamma}{\gamma}} \geq (b-a)^{\frac{-\beta+1}{\gamma}}. \tag{2.4}$$

Multiplying (2.3) and (2.4) we get (2.2). □

Proof of Result 3 (i). $\mathrm{M}^{[\alpha]}(f) \geq 1$ because $f(x) \geq 1$, so from $\frac{\alpha\beta}{\gamma} \geq 1$ and Lemma 2.1 it follows

$$\left(\mathrm{M}^{[\alpha]}(f)\right)^{\frac{\alpha\beta}{\gamma}} \geq \mathrm{M}^{[\alpha]}(f) \geq \mathrm{M}^{[\gamma]}(f). \tag{2.5}$$

By multiplication of (2.5) and the condition $(b-a)^{\frac{\beta-1}{\gamma}} \geq 1$ we get (2.2). □

Proof of Result 5. From

$$\mathrm{M}^{[\alpha]}(f) \geq \mathrm{M}^{[-\infty]}(f) \geq (b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}} \left(\frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}}\right)^{\frac{\gamma}{\alpha\beta-\gamma}}$$

and $\frac{\alpha\beta-\gamma}{\gamma} > 0$ we obtain

$$\left(\mathrm{M}^{[\alpha]}(f)\right)^{\frac{\alpha\beta-\gamma}{\gamma}} (b-a)^{\frac{\beta-1}{\gamma}} \geq \frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}}. \tag{2.6}$$

According to Lemma 2.2:

$$\mathrm{M}^{[\alpha]}(f) \frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}} \geq \mathrm{M}^{[\gamma]}(f). \tag{2.7}$$

From (2.6) and (2.7), by multiplication, we arrive at (2.2). □

Proof of Result 7. Since

$$f(x) \geq \frac{(b-a)^{\frac{\alpha-\gamma}{\alpha(\alpha\beta-\gamma)}}}{\left(b^{\alpha+1}-a^{\alpha+1}\right)^{1/\alpha}} \cdot \frac{(\alpha+1)^{\frac{\beta}{\alpha\beta-\gamma}}}{(\gamma+1)^{\frac{1}{\alpha\beta-\gamma}}} x$$

by integration it follows that

$$\mathrm{M}^{[\alpha]}(f) \geq (b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}} \left(\frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}}\right)^{\frac{\gamma}{\alpha\beta-\gamma}}. \tag{2.8}$$

However, Lemma 2.3 implies inequality (2.7). Thus, from (2.8) and (2.7) we finally find that inequality (2.2) is valid. □

3. CONDITIONS ASSOCIATED WITH THE FUNCTIONS WITH BOUNDED DERIVATIVE

In this section we will prove inequality (1.1) under different assumptions including the differentiability of f and boundedness of its derivative.

J. Pečarić and W. Janous proved in [4] the following theorem.

Theorem 3.1. *Let $1 < p \leq 2$ and $r \geq 3$. The differentiable function $f : [0, c] \to \mathbb{R}$ satisfies $f(0) = 0$ and $0 \leq f'(x) \leq M$ for all $0 \leq x \leq c$, c subject to*

$$0 < c \leq \left(\frac{p(p-1)2^{2-p}M^{p-r}}{r-1} \right)^{\frac{1}{r-2p+1}}. \tag{3.1}$$

Then

$$\left(\int_0^c f(x)\mathrm{d}x \right)^p \geq \int_0^c f(x)^r \mathrm{d}x.$$

(If $f'(x) \geq M$ the reverse inequality holds true under the condition that the second inequality in (3.1) is reversed.*)*

Remark 3.2. The emphasized words were left out of [4].

The following generalization will be proved:

Theorem 3.3. *Let $\alpha > 0$, $1 < \beta \leq 2$ and $\gamma \geq 2\alpha + 1$. The differentiable function $f : [0, c] \to \mathbb{R}$ satisfies $f(0) = 0$ and $0 \leq f'(x) \leq M$ for all $0 \leq x \leq c$, c subject to*

$$0 < c \leq \left(\frac{\beta(\beta-1)(\alpha+1)^{2-\beta}M^{\alpha\beta-\gamma}}{\gamma-\alpha} \right)^{\frac{1}{\gamma-\alpha\beta-\beta+1}}. \tag{3.2}$$

Then

$$\left(\int_0^c f(x)^\alpha \mathrm{d}x \right)^\beta \geq \int_0^c f(x)^\gamma \mathrm{d}x.$$

Remark 3.4. For $\alpha = 1$, $\beta = p$, $\gamma = r$, we get Theorem 3.1.

Proof. From $f(0) = 0$ and $0 \leq f'(x) \leq M$ we obtain

$$0 \leq f(x)^\alpha \leq M^\alpha x^\alpha \quad \text{and} \quad 0 \leq \int_0^x f(t)^\alpha \mathrm{d}t \leq \frac{M^\alpha x^{\alpha+1}}{\alpha+1} \quad \text{for} \quad 0 \leq x \leq c.$$

Now we define

$$F(x) := \left(\int_0^x f(t)^\alpha \mathrm{d}t \right)^\beta - \int_0^x f(t)^\gamma \mathrm{d}t.$$

Then $F(0) = 0$ and $F'(x) = f(x)^\alpha g(x)$, where

$$g(x) := \beta \left(\int_0^x f(t)^\alpha \mathrm{d}t \right)^{\beta-1} - f(x)^{\gamma-\alpha}.$$

Clearly, $g(0) = 0$ and $g'(x) = f(x)^\alpha h(x)$, where

$$h(x) := \beta(\beta-1) \left(\int_0^x f(t)^\alpha \mathrm{d}t \right)^{\beta-2} - (\gamma-\alpha) f(x)^{\gamma-2\alpha-1} f'(x).$$

From the conditions of the theorem we have

$$\begin{aligned} h(x) &\geq \beta(\beta-1) \left(\frac{M^\alpha x^{\alpha+1}}{\alpha+1} \right)^{\beta-2} - (\gamma-\alpha)(Mx)^{\gamma-2\alpha-1} M \\ &= M^{\gamma-2\alpha} x^{(\alpha+1)(\beta-2)} \left(\beta(\beta-1)(\alpha+1)^{2-\beta} M^{\alpha\beta-\gamma} - (\gamma-\alpha) x^{\gamma-\alpha\beta-\beta+1} \right) \end{aligned}$$

Thus, since (3.2) is equivalent to

$$\beta(\beta-1)(\alpha+1)^{2-\beta}M^{\alpha\beta-\gamma} \geq (\gamma-\alpha)x^{\gamma-\alpha\beta-\beta+1}, \quad x \in [0,c],$$

we have $h(x) \geq 0$, $g'(x) \geq 0$, $g(x) \geq 0$, $F'(x) \geq 0$ and finally $F(x) \geq 0$. So $F(c) \geq 0$. □

Substituting $c = a - b$ and translating function f a units to the right ($f(x) \mapsto f(x-a)$) we obtain the following theorem.

Theorem 3.5. *Let* $\alpha > 0$, $1 < \beta \leq 2$ *and* $\gamma \geq 2\alpha + 1$. *The differentiable function* $f : [a,b] \to \mathbb{R}$ *satisfies* $f(a) = 0$ *and* $0 \leq f'(x) \leq M$ *for all* $a \leq x \leq b$, *where*

$$0 < b - a \leq \left(\frac{\beta(\beta-1)(\alpha+1)^{2-\beta}M^{\alpha\beta-\gamma}}{\gamma-\alpha} \right)^{\frac{1}{\gamma-\alpha\beta-\beta+1}}. \tag{3.3}$$

Then the inequality (1.1) *holds.*

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