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# EQUATIONS AND INEQUALITIES INVOLVING $v_p(n!)$

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ABSTRACT. In this paper we study $v_p(n!)$, the greatest power of prime $p$ in factorization of $n!$. We find some lower and upper bounds for $v_p(n!)$, and we show that $v_p(n!) = \frac{n}{p-1} + O(\ln n)$. By using the afore mentioned bounds, we study the equation $v_p(n!) = v$ for a fixed positive integer $v$. Also, we study the triangle inequality about $v_p(n!)$, and show that the inequality $p^{v_p(n!)} > q^{v_q(n!)}$ holds for primes $p < q$ and sufficiently large values of $n$.

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## 1. INTRODUCTION

As we know, for every $n \in \mathbb{N}$, $n! = 1 \times 2 \times 3 \times \cdots \times n$. Let $v_p(n!)$ be the highest power of prime $p$ in factorization of $n!$ to prime numbers. It is well-known that (see [3] or [5])

$$v_p(n!) = \sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right] = \sum_{k=1}^{\left[\frac{\ln n}{\ln p}\right]} \left[\frac{n}{p^k}\right], \tag{1.1}$$

in which $[x]$ is the largest integer less than or equal to $x$. An elementary problem about $n!$ is finding the number of zeros at the end of it, in which clearly its answer is $v_5(n!)$. The inverse of this problem is very nice; for example finding values of $n$ in which $n!$ terminates in 37 zeros [3], and generally finding values of $n$ such that $v_p(n!) = v$. We show that if $v_p(n!) = v$ has a

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solution then it has exactly $p$ solutions. For doing these, we need some properties of $[x]$, such as

$$[x] + [y] \leq [x + y] \qquad (x, y \in \mathbb{R}), \tag{1.2}$$

and

$$\left[\frac{x}{n}\right] = \left[\frac{[x]}{n}\right] \qquad (x \in \mathbb{R}, n \in \mathbb{N}). \tag{1.3}$$

## 2. ESTIMATING $v_p(n!)$

**Theorem 2.1.** *For every $n \in \mathbb{N}$ and prime $p$, such that $p \leq n$, we have:*

$$\frac{n-p}{p-1} - \frac{\ln n}{\ln p} < v_p(n!) \leq \frac{n-1}{p-1}. \tag{2.1}$$

*Proof.* According to the relation (1.1), we have $v_p(n!) = \sum_{k=1}^{m} \left[\frac{n}{p^k}\right]$ in which $m = \left[\frac{\ln n}{\ln p}\right]$, and since $x - 1 < [x] \leq x$, we obtain

$$n \sum_{k=1}^{m} \frac{1}{p^k} - m < v_p(n!) \leq n \sum_{k=1}^{m} \frac{1}{p^k},$$

considering $\sum_{k=1}^{m} \frac{1}{p^k} = \frac{1 - \frac{1}{p^m}}{p-1}$, we obtain

$$\frac{n}{p-1}\left(1 - \frac{1}{p^m}\right) - m < v_p(n!) \leq \frac{n}{p-1}\left(1 - \frac{1}{p^m}\right),$$

and combining this inequality with $\frac{\ln n}{\ln p} - 1 < m \leq \frac{\ln n}{\ln p}$ completes the proof. □

**Corollary 2.2.** *For every $n \in \mathbb{N}$ and prime $p$, such that $p \leq n$, we have:*

$$v_p(n!) = \frac{n}{p-1} + O(\ln n).$$

*Proof.* By using (2.1), we have

$$0 < \frac{\frac{n}{p-1} - v_p(n!)}{\ln n} < \frac{1}{\ln p} + O\left(\frac{1}{\ln n}\right),$$

and this yields the result. □

Note that the above corollary asserts that $n!$ ends approximately in $\frac{n}{4}$ zeros [1].

**Corollary 2.3.** *For every $n \in \mathbb{N}$ and prime $p$, such that $p \leq n$, and for all $a \in (0, \infty)$ we have:*

$$\frac{n-p}{p-1} - \frac{1}{\ln p}\left(\frac{n}{a} + \ln a - 1\right) < v_p(n!). \tag{2.2}$$

*Proof.* Consider the function $f(x) = \ln x$. Since, $f''(x) = -\frac{1}{x^2}$, $\ln x$ is a concave function and so, for every $a \in (0, +\infty)$ we have

$$\ln x \leq \ln a + \frac{1}{a}(x - a),$$

combining this with the left hand side of (2.1) completes the proof. □

## 3. STUDY OF THE EQUATION $v_p(n!) = v$

Suppose $v \in \mathbb{N}$ is given. We are interested in finding the values of $n$ such that in factorization of $n!$, the highest power of $p$, is equal to $v$. First, we find some lower and upper bounds for these $n$'s.

**Lemma 3.1.** *Suppose $v \in \mathbb{N}$ and $p$ is a prime and $v_p(n!) = v$, then we have*

$$1 + (p-1)v \leq n < \frac{v + \frac{p}{p-1} + \frac{\ln(1+(p-1)v)}{\ln p} - \frac{1}{\ln p}}{\frac{1}{p-1} - \frac{1}{(1+(p-1)v)\ln p}}. \tag{3.1}$$

*Proof.* For proving the left hand side of (3.1), use right hand side of (2.1) with the assumption $v_p(n!) = v$, and for proving the right hand side of (3.1), use (2.2) with $a = 1 + (p-1)v$. □

Lemma 3.1 suggests an interval for the solution of $v_p(n!) = v$. In the next lemma we show that it is sufficient for one to check only multiples of $p$ in above interval.

**Lemma 3.2.** *Suppose $m \in \mathbb{N}$ and $p$ is a prime, then we have*

$$v_p((pm+p)!) - v_p((pm)!) \geq 1. \tag{3.2}$$

*Proof.* By using (1.1) and (1.2) we have

$$\begin{aligned} v_p((pm+p)!) &= \sum_{k=1}^{\infty} \left[\frac{pm+p}{p^k}\right] \\ &\geq \sum_{k=1}^{\infty} \left[\frac{pm}{p^k}\right] + \sum_{k=1}^{\infty} \left[\frac{p}{p^k}\right] \\ &= 1 + v_p((pm)!), \end{aligned}$$

and this completes the proof. □

In the next lemma, we show that if $v_p(n!) = v$ has a solution, then it has exactly $p$ solutions. In fact, the next lemma asserts that if $v_p((mp)!) = v$ holds, then for all $0 \leq r \leq p-1$, $v_p((mp+r)!) = v$ also holds.

**Lemma 3.3.** *Suppose $m \in \mathbb{N}$ and $p$ is a prime, then we have*

$$v_p((m+1)!) \geq v_p(m!), \tag{3.3}$$

*and*

$$v_p((pm+p-1)!) = v_p((pm)!). \tag{3.4}$$

*Proof.* For proving (3.3), use (1.1) and (1.2) as follows

$$\begin{aligned} v_p((m+1)!) &= \sum_{k=1}^{\infty}\left[\frac{m+1}{p^k}\right] \\ &\geq \sum_{k=1}^{\infty}\left[\frac{m}{p^k}\right] + \sum_{k=1}^{\infty}\left[\frac{1}{p^k}\right] \\ &= \sum_{k=1}^{\infty}\left[\frac{m}{p^k}\right] = v_p(m!). \end{aligned}$$

For proving (3.4), it is enough to show that for all $k \in \mathbb{N}$, $\left[\frac{pm+p-1}{p^k}\right] = \left[\frac{pm}{p^k}\right]$ and we do this by induction on $k$; for $k = 1$, clearly $\left[\frac{pm+p-1}{p}\right] = \left[\frac{pm}{p}\right]$. Now, by using (1.3) we have

$$\begin{aligned} \left[\frac{pm+p-1}{p^{k+1}}\right] &= \left[\frac{\frac{pm+p-1}{p^k}}{p}\right] = \left[\frac{\left[\frac{pm+p-1}{p^k}\right]}{p}\right] = \left[\frac{\left[\frac{pm}{p^k}\right]}{p}\right] \\ &= \left[\frac{\frac{pm}{p^k}}{p}\right] = \left[\frac{pm}{p^{k+1}}\right]. \end{aligned}$$

This completes the proof. □

So, we have proved that

**Theorem 3.4.** *Suppose $v \in \mathbb{N}$ and $p$ is a prime. For solving the equation $v_p(n!) = v$, it is sufficient to check the values $n = mp$, in which $m \in \mathbb{N}$ and*

$$\left[\frac{1+(p-1)v}{p}\right] \leq m \leq \left[\frac{v + \frac{p}{p-1} + \frac{\ln(1+(p-1)v)}{\ln p} - \frac{1}{\ln p}}{\frac{p}{p-1} - \frac{p}{(1+(p-1)v)\ln p}}\right]. \tag{3.5}$$

*Also, if $n = mp$ is a solution of $v_p(n!) = v$, then it has exactly $p$ solutions $n = mp + r$, in which $0 \leq r \leq p-1$.*

**Note and Problem 1.** As we see, there is no guarantee of the existence of a solution for $v_p(n!) = v$. In fact we need to show that $\{v_p(n!) | n \in \mathbb{N}\} = \mathbb{N}$; however, computational observations suggest that $n = p\left\|\frac{1+(p-1)v}{p}\right\|$ usually is a solution, such that $||x||$ is the nearest integer to $x$, but we cannot prove it.

**Note and Problem 2.** Other problems can lead us to other equations involving $v_p(n!)$; for example, suppose $n, v \in \mathbb{N}$ given, find the value of prime $p$ such that $v_p(n!) = v$.
Or, suppose $p$ and $q$ are primes and $f : \mathbb{N}^2 \to \mathbb{N}$ is a prime value function, for which $n$'s do we have $v_p(n!) + v_q(n!) = v_{f(p,q)}(n!)$? And many other problems!

## 4. TRIANGLE INEQUALITY CONCERNING $v_p(n!)$

In this section we are going to compare $v_p((m+n)!)$ and $v_p(m!) + v_p(n!)$.

**Theorem 4.1.** *For every* $m, n \in \mathbb{N}$ *and prime* $p$*, such that* $p \le \min\{m, n\}$*, we have*

$$v_p((m+n)!) \ge v_p(m!) + v_p(n!), \tag{4.1}$$

*and*

$$v_p((m+n)!) - v_p(m!) - v_p(n!) = O(\ln(mn)). \tag{4.2}$$

*Proof.* By using (1.1) and (1.2), we have

$$\begin{aligned} v_p((m+n)!) &= \sum_{k=1}^{\infty} \left[\frac{m+n}{p^k}\right] \\ &\ge \sum_{k=1}^{\infty} \left[\frac{m}{p^k}\right] + \sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right] \\ &= v_p(m!) + v_p(n!). \end{aligned}$$

Also, by using (2.1) and (4.1) we obtain

$$\begin{aligned} 0 &\le v_p((m+n)!) - v_p(m!) - v_p(n!) \\ &< \frac{2p-1}{p-1} + \frac{\ln(mn)}{\ln p} \le 3 + \frac{\ln(mn)}{\ln 2}, \end{aligned}$$

this completes the proof. □

More generally, if $n_1, n_2, \dots, n_t \in \mathbb{N}$ and $p$ is a prime, in which $p \le \min\{n_1, n_2, \dots, n_t\}$, by using an extension of (1.2), we obtain

$$v_p\left(\left(\sum_{k=1}^{t} n_k\right)!\right) \ge \sum_{k=1}^{t} v_p(n_k!),$$

and by using this inequality and (2.1), we obtain

$$\begin{aligned} 0 &\le v_p\left(\left(\sum_{k=1}^{t} n_k\right)!\right) - \sum_{k=1}^{t} v_p(n_k!) \\ &< \frac{tp-1}{p-1} + \frac{\ln(n_1 n_2 \cdots n_t)}{\ln p} \\ &\le 2t - 1 + \frac{\ln(n_1 n_2 \cdots n_t)}{\ln 2}, \end{aligned}$$

and consequently we have

$$v_p\left(\left(\sum_{k=1}^{t} n_k\right)!\right) - \sum_{k=1}^{t} v_p(n_k!) = O(\ln(n_1 n_2 \cdots n_t)).$$

**Note and Problem 3.** Suppose $f : \mathbb{N}^t \to \mathbb{N}$ is a function and $p$ is a prime. For which $n_1, n_2, \dots, n_t \in \mathbb{N}$, do we have

$$v_p((f(n_1, n_2, \dots, n_t)!) \ge f(v_p(n_1!), v_p(n_2!), \dots, v_p(n_t!))?$$

Also, we can consider the above question in other view points.

## 5. THE INEQUALITY $p^{v_p(n!)} > q^{v_q(n!)}$

Suppose $p$ and $q$ are primes and $p < q$. Since $v_p(n!) \geq v_q(n!)$, comparing $p^{v_p(n!)}$ and $q^{v_q(n!)}$ becomes a nice problem. In [2], by using elementary properties about $[x]$, the inequality $p^{v_p(n!)} > q^{v_q(n!)}$ was considered for some special cases. In addition, it was shown that $2^{v_2(n!)} > 3^{v_3(n!)}$ holds for all $n \geq 4$. In this section we study $p^{v_p(n!)} > q^{v_q(n!)}$ in the more general case and also reprove $2^{v_2(n!)} > 3^{v_3(n!)}$.

**Lemma 5.1.** *Suppose $p$ and $q$ are primes and $p < q$, then*

$$p^{q-1} > q^{p-1}.$$

*Proof.* Consider the function

$$f(x) = x^{\frac{1}{x-1}} \qquad (x \geq 2).$$

A simple calculation yields that for $x \geq 2$ we have

$$f'(x) = -\frac{x^{\frac{2-x}{x-1}}(x \ln x - x + 1)}{(x-1)^2} < 0,$$

so, $f$ is strictly decreasing and $f(p) > f(q)$. This completes the proof. □

**Theorem 5.2.** *Suppose $p$ and $q$ are primes and $p < q$, then for sufficiently large $n$'s we have*

$$p^{v_p(n!)} > q^{v_q(n!)}. \tag{5.1}$$

*Proof.* Since $p < q$, Lemma 5.1 yields that $\frac{p^{q-1}}{q^{p-1}} > 1$ and so, there exits $N \in \mathbb{N}$ such that for $n > N$ we have

$$\left(\frac{p^{q-1}}{q^{p-1}}\right)^n \geq \frac{p^{p(q-1)}}{q^{p-1}} n^{(p-1)(q-1)}.$$

Thus,

$$\frac{p^{n(q-1)}}{n^{(p-1)(q-1)}p^{p(q-1)}} \geq \frac{q^{n(p-1)}}{q^{p-1}},$$

and therefore,

$$\frac{p^{\frac{n}{p-1}}}{np^{\frac{p}{p-1}}} \geq \frac{q^{\frac{n}{q-1}}}{q^{\frac{1}{q-1}}}.$$

So, we obtain

$$p^{\frac{n-p}{p-1} - \frac{\ln n}{\ln p}} \geq q^{\frac{n-1}{q-1}},$$

and considering this inequality with (2.1), completes the proof. □

**Corollary 5.3.** *For $n = 2$ and $n \geq 4$ we have*

$$2^{v_2(n!)} > 3^{v_3(n!)}. \tag{5.2}$$

*Proof.* It is easy to see that for $n \geq 30$ we have

$$\left(\frac{4}{3}\right)^n \geq \frac{16}{3} n^2,$$

and by Theorem 5.2, we yield (5.2) for $n \geq 30$. For $n = 2$ and $4 \leq n < 30$ check it using a computer. □

**A Computational Note.** In Theorem 5.2, the relation (5.1) holds for $n > N$ (see its proof). We can check (5.1) for $n \leq N$ at most by checking the following number of cases:

$$R(N) := \#\left\{(p,q,n) \mid p,q \in \mathbb{P}, n = 3,4,\dots,N, \text{ and } p < q \leq N\right\},$$

in which $\mathbb{P}$ is the set of all primes. If, $\pi(x) =$ The number of primes $\leq x$, then we have

$$R(N) = \sum_{n=3}^{N} \#\left\{(p,q) \mid p,q \in \mathbb{P}, \text{ and } p < q \leq n\right\} = \frac{1}{2}\sum_{n=3}^{N} \pi(n)(\pi(n)-1).$$

But, clearly $\pi(n) < n$ and this yields that

$$R(N) < \frac{N^3}{6}.$$

Of course, we have other bounds for $\pi(n)$ sharper than $n$ such as [4]

$$\pi(n) \leq \frac{n}{\ln n}\left(1 + \frac{1}{\ln n} + \frac{2.25}{\ln^2 n}\right) \qquad (n \geq 355991),$$

and by using this bound we can find sharper bounds for $R(N)$.

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