Journal of Inequalities in Pure and Applied Mathematics

CERTAIN INEQUALITIES CONCERNING BICENTRIC QUADRILATERALS, HEXAGONS AND OCTAGONS

MIRKO RADIĆ

University of Rijeka
Faculty of Philosophy
Department of Mathematics
51000 Rijeka, Omladinska 14, Croatia.

EMail: mradic@pefri.hr

volume 6, issue 1, article 1, 2005.

Received 15 April, 2004; accepted 24 November, 2004.

Communicated by: J. Sándor

ISSN (electronic): 1443-5756
118-04

Abstract

In this paper we restrict ourselves to the case when conics are circles one completely inside of the other. Certain inequalities concerning bicentric quadrilaterals, hexagons and octagons in connection with Poncelet's closure theorem are established.

2000 Mathematics Subject Classification: 51E12
Key words: Bicentric Polygon, Inequality.

Contents

1. Introduction

A polygon which is both chordal and tangential is briefly called a bicentric polygon. The following notation will be used.

If $A_1 \cdots A_n$ is considered to be a bicentric n-gon, then its incircle is denoted by C_1, circumcircle by C_2, radius of C_1 by r, radius of C_2 by R, center of C_1 by I, center of C_2 by O, distance between I and O by d.

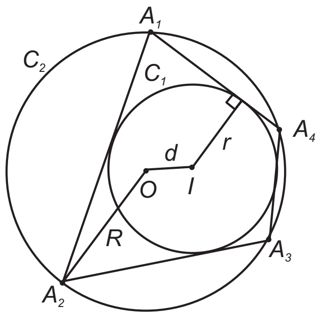

Figure 1.1

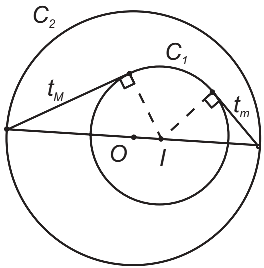

Figure 1.2

The first person who was concerned with bicentric polygons was the German mathematician Nicolaus Fuss (1755-1826). He found that C_1 is the incircle and C_2 the circumcircle of a bicentric quadrilateral $A_1A_2A_3A_4$ iff

$$(R^2 - d^2)^2 = 2r^2(R^2 + d^2), \tag{1.1}$$

(see [4]). The problem of finding this relation has been mentioned in [3] as one of 100 great problems of elementary mathematics.

Fuss also found the corresponding relations (conditions) for bicentric pentagon, hexagon, heptagon and octagon [5]. For bicentric hexagons and octagons these relations are

$$3p^4q^4 - 2p^2q^2r^2(p^2+q^2) = r^4(p^2-q^2)^2 \tag{1.2}$$

and

$$[r^2(p^2+q^2) - p^2q^2]^4 = 16p^4q^4r^4(p^2-r^2)(q^2-r^2), \tag{1.3}$$

where $p = R + d$, $q = R - d$.

The very remarkable theorem concerning bicentric polygons was given by the French mathematician Poncelet (1788-1867). This theorem is known as *Poncelet's closure theorem*. For the case when conics are circles, one inside the other, this theorem can be stated as follows:

If there is one bicentric n-gon whose incircle is C_1 and circumcircle C_2, then there are infinitely many bicentric n-gons whose incircle is C_1 and circumcircle is C_2. For every point P_1 on C_2 there are points $P_2, \ldots, P_n$ on C_2 such that $P_1 \cdots P_n$ are bicentric n-gons whose incircle is C_1 and circumcircle is C_2.

Although the famous Poncelet's closure theorem dates from the nineteenth century, many mathematicians have been working on a number of problems in connection with it. Many contributions have been made, and much interesting information can be found concerning it in the references [1] and [6].

An important role in the following will have the least and the largest tangent that can be drawn from C_2 to C_1. As can be seen from Figure 1.2, the following holds

$$t_m = \sqrt{(R-d)^2 - r^2}, \quad t_M = \sqrt{(R+d)^2 - r^2}. \tag{1.4}$$

2. Certain Inequalities Concerning Bicentric Quadrilaterals

Let $A_1A_2A_3A_4$ be any given bicentric quadrilateral whose incircle is C_1 and circumcircle C_2 and let

$$t_i + t_{i+1} = |A_iA_{i+1}|,\ i = 1, 2, 3, 4. \tag{2.1}$$

(Indices are calculated modulo 4.) In [8, Theorem 3.1 and Theorem 3.2] it is proven that the following hold

$$t_1t_3 = t_2t_4 = r^2, \tag{2.2}$$

and

$$t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 = 2(R^2 - d^2). \tag{2.3}$$

Reversely, if t_1, t_2, t_3, t_4 are such that (2.2) and (2.3) hold, then there is a bicentric quadrilateral such that (2.1) holds.

Theorem 2.1. *The tangent-lengths t_1, t_2, t_3, t_4 given by (2.1) satisfy the following inequalities*

$$2r \le t_1 + t_3 \le t_m + t_M, \tag{2.4}$$

$$2r \le t_2 + t_4 \le t_m + t_M, \tag{2.5}$$

$$4r \leq t_1 + t_2 + t_3 + t_4 \leq 4r \cdot \frac{R^2 + d^2}{R^2 - d^2}, \tag{2.6}$$

$$4r^2 \leq t_1^2 + t_2^2 + t_3^2 + t_4^2 \leq 4(R^2 + d^2 - r^2), \tag{2.7}$$

and

$$t_1^{2k} + t_2^{2k} + t_3^{2k} + t_4^{2k} \geq 4r^{2k}, \quad k \in \mathbb{N}. \tag{2.8}$$

The equalities hold only if $d = 0$.

Proof. First let us remark that $t_m t_M = r^2$ since there is a bicentric quadrilateral as shown in Figure 2.1. Now, let C denote a circle whose diameter is $t_m + t_M$ (Figure 2.2). Then for each t_i, $i = 1, 2, 3, 4$, since $t_m \leq t_i \leq t_M$, there are points Q and R on C such that

$$t_i = |PQ|, \quad t_{i+2} = |PR|, \tag{2.9}$$

where $|PQ| + |PR| = |QR|$. In this connection let us remark that the power of the circle C at P is $t_m t_M$. Therefore $|PQ||PR| = t_m t_M$.

Obviously $t_i + t_{i+2} \leq t_m + t_M$ since $t_m + t_M$ is a diameter of C. Also it is clear that $t_i + t_{i+2} \geq 2r$ since $r^2 = t_m t_M$.

This proves (2.4) and (2.5).

In the proof that (2.6) holds we shall use the relations

$$t_m = r \cdot \frac{R - d}{R + d}, \quad t_M = r \cdot \frac{R + d}{R - d}. \tag{2.10}$$

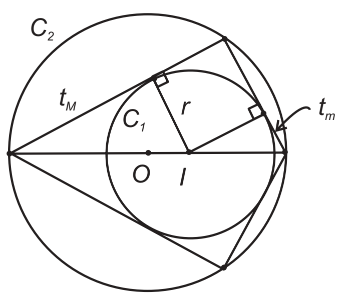

Figure 2.1

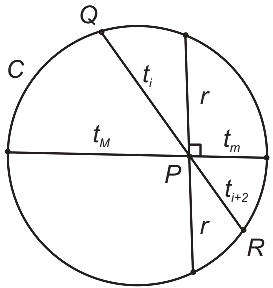

Figure 2.2

It is easy to show that each of the above relations is equivalent to the Fuss relation (1.1). So, for the first of them we can write

$$\begin{aligned}
&(R-d)^2 - r^2 = r^2\left(\frac{R-d}{R+d}\right)^2 \\
&(R^2-d^2)^2 - r^2(R+d)^2 = r^2(R-d)^2 \\
&(R^2-d^2)^2 = 2r^2(R^2+d^2).
\end{aligned}$$

The proof that (2.6) holds can be written as

$$2r + 2r \le t_1 + t_3 + t_2 + t_4,$$
$$t_1 + t_3 + t_2 + t_4 \le 2(t_m + t_M) = 2r\left(\tfrac{R-d}{R+d} + \tfrac{R+d}{R-d}\right) = 4r \cdot \tfrac{R^2+d^2}{R^2-d^2}.$$

The proof that (2.7) holds is as follows.

Since $2r \le t_1 + t_3$, $2r \le t_2 + t_4$, we have

$$4r^2 \le t_1^2 + t_3^2 + 2t_1t_3, \quad 4r^2 \le t_2^2 + t_4^2 + 2t_2t_4$$

or, since $2t_1t_3 = 2t_2t_4 = 2r^2$,

$$2r^2 \le t_1^2 + t_3^2, \quad 2r^2 \le t_2^2 + t_4^2.$$

Thus, $4r^2 \le t_1^2 + t_2^2 + t_3^2 + t_4^2$.

From $t_1 + t_3 \le t_m + t_M$, $t_2 + t_4 \le t_m + t_M$ it follows that

$$t_1^2 + t_3^2 \le t_m^2 + t_M^2, \quad t_2^2 + t_4^2 \le t_m^2 + t_M^2$$

since $2t_1t_3 = 2t_2t_4 = 2t_mt_M$. Thus, we obtain

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 \le 2(t_m^2 + t_M^2),$$

where $t_m^2 + t_M^2 = (R-d)^2 - r^2 + (R+d)^2 - r^2 = 2(R^2 + d^2 - r^2)$.

In the same way it can be proved that (2.8) holds. So, starting from $2r \le t_1 + t_3$, since $2t_1^kt_3^k = 2r^{2k}$, we can write

$$\begin{aligned} &2r^2 \le t_1^2 + t_3^2, \\ &4r^4 \le t_1^4 + t_3^4 + 2t_1^2t_3^2 \text{ or } 2r^4 \le t_1^4 + t_3^4 \end{aligned}$$

and so on.

Starting from $t_1 + t_3 \le t_m + t_M$ it can be written

$$\begin{aligned} t_1^2 + t_3^2 &\le t_m^2 + t_M^2, \\ t_1^4 + t_3^4 &\le t_m^4 + t_M^4, \end{aligned}$$

and so on.

Since $t_m = t_M$ only if $d = 0$, it is clear that the relations (2.4) – (2.8) become equalities only if $d = 0$. Thus, if $d \neq 0$, then in the above relations instead of $\leq$ we may put $<$.

Theorem 2.1 is thus proved. □

Corollary 2.2. *The following holds*

$$\frac{4}{r} \leq \sum_{i=1}^{4} \frac{1}{t_i} \leq \frac{4}{r} \cdot \frac{R^2 + d^2}{R^2 - d^2}. \tag{2.11}$$

Proof. From $2r \leq t_1 + t_3$ it follows that $\frac{2}{r} \leq \frac{1}{t_1} + \frac{1}{t_3}$, since

$$\frac{1}{t_1} + \frac{1}{t_3} = \frac{t_1 + t_3}{t_1 t_3} = \frac{t_1 + t_3}{r^2} \geq \frac{2r}{r^2} = \frac{2}{r}.$$

From $t_1 + t_3 \leq t_m + t_M$ it follows that $\frac{1}{t_1} + \frac{1}{t_3} \leq \frac{1}{t_m} + \frac{1}{t_M}$, since

$$\frac{1}{t_1} + \frac{1}{t_3} = \frac{t_1 + t_3}{r^2}, \quad \frac{1}{t_m} + \frac{1}{t_M} = \frac{t_m + t_M}{r^2}.$$

□

Corollary 2.3. *Let* $a = t_1 + t_2$, $b = t_2 + t_3$, $c = t_3 + t_4$, $d = t_4 + t_1$.*Then*

$$8r \leq a + b + c + d \leq 8r \cdot \frac{R^2 - d^2}{R^2 + d^2}. \tag{2.12}$$

Corollary 2.4. *Let a, b, c, d be as in Corollary 2.3. Then*

$$4(R^2 - d^2 + 2r^2) \le a^2 + b^2 + c^2 + d^2 \le 4(3R^2 - 2r^2). \tag{2.13}$$

Proof. Using relation (2.3) we can write

$$a^2 + b^2 + c^2 + d^2 = 2(t_1^2 + t_2^2 + t_3^2 + t_4^2) + 4(R^2 - d^2).$$

Now, using relations (2.7) we can write relations (2.13). □

Corollary 2.5. *The following holds*

$$2r^2 + d^2 \le R^2 \le 2r^2 + d^2 + 2rd. \tag{2.14}$$

Proof. Since $t_1 + t_3 \ge 2r$, $t_2 + t_4 \ge 2r$, we can write

$$\begin{aligned} (t_1 + t_3)(t_2 + t_4) &\ge 4r^2, \\ t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 &\ge 4r^2, \\ 2(R^2 - d^2) &\ge 4r^2, \\ R^2 - d^2 &\ge 2r^2. \end{aligned}$$

The fact that $R^2 \le 2r^2 + d^2 + 2rd$ is clear from the quadratic function

$$f(d) = d^2 + 2rd + R^2 - 2r^2.$$

If $d = 0$, then $f(d) = 0$, but if $d > 0$, then $f(d) > 0$. □

Remark 1. *It may be interesting that relations (2.14) can be obtained directly from Fuss' relation (1.1). It was done by L. Fejes Toth in [11]. Namely, relation (1.1) implies*

$$d^2 = r^2 + R^2 - r\sqrt{r^2 + 4R^2}, \tag{2.15}$$

so the left side inequality of (2.14) becomes equivalent to $2r^2 \leq R^2$ or

$$r\sqrt{2} \leq R. \tag{2.16}$$

The right side of (2.14) is equivalent to (quadratic polynomial inequality in d)

$$d \geq -r + \sqrt{R^2 - r^2}$$

or by using (2.15), after some simple computations, to (2.16), again.

Concerning the sign $\leq$ in the relations (2.11) – (2.14), it is clear that in the case when $d \neq 0$, that is, when $t_m \neq t_M$, then instead of $\leq$ may be put $<$.

In connection with Theorem 2.1, the following theorem is of some interest.

Theorem 2.6. *Let $\underline{P} = P_1P_2P_3P_4$ and $\underline{Q} = Q_1Q_2Q_3Q_4$ be axially symmetric bicentric quadrilaterals whose incircle is C_1 and circumcircle C_2 (Figure 2.3). Denote by $2p_M$ and $2p_m$ respectively the perimeters of $\underline{P}$ and $\underline{Q}$. Then for every bicentric quadrilateral $\underline{A} = A_1A_2A_3A_4$ whose incircle is C_1 and circumcircle C_2 it holds that*

$$p_m \leq \sum_{i=1}^{4} t_i \leq p_M, \tag{2.17}$$

where $t_i + t_{i+1} = |A_i A_{i+1}|$, $i = 1, 2, 3, 4$. *Also, if* $d \neq 0$, *then* $p_m < p_M$ *and*

$$\sum_{i=1}^{4} t_i = p_M \text{ iff } \underline{A} = \underline{P}, \quad \sum_{i=1}^{4} t_i = p_m \text{ iff } \underline{A} = \underline{Q}. \tag{2.18}$$

Proof. First we see that

$$p_M = t_m + 2r + t_M, \quad p_m = 2(\hat{t}_1 + \hat{t}_3), \tag{2.19}$$

where $r = |P_2 H|$ and

$$t_m = |P_1 G| = \sqrt{(R-d)^2 - r^2}, \quad t_M = |P_3 H| = \sqrt{(R+d)^2 - r^2}, \tag{2.20}$$

$$\hat{t}_1 = |EQ_1| = \sqrt{R^2 - (r+d)^2}, \quad \hat{t}_3 = |FQ_3| = \sqrt{R^2 - (r-d)^2}. \tag{2.21}$$

According to Theorem 3.3 in [8], the tangent lengths t_2, t_3, t_4 can be expressed by t_1 as follows:

$$t_2 = \frac{(R^2 - d^2)t_1 + \sqrt{D}}{r^2 + t_1^2}, \quad t_3 = \frac{r^2}{t_1}, \quad t_4 = \frac{r^2}{t_2}, \tag{2.22}$$

where

$$D = (R^2 - d^2)^2 t_1^2 + r^2(r^2 + t_1^2)^2. \tag{2.23}$$

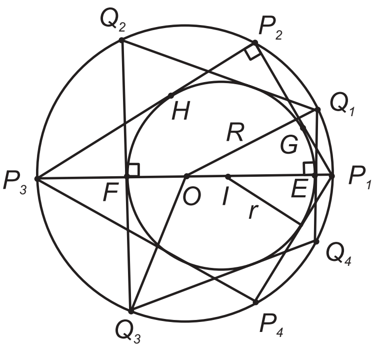

Figure 2.3

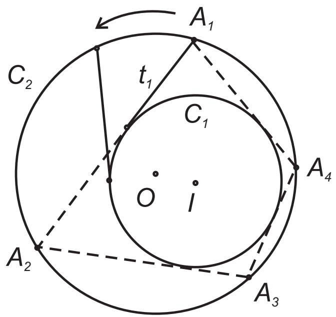

Figure 2.4

In this connection let us remark that for every point A_1 on C_2 there is a tangent t_1 drawn from C_2 to C_1 (Figure 2.4). If t_1 is given, then quadrilateral $A_1A_2A_3A_4$ is completely determined by t_1, and t_2, t_3, t_4 can be calculated using expressions (2.22).

Let the sum $\sum_{i=1}^{4} t_i$, where t_2, t_3, t_4 are expressed by t_1, be denoted by $s(t_1)$. It can be easily found that $\frac{\mathrm{d}}{\mathrm{d}t_1}s(t_1) = 0$ can be written as

$$(t_1^2 - r^2)[t_1^4 - 2(R^2 - d^2 - r^2)t_1^2 + r^4] = 0,$$

from which it follows that

$$(t_1^2)_1 = r^2, \quad (t_1^2)_2 = \hat{t}_1^2, \quad (t_1^2)_3 = \hat{t}_3^2,$$

where $\hat{t}_1$ and $\hat{t}_3$ are given by (2.21). In this connection let us remark that

$$\pm\sqrt{(R^2 - d^2 - r^2)^2 - r^4} = \pm 2dr$$

since, using Fuss' relation (1.1), we can write

$$\begin{aligned}(R^2-d^2-r^2)^2-r^4 &= (R^2-d^2)^2-2(R^2-d^2)r^2\\ &= 2r^2(R^2+d^2)-2(R^2-d^2)r^2=4d^2r^2.\end{aligned}$$

The part of the expression $\frac{\mathrm{d}^2}{\mathrm{d}t_1^2}s(t_1)$ important for discussion can be expressed as

$$t_1^4-2(R^2-d^2-r^2)t_1^2+r^4+2(t_1^2-r^2)[t_1^2-(R^2-d^2-r^2)].$$

For brevity, let the above expression be denoted by $S(t_1)$. It is easy to find that

$$S(r)=-R^2+2r^2+d^2<0, \tag{2.24}$$

$$S(\hat{t}_1)=2dr>0, \tag{2.25}$$

$$S(\hat{t}_3)=(R^2-2r^2-d^2-2rd)(-2dr)>0, \tag{2.26}$$

where the relations (2.14) are used.

In this connection let us remark that by Theorem 3.3 in [8] the following holds:

$$\text{if } t_1 = r \text{ and } t_2, t_3, t_4 \text{ are given by (2.22), then } \sum_{i=1}^{4} t_i = p_M,$$

$$\text{if } t_1 = \hat{t}_1 \text{ and } t_2, t_3, t_4 \text{ are given by (2.22), then } \sum_{i=1}^{4} t_i = p_m,$$

$$\text{if } t_1 = \hat{t}_3 \text{ and } t_2, t_3, t_4 \text{ are given by (2.22), then } \sum_{i=1}^{4} t_i = p_m.$$

Theorem 2.6 is thus proved. □

Corollary 2.7. *Let $\underline{A}$ be as in Theorem 2.6, that is, $\underline{A}$ is any given bicentric quadrilateral whose incircle is C_1 and circumcircle C_2. Then*

$$\textit{area of } \underline{Q} \leq \textit{ area of } \underline{A} \leq \textit{ area of } \underline{P}.$$

Proof. From (2.17) it follows that

$$rp_m \leq r(t_1 + t_2 + t_3 + t_4) \leq rp_M. \tag{2.27}$$

Using relations (2.22) and denoting the area of $\underline{A}$ by $J(t_1)$, the inequalities (2.27) can be written as

$$J(\hat{t}_1) \leq J(t_1) \leq J(t_m),$$

where

$$J(t_1) = r\left(t_1 + \frac{r^2}{t_1} + \frac{(R^2-d^2)t_1 + \sqrt{D}}{r^2 + t_1^2} + \frac{r^2(r^2+t_1^2)}{(R^2-d^2)t_1 + \sqrt{D}}\right).$$

□

Since, according to Theorem 3.3 in [8], we have

$$J(t_m) = J(r) = J(t_M),$$
$$J(\hat{t}_1) = J(\hat{t}_3),$$

the graph of $J(t_1)$ is like that shown in Figure 2.5.

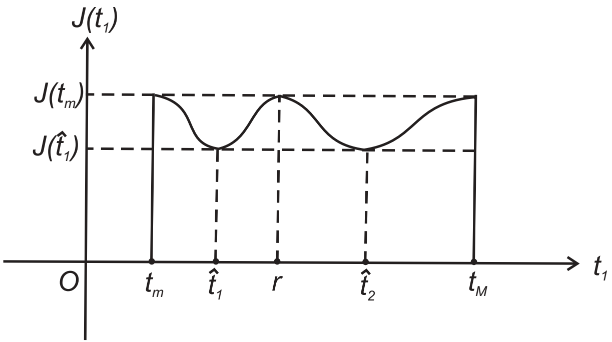

Figure 2.5

Of course, $J(t_m) = r(t_m + 2r + t_M)$, $J(\hat{t}_2) = 2r(\hat{t}_1 + \hat{t}_2)$. Let us remark that $\hat{t}_2 = \hat{t}_3$. (See Figure 2.3.)

Corollary 2.8. *The following holds*

$$\frac{p_m}{r^2} \leq \sum_{i=1}^{4} \frac{1}{t_i} \leq \frac{p_M}{r^2}.$$

Proof. Since $t_1t_3 = t_2t_4 = r^2$, we have

$$\sum_{i=1}^{4} \frac{1}{t_i} = \frac{t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2}{t_1t_2t_3t_4} = \frac{t_1 + t_2 + t_3 + t_4}{r^2}. \tag{2.28}$$

□

From the proof it is clear that

$$\sum_{i=1}^{4} \frac{1}{t_i} = \text{ maximum (minimum) iff } \sum_{i=1}^{4} t_i = \text{ maximum (minimum).}$$

Corollary 2.9. *The following holds*

$$p_m^2 - 4(R^2 - d^2) \leq \sum_{i=1}^{4} t_i^2 \leq p_M^2 - 4(R^2 - d^2).$$

Proof. Since $(t_1 + t_2 + t_3 + t_4)^2 = \sum_{i=1}^{4} t_i^2 + 4(R^2 - d^2)$, we have

$$p_m^2 \leq \sum_{i=1}^{4} t_i^2 + 4(R^2 - d^2) \leq p_M^2.$$

□

From the proof it is clear that

$$\sum_{i=1}^{4} t_i^2 = \text{ maximum (minimum) iff } \sum_{i=1}^{4} t_i = \text{ maximum (minimum).}$$

Corollary 2.10. *When the arithmetic mean $A(t_1, t_2, t_3, t_4)$ is maximum, then the harmonic mean $H(t_1, t_2, t_3, t_4)$ is minimum and vice versa.*

Proof. From (2.28) it follows that

$$A(t_1, t_2, t_3, t_4) \cdot H(t_1, t_2, t_3, t_4) = r^2.$$

$\square$

Corollary 2.11. *Let t_1 be given such that $t_m \leq t_1 \leq t_M$. Then the equation*

$$J(t_1)J(x) = J(t_m)J(\hat{t}_1)$$

has four positive roots x_1, x_2, x_3, x_4 and we have

$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = 2(R^2 - d^2), \; x_1x_2x_3x_4 = r^4.$$

Proof. There is a bicentric quadrilateral $X_1X_2X_3X_4$ whose incircle is C_1 and circumcircle C_2 such that

$$\begin{aligned} &\text{area of } A_1A_2A_3A_4 \cdot \text{ area of } X_1X_2X_3X_4 = J(t_m)J(\hat{t}_1), \\ &x_i + x_{i+1} = |X_iX_{i+1}|, \; i = 1, 2, 3, 4. \end{aligned}$$

$\square$

In connection with the sum $t_1^v + t_2^v + t_3^v + t_4^v$, where v is a real number, the following theorem will be proved.

Theorem 2.12. *If there is a bicentric quadrilateral whose tangent lengths are t_1, t_2, t_3, t_4, then there is a bicentric quadrilateral whose tangent lengths are $t_1^v, t_2^v, t_3^v, t_4^v$, where v may be any given real number.*

Proof. Let $\underline{A} = A_1A_2A_3A_4$ be a bicentric quadrilateral whose incircle is C_1 and circumcircle C_2 and let $|A_iA_{i+1}| = t_i + t_{i+1}$, $i = 1, 2, 3, 4$. Then

$$t_1^v t_3^v = t_2^v t_4^v = (r^v)^2. \tag{2.29}$$

According to what we said in connection with the relations (2.2) and (2.3) there is a bicentric quadrilateral $\underline{A}^{(v)} = A_1^{(v)}A_2^{(v)}A_3^{(v)}A_4^{(v)}$ such that

$$A_i^{(v)}A_{i+1}^{(v)} = t_i^v + t_{i+1}^v, \ \ i = 1, 2, 3, 4.$$

Let its incircle and circumcircle be denoted respectively by $C_1^{(v)}$ and $C_2^{(v)}$ and let

$$\begin{aligned} r_v &= \text{ radius of } C_1^{(v)}, \\ R_v &= \text{ radius of } C_2^{(v)}, \\ d_v &= \text{ distance between the centers of } C_1^{(v)} \text{ and } C_2^{(v)}. \end{aligned}$$

From (2.29) we see that

$$r_v = r^v. \tag{2.30}$$

In order to obtain R_v and d_v we shall use relations

$$t_1^v t_2^v + t_2^v t_3^v + t_3^v t_4^v + t_4^v t_1^v = 2(R_v^2 - d_v^2), \tag{2.31}$$

$$(R_v^2 - d_v^2)^2 = 2r_v^2(R_v^2 + d_v^2), \tag{2.32}$$

where the second is Fuss' relation. If, for brevity, the left-hand side of (2.31) is denoted by s, we can write

$$\frac{s}{2} = R_v^2 - d_v^2, \quad \frac{s^2}{4} = 2r_v^2(R_v^2 + d_v^2)$$

from which follows that

$$R_v = \frac{\sqrt{s^2 + 4sr_v^2}}{4r_v}, \quad d_v = \frac{\sqrt{s^2 - 4sr_v^2}}{4r_v}. \tag{2.33}$$

Theorem 2.12 is thus proved. □

Before we state some of its corollaries here are some examples.

Example 2.1. *If* $v = 0$, *then* $s = 4$, $r_v = 1$, $R_v = \sqrt{2}$, $d_v = 0$.

Example 2.2. *If* $v = -1$, *then* $s = \frac{2(R^2 - d^2)}{r^4}$, $r_v = \frac{1}{r}$, $R_v = \frac{R}{r^2}$, $d_v = \frac{d}{r^2}$.

Corollary 2.13. *The following holds*

$$2(t_{1,v} + t_{3,v}) \le \sum_{i=1}^{4} t_i^v \le t_{m,v} + t_{M,v} + 2r_v,$$

where

$$t_{m,v}^2 = (R_v - d_v)^2 - r_v^2, \quad t_{M,v}^2 = (R_v + d_v)^2 - r_v^2,$$
$$t_{1,v}^2 = R_v^2 - (d_v + r_v)^2, \quad t_{3,v}^2 = R_v^2 - (d_v - r_v)^2.$$

This corollary is analogous to Theorem 2.6. (See (2.17).)

Corollary 2.14. *The following holds*

$$\begin{aligned} 2r_v &\le t_1^v + t_3^v \le t_{m,v} + t_{M,v} \\ 2r_v &\le t_2^v + t_4^v \le t_{m,v} + t_{M,v} \\ 4r_v &\le t_1^v + t_2^v + t_3^v + t_4^v \le 4r_v \cdot \frac{R_v^2 + d_v^2}{R_v^2 - d_v^2}. \end{aligned}$$

The proof is analogous to the proof that (2.4) – (2.6) hold. We can imagine that in Figure 2.2 instead of $t_i, t_{i+2}, t_m, t_M, r$ there are $t_i^v, t_{i+2}^v, t_{m,v}, t_{M,v}, r_v$.

Corollary 2.15. *The following holds*

$$A(t_1^v, t_2^v, t_3^v, t_4^v) \cdot H(t_1^v, t_2^v, t_3^v, t_4^v) = r_v^2. \tag{2.34}$$

This corollary is analogous to Corollary 2.10.

Theorem 2.16. *Each of the following six sums is maximum (minimum) iff the sum $\sum_{i=1}^4 t_i$ is maximum (minimum).*

$$a)\ \sum_{i=1}^4 t_i^2, \quad b)\ \sum_{i=1}^4 t_i^{-2}, \quad c)\ \sum_{i=1}^4 t_i^3, \quad d)\ \sum_{i=1}^4 t_i^{-3}, \quad e)\ \sum_{i=1}^4 t_i^4, \quad f)\ \sum_{i=1}^4 t_i^{-4}.$$

In other words,

$$2(\hat{t}_1^v + \hat{t}_3^v) \le \sum_{i=1}^4 t_i^v \le t_m^v + 2r^v + t_M^v, \quad v = 2, -2, 3, -3, 4, -4, \tag{2.35}$$

where $t_m, t_M, \hat{t}_1, \hat{t}_3$ are given by (2.20) and (2.21).

Proof. a) It holds

$$(t_1+t_2+t_3+t_4)^2 = \sum_{i=1}^{4} t_i^2 + 4(R^2-d^2).$$

b) Since $t_1t_3 = t_2t_4 = r^2$, we can write

$$\sum_{i=1}^{4} t_i^{-2} = \frac{r^4(t_1^2+t_2^2+t_3^2+t_4^2)}{r^8} = \frac{t_1^2+t_2^2+t_3^2+t_4^2}{r^4}.$$

c) From

$$(t_1+t_2+t_3+t_4)^3 = (t_1+t_2+t_3+t_4)^2(t_1+t_2+t_3+t_4)$$

or

$$\left(\sum_{i=1}^{4} t_i\right)^3 = [t_1^2+t_2^2+t_3^2+t_4^2+4(R^2-d^2+r^2)](t_1+t_2+t_3+t_4)$$

follows

$$(t_1+t_2+t_3+t_4)[(t_1+t_2+t_3+t_4)^2 - 6(R^2-d^2) - 3r^2] = \sum_{i=1}^{4} t_i^3.$$

d) It holds

$$\sum_{i=1}^{4} t_i^{-3} = \frac{t_1^3+t_2^3+t_3^3+t_4^3}{r^6}.$$

e) From

$$\left(\sum_{i=1}^{4} t_i^2\right)^2 = \sum_{i=1}^{4} t_i^4 + 2\left(\sum_{i=1}^{4} t_i^2 t_{i+1}^2 + 2r^4\right),$$

since

$$\left(\sum_{i=1}^{4} t_i t_{i+1}\right)^2 = \sum_{i=1}^{4} t_i^2 t_{i+1}^2 + 2r^2\left(\sum_{i=1}^{4} t_i^2\right) + 4r^4, \tag{2.36}$$

we get

$$\sum_{i=1}^{4} t_i^4 = \left(\sum_{i=1}^{4} t_i^2\right)^2 + 4r^2\left(\sum_{i=1}^{4} t_i^2\right) - 8(R^2 - d^2)^2 + 4r^4.$$

f) It holds

$$\sum_{i=1}^{4} t_i^{-4} = \frac{t_1^4 + t_2^4 + t_3^4 + t_4^4}{r^8}.$$

Theorem 2.16 is proved. □

In connection with b), d), f) in this theorem let us remark that

$$\sum_{i=1}^{4} \frac{1}{t_i^k} = \sum_{i=1}^{4} \left(\frac{t_i}{r^2}\right)^k.$$

It is easy to see that this is equivalent to

$$A(t_1^k, t_2^k, t_3^k, t_4^k) \cdot H(t_1^k, t_2^k, t_3^k, t_4^k) = r^{2k}.$$

Corollary 2.17. *Let $f_i(t_1)$, $i = 1, 2, 3, 4$, be the functions given by*

$$f_1(t_1) = t_1, \quad f_2(t_1) = t_2, \quad f_3(t_1) = \frac{r^2}{t_1}, \quad f_4(t_1) = \frac{r^2}{t_2}$$

where t_2 is expressed in (2.22). Then each of the following two equations

$$\frac{d}{dt_1}\sum_{i=1}^{4} f_i(t_1) = 0, \quad \frac{d}{dt_1}\sum_{i=1}^{4} f_i^k(t_1) = 0, \quad k = 2, 3, 4 \tag{2.37}$$

has in the interval $[t_m, t_M]$ the same solutions $t_m, \hat{t}_1, r, \hat{t}_3, t_M$ given by (2.20) and (2.21).

Thus, the graph of the function $F(t_1) = \sum_{i=1}^{4} f_i^k(t_1)$ is like the graph of the function $J(t_1)$ shown in Figure 2.5.

If $f(t_1)$ and $g(t_1)$ are polynomials which respectively correspond to the equations given by (2.37), then $f(t_1)|g(t_1)$.

Remark 2. *We conjecture that Corollary 2.17 is valid for every real number $k \neq 0$.*

Corollary 2.18. $\sum_{i=1}^{4} t_i^2 t_{i+1}^2$ *is minimum when* $\sum_{i=1}^{4} t_i$ *is maximum and vice versa. In other words, the following holds*

$$4r^2(R^2 - r^2 + d^2) \leq \sum_{i=1}^{4} t_i^2 t_{i+1}^2 \leq 4(R^2 - r^2 - d^2)^2,$$

where

$$t_m^2 r^2 + r^2 t_M^2 + t_M^2 r^2 + r^2 t_m^2 = 4r^2(R^2 - r^2 + d^2),$$
$$\hat{t}_1^2 \hat{t}_2^2 + \hat{t}_2^2 \hat{t}_3^2 + \hat{t}_3^2 \hat{t}_4^2 + \hat{t}_4^2 \hat{t}_1^2 = 4(R^2 - r^2 - d^2)^2.$$

Proof. From (2.36), since $\sum_{i=1}^{4} t_i t_{i+1} = 2(R^2 - d^2)$, it follows that

$$4(R^2 - d^2)^2 - 4r^4 = \sum_{i=1}^{4} t_i^2 t_{i+1}^2 + 2r^2 \left(\sum_{i=1}^{4} t_i^2 \right).$$

□

In this connection let us remark that from

$$4r^2(R^2 - r^2 + d^2) \leq 4(R^2 - r^2 - d^2)^2,$$

using Fuss' relation (1.1), we get the following inequality

$$R^2 \leq 2r^2 + 3d^2. \tag{2.38}$$

(Cf. with (2.14). The equality holds only if $d = 0$.)

Remark 3. *W. J. Blundon and R. H. Eddy in [2] have proved that for semiperimeter s of bicentric polygons, the following inequalities hold*

$$s \leq r + \sqrt{r^2 + 4R^2}, \quad s^2 \geq 8r \left(\sqrt{r^2 + 4R^2} - r \right),$$

and two other inequalities in s. (Both inequalities are based upon (2.16) stated in Remark 1.)

Inequalities (2.38), using (2.15) stated in Remark 1, can also be proved.

3. Certain Inequalities Concerning Bicentric Hexagons

Let now, in this section, C_1 and C_2 be given circles such that there is a bicentric hexagon whose incircle is C_1 and circumcircle C_2 and let

$$\begin{aligned} r &= \text{ radius of } C_1, \quad R = \text{ radius of } C_2, \\ d &= \text{ distance between centers of } C_1 \text{ and } C_2, \end{aligned}$$

$$t_m = \sqrt{(R-d)^2 - r^2}, \quad t_M = \sqrt{(R+d)^2 - r^2}. \tag{3.1}$$

We shall use the following results given in [9, Theorem 1-2].

Let $\underline{A} = A_1 \cdots A_6$ be any given bicentric hexagon whose incircle is C_1 and circumcircle C_2 and let

$$t_i + t_{i+1} = |A_i A_{i+1}|, \ i = 1, \dots, 6. \tag{3.2}$$

Then

$$t_1t_3 + t_3t_5 + t_5t_1 = r^2, \tag{3.3}$$

$$t_2t_4 + t_4t_6 + t_6t_2 = r^2 \tag{3.4}$$

and

$$t_1t_4 = t_2t_5 = t_3t_6 = h, \tag{3.5}$$

where

$$h = t_m t_M. \tag{3.6}$$

If t_1 is given, then $t_2, \dots, t_6$ are given by

$$t_3 = \frac{a}{2} + \left(\frac{a}{2}\right)^2 - b, \qquad t_5 = \frac{b}{t_3}, \tag{3.7}$$

$$t_2 = \frac{h}{t_5}, \quad t_4 = \frac{h}{t_1}, \quad t_6 = \frac{h}{t_3} \tag{3.8}$$

where

$$a = \frac{(r^4 - h^2)t_1}{r^2 t_1^2 + h^2}, \quad b = \frac{h^2(r^2 + t_1^2)}{r^2 t_1^2 + h^2}. \tag{3.9}$$

Thus, for every t_1 such that $t_m \le t_1 \le t_M$ there is a bicentric hexagon whose tangent lengths are $t_1, t_2, \dots, t_6$, where $t_2, \dots, t_6$ are given by (3.7) and (3.8).

Theorem 3.1. *The following results hold*

$$2\sqrt{h} \le t_i + t_{i+3} \le t_m + t_M, \ i = 1, 2, 3 \tag{3.10}$$

$$6\sqrt{h} \le \sum_{i=1}^{6} t_i \le 3(t_m + t_M) \tag{3.11}$$

and

$$6h \leq \sum_{i=1}^{6} t_i^2 \leq 6(R^2 + d^2 - r^2 + h). \tag{3.12}$$

Proof. Analogous to the proof of Theorem 2.1. (Here vertices A_i and A_{i+3} are opposite and instead of r^2 we have h.) □

Theorem 3.2. *The following result holds*

$$r^2 \geq 3h, \tag{3.13}$$

where $r^2 = 3h$ *only if* $d = 0$.

Proof. Since from (3.5) we have

$$t_2 = \frac{h}{t_5}, \quad t_4 = \frac{h}{t_1}, \quad t_6 = \frac{h}{t_3}, \tag{3.14}$$

the relation (3.4) can be written as

$$t_1 + t_3 + t_5 = \left(\frac{r}{h}\right)^2 t_1 t_3 t_5. \tag{3.15}$$

Using this relation and relation (2.4) we can write

$$t_3 + t_5 = \left(\frac{r}{h}\right)^2 t_1 t_3 t_5 - t_1,$$
$$t_1(t_3 + t_5) + t_3 t_5 = r^2,$$

from which follows that

$$t_3 + t_5 = a, \quad t_3 t_5 = b, \tag{3.16}$$

where a and b are given by (3.9). Thus, we have the equation

$$t_3 + \frac{b}{t_3} = a \quad \text{or} \quad t_3^2 - at_3 + b = 0.$$

Let the discriminant of the above square equation in t_3 be denoted by D. Then we can write

$$D = -4h^2r^2t_1^4 + [(r^4 - h^2)^2 - 4h^4 - 4h^2r^4]t_1^2 - 4h^4r^2 \geq 0. \tag{3.17}$$

Now, the discriminant of the corresponding quadratic equation in t_1^2 is given by

$$D_1 = [(r^4 - h^2)^2 - 4h^4 - 4h^2r^4]^2 - 64h^6r^4.$$

Since $D_1 \geq 0$ must hold, we have the following inequality

$$(r^4 - h^2)^2 - 4h^4 - 4h^2r^4 - 8h^3r^2 \geq 0,$$

which can be written as

$$(r^2 - 3h)(r^2 + h) \geq 0.$$

Thus, $r^2 - 3h \geq 0$.

If $d = 0$, then $r = R\cos 30^\circ = \frac{R\sqrt{3}}{2}$, $t_m = t_M = R\sin 30^\circ = \frac{R}{2}$ and $r^2 = t_m t_M$.

Theorem 3.2 is proved. $\square$

In proving this theorem we also have proved that for every t_1 such that $t_m \leq t_1 \leq t_M$ the inequality (3.17) holds.

It may be of some interest to note that Theorem 3.2 can be readily proved using a connection between arithmetic and harmonic means. Namely, starting from

$$A(t_1, t_3, t_5) \geq H(t_1, t_3, t_5),$$

we can write

$$(t_1 + t_3 + t_5) \cdot \frac{t_1 t_3 + t_3 t_5 + t_5 t_1}{t_1 t_3 t_5} \geq 9$$

or, since $t_1 t_3 + t_3 t_5 + t_5 t_1 = r^2$,

$$r^2 \geq 9 \cdot \frac{t_1 t_3 t_5}{t_1 + t_3 + t_5}. \tag{3.18}$$

Also we have that the relation (3.15) can be written as

$$r^2 = h^2 \cdot \frac{t_1 + t_3 + t_5}{t_1 t_3 t_5}. \tag{3.19}$$

Now, using relations (3.18) and (3.19) we can write

$$r^4 \geq 9r^2 \frac{t_1 t_3 t_5}{t_1 + t_3 + t_5} = 9 \left(h^2 \cdot \frac{t_1 + t_3 + t_5}{t_1 t_3 t_5} \right) \frac{t_1 t_3 t_5}{t_1 + t_3 + t_5} = 9h^2$$

or

$$r^2 \geq 3h.$$

Corollary 3.3. *The following result holds*

$$t_1 t_3 + t_3 t_5 + t_5 t_1 \geq 3h, \quad t_2 t_4 + t_4 t_6 + t_6 t_2 \geq 3h.$$

Proof. Follows from (3.3), (3.4) and (3.13). □

Corollary 3.4. *It holds*

$$h \geq 3 \cdot \frac{t_1 t_3 t_5}{t_1 + t_3 + t_5}. \tag{3.20}$$

Proof. Since $\left(\frac{r}{h}\right)^2 \geq \frac{3}{h}$, the relation follows from (3.19). □

Corollary 3.5. *The following holds*

$$h \geq 3 \cdot \frac{t_2 t_4 t_6}{t_2 + t_4 + t_6}. \tag{3.21}$$

Proof. From (3.3), using (3.5), we get

$$r^2 = h^2 \cdot \frac{t_2 + t_4 + t_6}{t_2 t_4 t_6}.$$

□

Corollary 3.6. *The following result holds*

$$\sum_{i=1}^{6} t_i t_{i+1} \geq 6h. \tag{3.22}$$

Proof. Starting from $r^2 \geq 3h$, we can write

$$r^4 \geq 9h^2,$$
$$r^4 - 3h^2 \geq 6h^2,$$
$$\frac{r^4 - 3h^2}{h} \geq 6h.$$

In [9, Theorem 3] it is proved that $\sum_{i=1}^{6} t_i t_{1+1} = \frac{r^4 - 3h^2}{h}$. □

The following theorem is analogous to Theorem 2.6 but with much more involved calculation. Before its statement we have the following preliminary work.

In Figure 3.1a we have drawn an axially symmetric bicentric hexagon $P_1 \cdots P_6$. The marked tangent lengths $\bar{t}_2$ and $\bar{t}_3$ are given by

$$\bar{t}_2 = -t_M + R + d \tag{3.23}$$

and

$$\bar{t}_3 = -t_m + R - d. \tag{3.24}$$

The proof is as follows.

Since $\bar{t}_2 = t_6$ and $t_4 = t_M$, the relation (3.4) can be written as

$$(\bar{t}_2)^2 + 2t_M\bar{t}_2 - r^2 = 0,$$

from which follows (3.23).

Since $\bar{t}_3 = t_5$, $t_1 = t_m$, the relation (3.3) can be written as

$$(\bar{t}_3)^2 + 2t_m\bar{t}_3 - r^2 = 0,$$

from which follows (3.24).

In Figure 3.1b we have drawn an axially symmetric bicentric hexagon $Q_1 \cdots Q_6$. The marked tangent lengths $\hat{t}_1, \hat{t}_2, \hat{t}_3$ are given by

$$\hat{t}_1 = \sqrt{R^2 - (r+d)^2}, \tag{3.25}$$

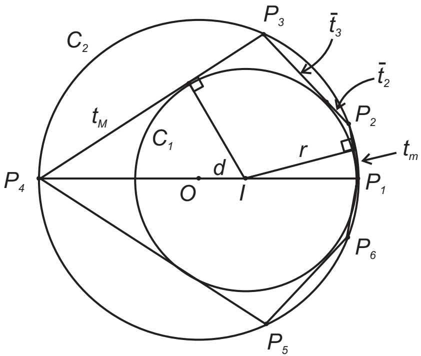

Figure 3.1a

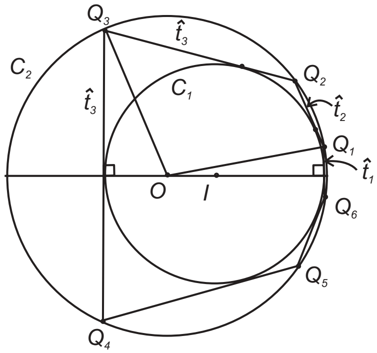

Figure 3.1b

$$\hat{t}_3 = \sqrt{R^2 - (r-d)^2}, \tag{3.26}$$

$$\hat{t}_2 = \frac{r^2 - \hat{t}_1\hat{t}_3}{\hat{t}_1 + \hat{t}_3}. \tag{3.27}$$

For $\hat{t}_1$ and $\hat{t}_3$ it is obvious from Figure 3.1b. Also, from Figure 3.1b we see that $\hat{t}_2 = t_5$, so relation (3.3) can be written as

$$t_1t_3 + t_3t_5 + t_5t_1 = r^2,$$

from which follows

$$t_5 = \frac{r^2 - \hat{t}_1\hat{t}_3}{\hat{t}_1 + \hat{t}_3}.$$

Now, let $A_1 \cdots A_6$ be a bicentric hexagon whose incircle is C_1 and circumcircle C_2 and let its area be denoted by $J(t_1)$, that is

$$J(t_1) = r(t_1 + t_2 + \cdots + t_6), \tag{3.28}$$

where $t_i + t_{i+1} = |A_i A_{i+1}|$, $i = 1, \ldots, 6$, and $t_2, \ldots, t_6$ are given by (3.7) and (3.8).

According to Theorem 2 in [9], we have

$$J(t_m) = J(\bar{t}_2) = J(\bar{t}_3) = J(t_M) \tag{3.29}$$

and

$$J(\hat{t}_1) = J(\hat{t}_2) = J(\hat{t}_3). \tag{3.30}$$

Theorem 3.7. *Let the perimeter of the hexagon $P_1 \cdots P_6$ shown in Figure 3.1a be denoted by $2p_M$, and let the perimeter of the hexagon $Q_1 \ldots Q_6$ shown in Figure 3.1b be denoted by $2p_m$. Then*

$$rp_m \le J(t_1) \le rp_M, \tag{3.31}$$

that is

$$\begin{aligned} J(t_1) &= \text{ maximum if } t_1 \in \left\{t_m, \bar{t}_2, \bar{t}_3, t_M\right\}, \\ J(t_1) &= \text{ minimum if } t_1 \in \left\{\hat{t}_1, \hat{t}_2, \hat{t}_3\right\}. \end{aligned}$$

Proof. The relation (3.28), using relations given by (3.7) and (3.8), can be written as

$$J(t_1) = \frac{r^2\left(h^4 + 2h^2r^2t_1^2 + r^4t_1^4 + ht_1^2(r^2+t_1^2)^2\right)}{ht_1(r^2+t_1^2)(h^2+r^2t_1^2)}.$$

From $\frac{\mathrm{d}}{\mathrm{d}t_1}J(t_1)=0$, we obtain the equation

$$\frac{r^2(h-t_1^2)[h(h^2+ht_1^2+t_1^4)+2hr^2t_1^2-r^4t_1^2][3h^2t_1^2+(h^2+t_1^4)r^2-r^4t_1^2]}{ht_1^2(r^2+t_1^2)^2(h^2+r^2t_1^2)^2}=0,$$

from which follow

$$(t_1^2)_1 = h, \tag{3.32}$$

$$(t_1^2)_{2,3}=\frac{(r^2-h)^2\pm\sqrt{(r^2-h)^4-4h^4}}{2h}, \tag{3.33}$$

and

$$(t_1^2)_{4,5}=\frac{r^4-3h^2\pm\sqrt{(r^4-3h^2)^2-4r^4h^2}}{2r^2}. \tag{3.34}$$

Since R,r,d satisfy Fuss' relation (1.2), it can be shown, using this relation, that

$$\begin{aligned}(t_1)_1 &= \hat{t}_2, \quad (t_1)_2=\hat{t}_3, \quad (t_1)_3=\hat{t}_1\\ (t_1)_4 &= \bar{t}_3, \quad (t_1)_5=\bar{t}_2.\end{aligned}$$

So, for example, it can be found that $t_3=(t_1)_4$ is equivalent to

$$64r^4(R-d+r)(-R+d+r)[3(R^2-d^2)^4-4r^2(R^2+d^2)(R^2-d^2)^2-16R^2r^4d^2]=0,$$

where only the last factor is equal to zero since Fuss' relation (1.2) holds, which can be written as

$$3(R^2-d^2)^4-4r^2(R^2+d^2)(R^2-d^2)^2-16R^2r^4d^2=0.$$

In the same way as in Theorem 2.6, only with somewhat more involved calculation, it can be shown that the graph of $J(t_1)$ is like that shown in Figure 3.2.

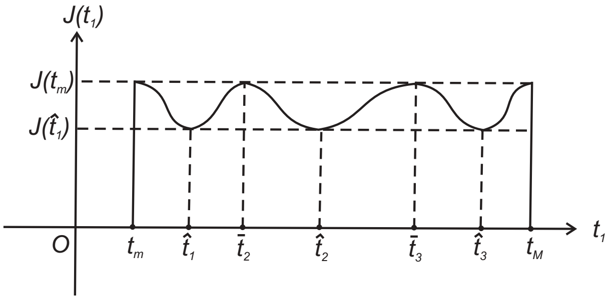

Figure 3.2

Theorem 3.7 is proved. □

Corollary 3.8. *Each of the following three sums*

$$a)\ \sum_{i=1}^{6} t_i^2,\quad b)\ \sum_{i=1}^{6}\frac{1}{t_i},\quad c)\ \sum_{i=1}^{6}\frac{1}{t_i^2}$$

has maximum if $t_1=t_m$ and minimum if $t_1=\hat{t}_1$.

Proof. a) We have

$$(t_1 + \ldots + t_6)^2 = t_1^2 + \ldots + t_6^2 + 2(t_1t_2 + t_2t_3 + \ldots + t_5t_6 + t_6t_1)$$
$$+ 2(t_1t_3 + t_3t_5 + t_5t_1) + 2(t_2t_4 + t_4t_6 + t_6t_2)$$
$$= \sum_{i=1}^{6} t_i^2 + 2 \cdot \frac{r^4 - 3h^2}{h} + 4r^2. \text{ (See [9, Theorem 3].)}$$

b) We have

$$\sum_{i=1}^{6} \frac{1}{t_i} = \frac{h^2(t_1 + \ldots + t_6)}{h^3} = \frac{t_1 + \ldots + t_6}{h}.$$

□

Corollary 3.9. *Let t_1 be given such that $t_m \leq t_1 \leq t_M$. Then the equation*

$$J(t_1)J(x) = J(t_m)J(\hat{t}_1)$$

has six positive roots $x_1, \ldots, x_6$ and we obtain

$$\sum_{i=1}^{6} x_i x_{i+1} = \frac{r^4 - 3h^2}{h},$$

$$\sum_{i=1}^{6} x_i x_{i+1} x_{i+2} x_{i+3} = r^4 - 3h^2,$$

$$x_1x_2x_3x_4x_5x_6 = h^3.$$

The proof is analogous to the proof of Corollary 2.11.

4. Certain Inequalities Concerning Bicentric Octagons

In this section, let C_1 and C_2 be given circles such that there is a bicentric octagon whose incircle is C_1 and circumcircle C_2 and let

$$\begin{aligned} r &= \text{ radius of } C_1,\ R = \text{ radius of } C_2, \\ d &= \text{ distance between centers of } C_1 \text{ and } C_2, \end{aligned}$$

$$t_m = \sqrt{(R-d)^2 - r^2}, \quad t_M = \sqrt{(R+d)^2 - r^2}. \tag{4.1}$$

We shall use some results given in [9, Theorem 4-5].

Let $\underline{A} = A_1 \cdots A_8$ be any given bicentric octagon whose incircle is C_1 and circumcircle C_2 and let

$$t_i + t_{i+1} = |A_i A_{i+1}|,\ i = 1, \ldots, 8. \tag{4.2}$$

Then

$$r^4 - r^2(t_1t_3 + t_3t_5 + t_5t_7 + t_7t_1 + t_1t_5 + t_3t_7) + t_1t_3t_5t_7 = 0, \tag{4.3}$$

$$r^4 - r^2(t_2t_4 + t_4t_6 + t_6t_8 + t_8t_2 + t_2t_6 + t_4t_8) + t_2t_4t_6t_8 = 0, \tag{4.4}$$

$$t_1t_5 = t_2t_6 = t_3t_7 = t_4t_8 = h, \tag{4.5}$$

$$h = t_m t_M. \tag{4.6}$$

A bicentric octagon may be convex or non-convex, but the relations (4.3) – (4.6) have the same form.

The theorem below will now be proved.

Theorem 4.1. *Let $A_1 \cdots A_8$ be a bicentric octagon. Then*

$$\left(r - \frac{h}{r}\right)^2 \geq 4h, \tag{4.7}$$

where equality holds only if $d = 0$.

Proof. The relation (4.3), using relations $t_1 t_5 = t_3 t_7 = h$, can be written as

$$(h + t_1^2)t_3^2 - \left(r - \frac{h}{r}\right)^2 t_1 t_3 + h(h + t_1^2) = 0. \tag{4.8}$$

The discriminant of the above quadratic equation in t_3 is

$$D = \left(r - \frac{h}{r}\right)^4 t_1^2 - 4h(h + t_1^2)^2.$$

Since $D \geq 0$, we obtain

$$-4ht_1^4 + \left[\left(r - \frac{h}{r}\right)^4 - 8h^2\right] t_1^2 - 4h^3 \geq 0.$$

We shall use the discriminant

$$D_1 = \left[\left(r - \frac{h}{r} \right)^4 - 8h^2 \right]^2 - 64h^4$$

of the corresponding quadratic equation in t_1^2 given by

$$-4ht_1^4 + \left[\left(r - \frac{h}{r} \right)^4 - 8h^2 \right] t_1^2 - 4h^3 = 0.$$

From $D_1 \geq 0$ it follows that

$$\left(r - \frac{h}{r} \right)^4 - 8h^2 \geq 8h^2,$$

i.e.

$$\left(r - \frac{h}{r} \right)^2 \geq 4h.$$

It remains to prove that $(r - \frac{h}{r})^2 = 4h$ only if $d = 0$. Since

$$r = R\cos 22.5^\circ, \;\; h = r^2 \sin^2 22.5^\circ \;\; \text{and}$$
$$\cos^2 22.5^\circ = \frac{2+\sqrt{2}}{4}, \;\; \sin^2 22.5^\circ = \frac{2-\sqrt{2}}{4}$$

we have

$$\left(r - \frac{h}{r} \right)^2 = r^2(2-\sqrt{2}), \quad 4h = r^2(2-\sqrt{2}).$$

Theorem 4.1 is proved. □

Corollary 4.2. *The following inequalities hold*

$$t_1t_3 + t_3t_5 + t_5t_7 + t_7t_1 \geq 4t_mt_M \tag{4.9}$$

and

$$t_2t_4 + t_4t_6 + t_6t_8 + t_8t_2 \geq 4t_mt_M. \tag{4.10}$$

Proof. The relations (4.3) and (4.4), using relations (4.5), can be written as

$$t_1t_3 + t_3t_5 + t_5t_7 + t_7t_1 = \left(r - \frac{h}{r}\right)^2,$$

$$t_2t_4 + t_4t_6 + t_6t_8 + t_8t_2 = \left(r - \frac{h}{r}\right)^2.$$

□

Remark 4. *It can be shown that for almost every property considered for bicentric quadrilaterals there are analogous properties for bicentric hexagons and octagons. But, since the number of the pages in the paper is limited, we omit some analogous theorems for bicentric hexagons and octagons. So that all we have stated about bicentric hexagons and octagons can be considered as some steps or an insight into possibilities for further research.*

References

[1] H.J.M. BOS, C. KERS, F. OORT AND D.W. RAVEN, Poncelet's closure theorem, *Exposition. Math.*, **5** (1987), 289, 364.

[2] W.J. BLUNDON AND R.H. EDDY, Problem 488, *Nieuw Arch. Wiskunde*, **26** (1978), 2321; solution in **26** (1978), 465.

[3] H. DÖRRIE, *100 Great Problems of Elementary Mathematics*, Dover, 1965.

[4] N. FUSS, De quadrilateris quibus circulum tam inscribere quam circumscribere licet, *Nova Acta Acad. Sci. Petrop.*, **10** (1797), 103–125.

[5] N. FUSS, De polygonis symmetrice irregularibus circulo simul inscriptis et circumscriptis, *Nova Acta Acad. Sci. Imp. Petrop.*, **13** (1802), 166–189.

[6] P. GRIFFITHS AND J. HARRIS, On Cayley's explicit solution to Poncelet's porism, *Enseign. Math.*, **24**(1-2) (1978), 31–40.

[7] J.V. PONCELET, Traité des propriétés projectives des figures, t.I, Paris 1865, (first ed. in 1822).

[8] M. RADIĆ, Some relations concerning triangles and bicentric quadrilaterals in connection with Poncelet's closure theorem, *Math. Maced.*, **1** (2003), 35–58.

[9] M. RADIĆ, Some properties and relations concerning bicentric hexagons and octagons in connection with Poncelet's closure theorem, to appear.

[10] M. RADIĆ, Extreme Areas of Triangles in Poncelet's Closure Theorem, *Forum Geometricorum*, **4** (2004), 23–26.

[11] L. FEJES TOTH, Inequalities concerning polygons and polyhedra, *Duke Math. J.*, **15**(1) (1948), 817–822.