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WEIGHTED GEOMETRIC MEAN INEQUALITIES OVER CONES IN \mathbb{R}^N

BABITA GUPTA, PANKAJ JAIN, LARS-ERIK PERSSON AND ANNA WEDESTIG

Department of Mathematics,
Shivaji College (University of Delhi),
Raja Garden,
Delhi-110 027 India.

EMail: babita74@hotmail.com

Department of Mathematics,
Deshbandhu College (University of Delhi),
New Delhi-110019, India.

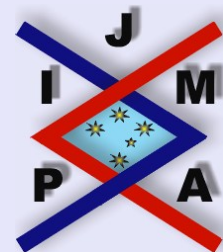
EMail: pankajkrjain@hotmail.com

Department of Mathematics,
Luleå University of Technology,
SE-971 87 Luleå, Sweden.

EMail: larserik@sm.luth.se

EMail: annaw@sm.luth.se

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Abstract

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Abstract

Let $0 < p \leq q < \infty$. Let A be a measurable subset of the unit sphere in \mathbb{R}^N , let $E = \{\mathbf{x} \in \mathbb{R}^N : \mathbf{x} = s\sigma, 0 \leq s < \infty, \sigma \in A\}$ be a cone in \mathbb{R}^N and let $S_{\mathbf{x}}$ be the part of E with 'radius' $\leq |\mathbf{x}|$. A characterization of the weights u and v on E is given such that the inequality

$$\left(\int_E \left(\exp \left(\frac{1}{|S_{\mathbf{x}}|} \int_{S_{\mathbf{x}}} \ln f(\mathbf{y}) d\mathbf{y} \right) \right)^q v(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_E f^p(\mathbf{x}) u(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}}$$

holds for all $f \geq 0$ and some positive and finite constant C . The inequality is obtained as a limiting case of a corresponding new Hardy type inequality. Also the corresponding companion inequalities are proved and the sharpness of the constant C is discussed.

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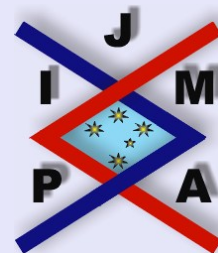
Key words: Inequalities, Multidimensional inequalities, Geometric mean inequalities, Hardy type inequalities, Cones in \mathbb{R}^N , Sharp constant.

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1. Introduction

In their paper [2] J.A. Cochran and C.S. Lee proved the inequality

$$(1.1) \quad \int_0^\infty \left[\exp \left(\varepsilon x^{-\varepsilon} \int_0^x y^{\varepsilon-1} \ln f(y) dy \right) \right] x^a dx \leq e^{\frac{a+1}{\varepsilon}} \int_0^\infty x^a f(x) dx,$$

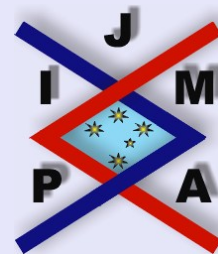
where a, ε are real numbers with $\varepsilon > 0$, f is a positive function defined on $(0, \infty)$ and the constant $e^{\frac{a+1}{\varepsilon}}$ is the best possible. This inequality, in fact, is a generalization of what sometimes is referred to as Knopp's inequality¹, which is obtained by taking $\varepsilon = 1$ and $a = 0$ in (1.1). Inequalities of the type (1.1) and its analogues have further been investigated and generalized by many authors e.g. see [1], [5] – [11], [14] and [16] – [21].

In particular, very recently A. Čižmešija, J. Pečarić and I. Perić [1, Th. 9, formula (23)] proved an N - dimensional analogue of (1.1) by replacing the interval $(0, \infty)$ by \mathbb{R}^N and the means are considered over the balls in \mathbb{R}^N centered at the origin. Their inequality reads:

$$(1.2) \quad \int_{\mathbb{R}^N} \left[\exp \left(\varepsilon |B_{\mathbf{x}}|^{-\varepsilon} \int_{B_{\mathbf{x}}} |B_{\mathbf{y}}|^{\varepsilon-1} \ln f(\mathbf{y}) d\mathbf{y} \right) \right] |B_{\mathbf{x}}|^a d\mathbf{x} \leq e^{\frac{a+1}{\varepsilon}} \int_{\mathbb{R}^N} f(\mathbf{x}) |B_{\mathbf{x}}|^a d\mathbf{x},$$

where $a \in \mathbb{R}$, $\varepsilon > 0$, f is a positive function on \mathbb{R}^N , $B_{\mathbf{x}}$ is a ball in \mathbb{R}^N with radius $|\mathbf{x}|$, $\mathbf{x} \in \mathbb{R}^N$, centered at the origin and $|B_{\mathbf{x}}|$ is its volume.

¹See e.g. [15, p. 143–144] and [12]. Note however that according to G.H. Hardy [4, p 156] this inequality was pointed out to him already in 1925 by G. Polya.



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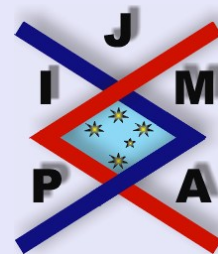
In this paper we prove a more general result, namely we characterize the weights u and v on \mathbb{R}^N such that for $0 < p \leq q < \infty$

$$\left(\int_{\mathbb{R}^N} \left[\exp \left(\frac{1}{|B_{\mathbf{x}}|} \int_{B_{\mathbf{x}}} \ln f(\mathbf{y}) d\mathbf{y} \right) \right]^q v(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^N} f^p(\mathbf{x}) u(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}}$$

holds for some finite positive constant C (See Corollary 3.2). In the case when $v(\mathbf{x}) = |S_{\mathbf{x}}|^a$ and $u(\mathbf{x}) = |S_{\mathbf{x}}|^b$ we obtain a genuine generalization of (1.2) (see Proposition 3.3 and Remark 3.4).

In this paper we also generalize the results in another direction, namely when the geometric averages over spheres in \mathbb{R}^N are replaced by such averages over spherical cones in \mathbb{R}^N (see notation below). This means in particular that our inequalities above and later on also hold when \mathbb{R}^N is replaced by \mathbb{R}_+^N or even more general cones in \mathbb{R}^N .

The paper is organized in the following way. In Section 2 we collect some preliminaries and prove a new Hardy inequality that averages functions over the cones in \mathbb{R}^N (see Theorem 2.1). In Section 3 we present and prove our main results concerning (the limiting) geometric mean operators (see Theorem 3.1 and Proposition 3.3). Finally, in Section 4 we present the corresponding companion inequalities (see Theorem 4.1, Corollary 4.2 and Proposition 4.3).



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2. Preliminaries

Let Σ^{N-1} be the unit sphere in \mathbb{R}^N , that is, $\Sigma^{N-1} = \{\mathbf{x} \in \mathbb{R}^N : |\mathbf{x}| = 1\}$, where $|\mathbf{x}|$ denotes the Euclidean norm of the vector $\mathbf{x} \in \mathbb{R}^N$. Let A be a measurable subset of Σ^{N-1} , and let $E \subseteq \mathbb{R}^N$ be a spherical cone, i.e.,

$$E = \{\mathbf{x} \in \mathbb{R}^N : \mathbf{x} = s\sigma, 0 \leq s < \infty, \sigma \in A\}.$$

Let $S_{\mathbf{x}}, \mathbf{x} \in \mathbb{R}^N$ denote the part of E with ‘radius’ $\leq |\mathbf{x}|$, i.e.,

$$S_{\mathbf{x}} = \{\mathbf{y} \in \mathbb{R}^N : \mathbf{y} = s\sigma, 0 \leq s \leq |\mathbf{x}|, \sigma \in A\}.$$

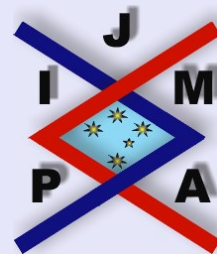
For $0 < p < \infty$ and a non-negative measurable function w on E , by $L_w^p := L_w^p(E)$ we denote the weighted Lebesgue space with the weight function w , consisting of all measurable functions f on E such that

$$\|f\|_{L_w^p} = \left(\int_E |f(\mathbf{x})|^p w(\mathbf{x}) dx \right)^{\frac{1}{p}} < \infty,$$

and make use of the abbreviations L^p and $\|f\|_{L^p}$ when $w(\mathbf{x}) \equiv 1$.

Let $S = S_{\mathbf{x}}, |\mathbf{x}| = 1$. The family of regions we shall average over is the collection of dilations of S . For $\mathbf{x} \in E \setminus \{0\}$ denote by $|S_{\mathbf{x}}|$ the Lebesgue measure of $S_{\mathbf{x}}$. Using polar coordinates we obtain ($d\sigma$ denotes the usual surface measure on Σ^{N-1})

$$|S_{\mathbf{x}}| = \int_0^{|\mathbf{x}|} \int_A s^{N-1} d\sigma ds = \frac{|\mathbf{x}|^N}{N} |A|.$$



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Moreover, we say that u is a weight function if it is a positive and measurable function on S . Throughout the paper, for any $p > 1$ we denote $p' = \frac{p}{p-1}$.

For later purposes but also of independent interest we now state and prove our announced Hardy inequality.

Theorem 2.1. *Let E be a cone in \mathbb{R}^N and $S_{\mathbf{x}}$, A be defined as above. Suppose that $1 < p \leq q < \infty$ and that u, v are weight functions on E . Then, the inequality*

$$(2.1) \quad \left(\int_E \left(\int_{S_{\mathbf{x}}} f(\mathbf{y}) d\mathbf{y} \right)^q v(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_E f^p(\mathbf{x}) u(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}}$$

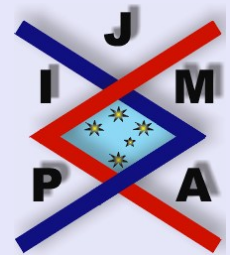
holds for all $f \geq 0$ if and only if

$$(2.2) \quad D := \sup_{t>0} \left(\int_{tS} u^{1-p'}(\mathbf{x}) d\mathbf{x} \right)^{-\frac{1}{p}} \times \left(\int_{tS} v(\mathbf{x}) \left(\int_{S_{\mathbf{x}}} u^{1-p'}(\mathbf{y}) d\mathbf{y} \right)^q d\mathbf{x} \right)^{\frac{1}{q}} < \infty.$$

Moreover, the best constant C in (2.1) can be estimated as follows:

$$D \leq C \leq p' D.$$

Remark 2.1. *Another weight characterization of (2.1) over balls in \mathbb{R}^N was proved by P. Drábek, H.P. Heinig and A. Kufner [3]. This result may be regarded as a generalization of the usual (Muckenaupt type) characterization in 1-dimension (see e.g. [13]) while our result may be seen as a higher dimensional version of another characterization by V.D. Stepanov and L.E. Persson (see [19], [20]).*



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Proof. By the duality principle (see e.g. [13]), it can be shown that the inequality (2.1) is equivalent to that the inequality

$$(2.3) \quad \left(\int_E \left(\int_{E \setminus S_x} g(\mathbf{y}) d\mathbf{y} \right)^{p'} u^{1-p'}(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p'}} \leq C \left(\int_E g^{q'}(\mathbf{x}) v^{1-q'}(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q'}}$$

holds for all $g \geq 0$ and with the same best constant C . First assume that (2.2) holds. Using polar coordinates and putting

$$(2.4) \quad \tilde{g}(t) = \int_A g(t\sigma) t^{N-1} d\sigma, \quad t \in (0, \infty)$$

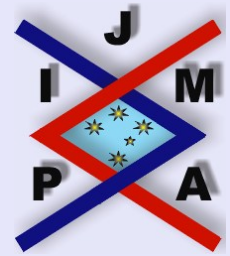
and

$$(2.5) \quad \tilde{u}(t) = \left(\int_A u^{1-p'}(t\tau) t^{N-1} d\tau \right)^{1-p}, \quad t \in (0, \infty)$$

we have

$$\begin{aligned} & \int_E \left(\int_{E \setminus S_x} g(\mathbf{y}) d\mathbf{y} \right)^{p'} u^{1-p'}(\mathbf{x}) d\mathbf{x} \\ &= \int_0^\infty \int_A \left(\int_t^\infty \int_A g(s\sigma) s^{N-1} d\sigma ds \right)^{p'} u^{1-p'}(t\tau) t^{N-1} d\tau dt \\ &= \int_0^\infty \left(\int_t^\infty \tilde{g}(s) ds \right)^{p'} \tilde{u}^{1-p'}(t) dt. \end{aligned}$$

Thus, using this, changing the order of integration and finally using Hölder's



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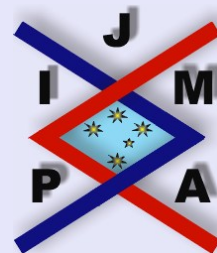
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inequality, we get

$$\begin{aligned}
 (2.6) \quad I &:= \int_E \left(\int_{E \setminus S_x} g(\mathbf{y}) d\mathbf{y} \right)^{p'} u^{1-p'}(\mathbf{x}) d\mathbf{x} \\
 &= \int_0^\infty \left(\int_t^\infty \tilde{g}(s) ds \right)^{p'} \tilde{u}^{1-p'}(t) dt \\
 &= \int_0^\infty \left(\int_z^\infty -\frac{d}{dt} \left(\int_t^\infty \tilde{g}(s) ds \right)^{p'} dt \right) \tilde{u}^{1-p'}(z) dz \\
 &= p' \int_0^\infty \left(\int_z^\infty \left(\int_t^\infty \tilde{g}(s) ds \right)^{p'-1} \tilde{g}(t) dt \right) \tilde{u}^{1-p'}(z) dz \\
 &= p' \int_0^\infty \left(\int_t^\infty \tilde{g}(s) ds \right)^{p'-1} \tilde{g}(t) \left(\int_0^t \tilde{u}^{1-p'}(z) dz \right) dt \\
 &= p' \int_0^\infty \int_A \left(\int_t^\infty \tilde{g}(s) ds \right)^{p'-1} \left(\int_0^t \tilde{u}^{1-p'}(s) ds \right) g(t\tau) t^{N-1} d\tau dt \\
 &\leq p' \left(\int_0^\infty \int_A g^{q'}(t\tau) v^{1-q'}(t\tau) t^{N-1} d\tau dt \right)^{\frac{1}{q'}} \\
 &\quad \times \left(\int_0^\infty \int_A \left(\int_t^\infty \tilde{g}(s) ds \right)^{(p'-1)q} \right. \\
 &\quad \times \left. \left(\int_0^t \tilde{u}^{1-p'}(s) ds \right)^q v(t\tau) t^{N-1} d\tau dt \right)^{\frac{1}{q}} \\
 &= p' \left(\int_E g^{q'}(\mathbf{x}) v^{1-q'}(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q'}} J^{\frac{1}{q}},
 \end{aligned}$$



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where

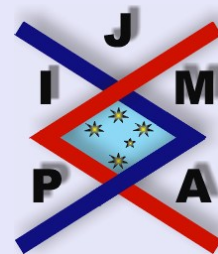
$$J = \int_0^\infty \left(\int_t^\infty \tilde{g}(s) ds \right)^{(p'-1)q} \left(\int_0^t \tilde{u}^{1-p'}(s) ds \right)^q \tilde{v}(t) dt$$

with

$$(2.7) \quad \tilde{v}(t) = \int_A v(t\tau) t^{N-1} d\tau.$$

Using Fubini's theorem, (2.2), (2.5) and (2.7), we get

$$\begin{aligned} J &= \int_0^\infty \int_t^\infty \frac{d}{dz} \left(- \left(\int_z^\infty \tilde{g}(s) ds \right)^{(p'-1)q} \right) dz \left(\int_0^t \tilde{u}^{1-p'}(s) ds \right)^q \tilde{v}(t) dt \\ &= \int_0^\infty \left[\frac{d}{dz} \left(- \left(\int_z^\infty \tilde{g}(s) ds \right)^{(p'-1)q} \right) \right] \int_0^z \left(\int_0^t \tilde{u}^{1-p'}(s) ds \right)^q \tilde{v}(t) dt dz \\ &= \int_0^\infty \left[\frac{d}{dz} \left(- \left(\int_z^\infty \tilde{g}(s) ds \right)^{(p'-1)q} \right) \right] \\ &\quad \times \left(\int_0^z \int_A \left(\int_0^t \int_A u^{1-p'}(s\sigma) s^{N-1} d\sigma ds \right)^q v(t\tau) t^{N-1} d\tau dt \right) dz \\ &= \int_0^\infty \left[\frac{d}{dz} \left(- \left(\int_z^\infty \tilde{g}(s) ds \right)^{(p'-1)q} \right) \right] \\ &\quad \times \left(\int_{zS} \left(\int_{S_x} u^{1-p'}(\mathbf{y}) d\mathbf{y} \right)^q v(\mathbf{x}) d\mathbf{x} \right) dz \end{aligned}$$



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$$\begin{aligned} &\leq D^q \int_0^\infty \left[\frac{d}{dz} \left(- \left(\int_z^\infty \tilde{g}(s) ds \right)^{(p'-1)q} \right) \right] \left(\int_{zS} u^{1-p'}(\mathbf{x}) d\mathbf{x} \right)^{\frac{q}{p}} dz \\ &= D^q \int_0^\infty \left[\frac{d}{dz} \left(- \left(\int_z^\infty \tilde{g}(s) ds \right)^{(p'-1)q} \right) \right] \left(\int_0^z \tilde{u}^{1-p'}(t) dt \right)^{\frac{q}{p}} dz. \end{aligned}$$

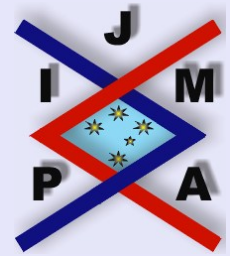
Thus, using Minkowski's integral inequality, (2.4) and (2.5) we have

$$\begin{aligned} J &\leq D^q \left(\int_0^\infty \left(\int_t^\infty \left[\frac{d}{dz} \left(- \left(\int_z^\infty \tilde{g}(s) ds \right)^{(p'-1)q} \right) \right] dz \right)^{\frac{p}{q}} \tilde{u}^{1-p'}(t) dt \right)^{\frac{q}{p}} \\ &= D^q \left(\int_0^\infty \left(\int_t^\infty \tilde{g}(s) ds \right)^{p'} \tilde{u}^{1-p'}(t) dt \right)^{\frac{q}{p}} \\ &= D^q \left(\int_E \left(\int_{E \setminus S_x} g(\mathbf{y}) d\mathbf{y} \right)^{p'} u^{1-p'}(\mathbf{x}) d\mathbf{x} \right)^{\frac{q}{p}}. \end{aligned}$$

Assume first that in (2.6) $I < \infty$. Then

$$\left(\int_E \left(\int_{E \setminus S_x} g(\mathbf{y}) d\mathbf{y} \right)^{p'} u^{1-p'}(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p'}} \leq p' D \left(\int_E g^{q'}(\mathbf{x}) v^{1-q'}(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q'}}$$

i.e., (2.3) holds for all $g \geq 0$ and also the constant C in (2.3) satisfies $C \leq p'D$. For the case $I = \infty$ replace $g(\mathbf{y})$ by an approximating sequence $g_n(\mathbf{y}) \leq g(\mathbf{y})$ (such that the corresponding $I_n < \infty$) and use the Monotone Convergence Theorem to obtain the result.



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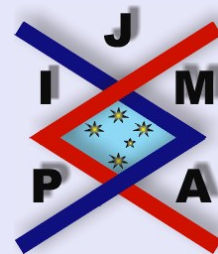
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Conversely, suppose that (2.1) holds for all $f \geq 0$. In this inequality, taking for any fixed $t > 0$ the function $f_t = \chi_{tS} u^{1-p'}$, we find that

$$\begin{aligned} C &\geq \left(\int_E \left(\int_{S_{\mathbf{x}}} f_t(\mathbf{y}) d\mathbf{y} \right)^q v(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \left(\int_E f_t^p(\mathbf{x}) u(\mathbf{x}) d\mathbf{x} \right)^{-\frac{1}{p}} \\ &\geq \left(\int_{tS} \left(\int_{S_{\mathbf{x}}} u^{1-p'}(\mathbf{y}) d\mathbf{y} \right)^q v(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \left(\int_{tS} u^{1-p'}(\mathbf{x}) d\mathbf{x} \right)^{-\frac{1}{p}}. \end{aligned}$$

By taking the supremum we find that (2.2) holds and, moreover, $D \leq C$. The proof is complete. \square



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3. Geometric Mean Inequalities

Here we prove our main geometric mean inequality by making a limit procedure in Theorem 2.1.

Theorem 3.1. *Let $0 < p \leq q < \infty$ and suppose that all other assumptions of Theorem 2.1 are satisfied. Then the inequality*

$$(3.1) \quad \left(\int_E \left(\exp \left(\frac{1}{|S_{\mathbf{x}}|} \int_{S_{\mathbf{x}}} \ln f(\mathbf{y}) d\mathbf{y} \right) \right)^q v(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_E f^p(\mathbf{x}) u(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}}$$

holds for all $f > 0$ if and only if

$$D_1 := \sup_{t>0} |tS|^{-\frac{1}{p}} \left(\int_{tS} w(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} < \infty,$$

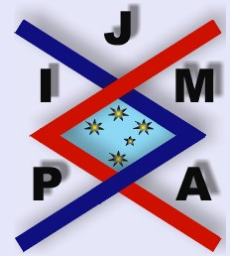
where

$$(3.2) \quad w(\mathbf{t}) := v(\mathbf{x}) \left(\exp \left(\frac{1}{|S_{\mathbf{x}}|} \int_{S_{\mathbf{x}}} \ln \frac{1}{u(\mathbf{y})} d\mathbf{y} \right) \right)^{\frac{q}{p}} < \infty.$$

Moreover, the best constant C satisfies $D_1 \leq C \leq e^{\frac{1}{p}} D_1$.

Proof. It is easy to see that (3.1) is equivalent to

$$\left(\int_E \left(\exp \left(\frac{1}{|S_{\mathbf{x}}|} \int_{S_{\mathbf{x}}} \ln f(\mathbf{y}) d\mathbf{y} \right) \right)^q w(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_E f^p(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}}$$



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with $w(\mathbf{x})$ defined by (3.2). Let $v(\mathbf{x}) = w(\mathbf{x}) |S_{\mathbf{x}}|^{-q}$ and $u(\mathbf{x}) = 1$ in Theorem 2.1 and choose an α such that $0 < \alpha < p \leq q < \infty$. Then $1 < \frac{p}{\alpha} \leq \frac{q}{\alpha} < \infty$. Now, replacing f, p, q and $v(\mathbf{x})$ by $f^\alpha, \frac{p}{\alpha}, \frac{q}{\alpha}$ in Theorem 2.1, we find that the inequality

$$(3.3) \quad \left(\int_E \left(\frac{1}{|S_{\mathbf{x}}|} \int_{S_{\mathbf{x}}} f^\alpha(\mathbf{y}) d\mathbf{y} \right)^{\frac{q}{\alpha}} w(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C_\alpha \left(\int_E f^p(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}}$$

holds for all functions $f > 0$ if and only if D_1 holds. Moreover, it is easy to see that (c.f. [20])

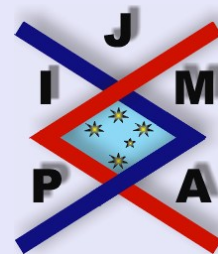
$$(3.4) \quad D_1 \leq C_\alpha \leq \left(\frac{p}{p-\alpha} \right)^{\frac{1}{\alpha}} D_1.$$

By letting $\alpha \rightarrow 0^+$ in (3.3) and (3.4) we find that $\left(\frac{p}{p-\alpha} \right)^{\frac{1}{\alpha}} \rightarrow e^{\frac{1}{p}}$ and

$$\left(\frac{1}{|S_{\mathbf{x}}|} \int_{S_{\mathbf{x}}} f^\alpha(\mathbf{y}) d\mathbf{y} \right)^{\frac{1}{\alpha}} \rightarrow \exp \left(\frac{1}{|S_{\mathbf{x}}|} \int_{S_{\mathbf{x}}} \ln f(\mathbf{y}) d\mathbf{y} \right),$$

i.e. the scale of power means converge to the geometric mean, and the proof follows. \square

Remark 3.1. *Our proof above shows that (3.1) in Theorem 3.1 may be regarded as a natural limiting case of Hardy's inequality (2.1) as it is in the classical one-dimensional situation. This fact indicates that our formulation of Hardy's inequality in Theorem 2.1 is very natural from this point of view.*



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As a special case, if we take $E = \mathbb{R}^N$ and $S_{\mathbf{x}} = B_{\mathbf{x}}$ the ball centered at the origin and with radius $|\mathbf{x}|$, and $|B_{\mathbf{x}}|$ its volume, then we immediately obtain the following corollary to Theorem 3.1 that averages functions over balls in \mathbb{R}^N :

Corollary 3.2. *Let $0 < p \leq q < \infty$ and u, v be weight functions in \mathbb{R}^N . Then the inequality*

$$\left(\int_{\mathbb{R}^N} \left(\exp \left(\frac{1}{|B_{\mathbf{x}}|} \int_{B_{\mathbf{x}}} \ln f(\mathbf{y}) d\mathbf{y} \right) \right)^q v(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^N} f^p(\mathbf{x}) u(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}}$$

holds for all $f > 0$ if and only if

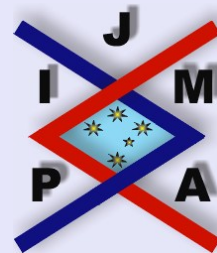
$$D_2 := \sup_{\mathbf{z} \in \mathbb{R}^N \setminus \{0\}} |B_{\mathbf{z}}|^{-\frac{1}{p}} \left(\int_{B_{\mathbf{z}}} v(\mathbf{x}) \left(\exp \left(\frac{1}{|B_{\mathbf{x}}|} \int_{B_{\mathbf{x}}} \ln \frac{1}{u(\mathbf{y})} d\mathbf{y} \right) \right)^{\frac{q}{p}} d\mathbf{x} \right)^{\frac{1}{q}} < \infty.$$

Moreover, the best constant C satisfies $D_2 \leq C \leq e^{\frac{1}{p}} D_2$.

Remark 3.2. *Corollary 3.2 extends a result of P. Drábek, H.P. Heinig and A. Kufner [3, Theorem 4.1], who obtained it for the case $p = q = 1$ and with a completely different proof.*

Remark 3.3. *Setting $E = \mathbb{R}_+^N = \{(x_1, \dots, x_N) \in \mathbb{R}^N, x_1 \geq 0, \dots, x_N \geq 0\}$ in Theorem 3.1 we obtain that Corollary 3.2 holds also for \mathbb{R}_+^N instead of \mathbb{R}^N and $B_{\mathbf{x}} \cap \mathbb{R}_+^N$ instead of $B_{\mathbf{x}}$.*

We shall now consider the special weights discussed in our introduction and in [1].



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Proposition 3.3. Let $0 < p \leq q < \infty$, $a, b \in \mathbb{R}$, $\varepsilon \in \mathbb{R}_+$, and E, S_x be defined as in Theorem 2.1. Then

$$(3.5) \quad \left(\int_E \left[\exp \left(\varepsilon |S_x|^{-\varepsilon} \int_{S_x} |S_y|^{\varepsilon-1} \ln f(\mathbf{y}) d\mathbf{y} \right) \right]^q |S_x|^a d\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_E f^p(\mathbf{x}) |S_x|^b d\mathbf{x} \right)^{\frac{1}{p}}$$

holds for all positive functions f for some finite constant C if and only if

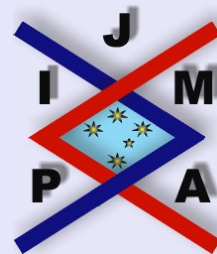
$$(3.6) \quad \frac{a+1}{q} = \frac{b+1}{p}$$

and the least constant C in (3.5) satisfies

$$\left(\frac{p}{q} \right)^{\frac{1}{q}} \varepsilon^{\frac{1}{p}-\frac{1}{q}} e^{\frac{b+1}{\varepsilon p}-\frac{1}{p}} \leq C \leq \left(\frac{p}{q} \right)^{\frac{1}{q}} \varepsilon^{\frac{1}{p}-\frac{1}{q}} e^{\frac{b+1}{\varepsilon p}}.$$

Proof. By writing (3.5) in polar coordinates we find that

$$\begin{aligned} & \left(\int_0^\infty \int_A \left[\exp \frac{\varepsilon N^\varepsilon}{t^{N\varepsilon} |A|^\varepsilon} \right. \right. \\ & \quad \times \left. \int_0^t \int_A \left(\frac{|A|}{N} \right)^{\varepsilon-1} s^{N\varepsilon-1} \ln f(s\sigma) d\sigma ds \right]^q t^{Na+N-1} \left(\frac{|A|}{N} \right)^a d\tau dt \Big)^{\frac{1}{q}} \\ & \leq \left(\int_0^\infty \int_A f^p(t\tau) \left(\frac{|A|}{N} \right)^b t^{Nb+N-1} d\tau dt \right)^{\frac{1}{p}}. \end{aligned}$$



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Exchanging variables, $s = r^{\frac{1}{\varepsilon}}$ and $t = z^{\frac{1}{\varepsilon}}$ we find that this inequality can be rewritten as

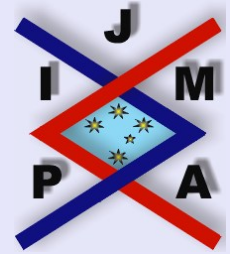
$$\begin{aligned} & \left(\int_0^\infty \int_A \left(\exp \left(\frac{N}{|A|} \int_0^z \int_A \ln f \left(r^{\frac{1}{\varepsilon}} \sigma \right) r^{N-1} d\sigma dr \right) \right)^q \\ & \quad \times \left(\frac{|A|}{N} \right)^a z^{N \left(\frac{a+1}{\varepsilon} - 1 \right)} z^{N-1} \frac{1}{\varepsilon} d\tau dz \right)^{\frac{1}{q}} \\ & \leq C \left(\int_0^\infty \int_A f^p \left(z^{\frac{1}{\varepsilon}} \tau \right) \left(\frac{|A|}{N} \right)^b z^{N \left(\frac{b+1}{\varepsilon} - 1 \right)} z^{N-1} \frac{1}{\varepsilon} d\tau dz \right)^{\frac{1}{p}}, \end{aligned}$$

that is,

$$\begin{aligned} (3.7) \quad & \left(\int_E \left(\exp \left(\frac{1}{|S_{\mathbf{x}}|} \int_{S_{\mathbf{x}}} \ln f_1(\mathbf{y}) d\mathbf{y} \right) \right)^q |S_{\mathbf{x}}|^{\frac{a+1}{\varepsilon} - 1} d\mathbf{x} \right)^{\frac{1}{q}} \\ & \leq C \left(\frac{|A|}{N} \right)^{\left(\frac{b+1}{p} - \frac{a+1}{q} \right) \left(1 - \frac{1}{\varepsilon} \right)} \varepsilon^{\frac{1}{q} - \frac{1}{p}} \left(\int_E f_1^p(\mathbf{x}) |S_{\mathbf{x}}|^{\frac{b+1}{\varepsilon} - 1} d\mathbf{x} \right)^{\frac{1}{p}}, \end{aligned}$$

where $f_1(r\sigma) = f(r^{\frac{1}{\varepsilon}}\sigma)$. This means that (3.5) is equivalent to (3.7) i.e., (3.1) holds with the weights $v(x) = |S_x|^{\frac{a+1}{\varepsilon} - 1}$ and $u(x) = |S_x|^{\frac{b+1}{\varepsilon} - 1}$. We note that for these weights we find after a direct calculation that the constant D_1 from Theorem 3.1 is

$$D_1 = \sup_{t>0} \frac{|tS|^{\frac{a+1}{\varepsilon q} - \frac{b+1}{\varepsilon p}} e^{\frac{1}{p} \left(\frac{b+1}{\varepsilon} - 1 \right)}}{\left(\frac{a+1}{\varepsilon} - \frac{q}{p} \left(\frac{b+1}{\varepsilon} - 1 \right) \right)^{\frac{1}{q}}}$$



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so we conclude that (3.6) must hold and then

$$D_1 = e^{\frac{1}{p} \left(\frac{b+1}{\varepsilon} - 1 \right)} \left(\frac{p}{q} \right)^{\frac{1}{q}}.$$

Thus, the proof follows from Theorem 3.1. \square

Remark 3.4. Setting $p = q = 1$, $a = b$, we have that (3.5) implies the estimate (1.2).

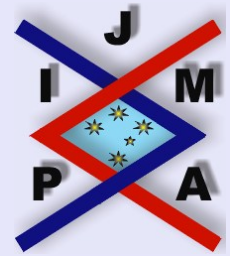
Remark 3.5 (Sharp Constant). In the above proposition, if we take $p = q$, then $a = b$. In this situation (3.5) holds with the constant $C = e^{(b+1)/p}$. Indeed, this constant is sharp. In order to show this for $\delta > 0$, we consider the function

$$f_\delta(x) = \begin{cases} e^{-\frac{b+1}{\varepsilon p}} |S|^{-(b+1)} |\mathbf{x}|^{-\frac{N}{p}(b+1-\varepsilon\delta)}, & x \in S, \\ e^{-\frac{b+1}{\varepsilon p}} |S|^{-(b+1)} |\mathbf{x}|^{-\frac{N}{p}(b+1+\varepsilon\delta)}, & x \in E \setminus S. \end{cases}$$

By using this function in (3.5), we find that

$$1 \leq \frac{RHS}{LHS} \leq e^{\frac{\delta}{p}} \rightarrow 1 \quad \text{as} \quad \delta \rightarrow 0$$

and consequently the constant is sharp. Note that the sharpness of the constant for $p = q$, in Proposition 3.3 has been proved in the more general setting than that in [1].



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4. The Companion Inequalities

We present the following result which is a companion of Theorem 3.1:

Theorem 4.1. *Let $0 < p \leq q < \infty$, $\varepsilon > 0$, and suppose that all other hypotheses of Theorem 3.1 are satisfied. Then the inequality*

$$(4.1) \quad \left(\int_E \left(\exp \left(\varepsilon |S_{\mathbf{x}}|^\varepsilon \int_{E \setminus S_{\mathbf{x}}} |S_{\mathbf{y}}|^{-\varepsilon-1} \ln f(\mathbf{y}) d\mathbf{y} \right) \right)^q v(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_E f^p(\mathbf{x}) u(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}}$$

holds for all $f > 0$ if and only if

$$D_3 := \sup_{t>0} |tS|^{-\frac{1}{p}} \left(\int_{tS} v_*(\mathbf{x}) \left(\exp \left(\frac{1}{|S_{\mathbf{x}}|} \int_{S_{\mathbf{x}}} \ln \frac{1}{u_*(\mathbf{y})} d\mathbf{y} \right) \right)^{\frac{q}{p}} d\mathbf{x} \right)^{\frac{1}{q}} < \infty,$$

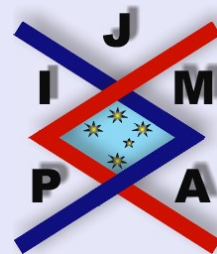
where

$$u_*(\mathbf{y}) := u(s^{-\frac{1}{\varepsilon}}\sigma) \frac{1}{\varepsilon} s^{-N(1+\frac{1}{\varepsilon})}, \quad v_*(\mathbf{y}) := v(s^{-\frac{1}{\varepsilon}}\sigma) \frac{1}{\varepsilon} s^{-N(1+\frac{1}{\varepsilon})}.$$

Moreover, the constant C satisfies $D_3 \leq C \leq e^{\frac{1}{p}} D_3$.

Proof. Note that for $x \in \mathbb{R}^N$

$$|S_{\mathbf{x}}| = \int_0^{|\mathbf{x}|} \int_A t^{N-1} d\tau dt = \frac{|\mathbf{x}|^N}{N} |A|.$$



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Now, using polar coordinates, (4.1) can be written as

$$\begin{aligned} & \left(\int_0^\infty \int_A \left(\exp \frac{\varepsilon |A|^\varepsilon t^{N\varepsilon}}{N} \right. \right. \\ & \quad \times \left. \left. \int_t^\infty \int_A \left(\frac{|A|}{N} \right)^{-\varepsilon-1} s^{-N\varepsilon-1} \ln f(s\sigma) d\sigma ds \right)^q v(t\tau) t^{N-1} d\tau dt \right)^{\frac{1}{q}} \\ & \leq C \left(\int_0^\infty \int_A f^p(t\tau) u(t\tau) t^{N-1} d\tau dt \right)^{\frac{1}{p}}. \end{aligned}$$

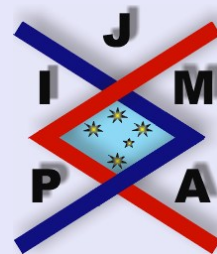
Using the exchange of variables $s = r^{-1/\varepsilon}$ and $t = z^{-1/\varepsilon}$ we obtain

$$\begin{aligned} & \left(\int_0^\infty \int_A \left[\exp \left(\frac{N}{|A| z^N} \int_A \int_0^z \ln f(r^{-\frac{1}{\varepsilon}} \sigma) r^{N-1} d\sigma dr \right) \right]^q \right. \\ & \quad \times \left. v(z^{-\frac{1}{\varepsilon}} \tau) z^{-N(1+\frac{1}{\varepsilon})} \frac{1}{\varepsilon} z^{N-1} d\tau dz \right)^{\frac{1}{q}} \\ & \leq C \left(\int_0^\infty \int_A f^p(z^{-\frac{1}{\varepsilon}} \tau) u(z^{-\frac{1}{\varepsilon}} \tau) z^{-N(1+\frac{1}{\varepsilon})} \frac{1}{\varepsilon} z^{N-1} d\tau dz \right)^{\frac{1}{p}} \end{aligned}$$

and put $f_*(t\tau) = f(t^{-\frac{1}{\varepsilon}} \tau)$. (4.1) can be equivalently rewritten as

$$\left(\int_E \left(\exp \left(\frac{1}{|S_{\mathbf{x}}|} \int_{S_{\mathbf{x}}} \ln f_*(\mathbf{y}) d\mathbf{y} \right) \right)^q v_*(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_E f_*^p(\mathbf{x}) u_*(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}}.$$

Now, the result is obtained by using Theorem 3.1. \square



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Analogously to Corollary 3.2, we can immediately obtain a special case of Theorem 4.1 that averages functions over balls in \mathbb{R}^N centered at origin.

Corollary 4.2. *Let $0 < p \leq q < \infty$, $\varepsilon > 0$, and u, v be weight functions in \mathbb{R}^N . Then the inequality*

$$(4.2) \quad \left(\int_{\mathbb{R}^N} \left(\exp \left(\varepsilon |B_{\mathbf{x}}|^\varepsilon \int_{\mathbb{R}^N \setminus B_{\mathbf{x}}} |B_{\mathbf{y}}|^{-\varepsilon-1} \ln f(\mathbf{y}) d\mathbf{y} \right) \right)^q v(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^N} f^p(\mathbf{x}) u(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}}$$

holds for all $f > 0$ if and only if

$$\tilde{B} := \sup_{z \in \mathbb{R}^N} |B_z|^{-\frac{1}{p}} \left(\int_{B_z} v_0(\mathbf{x}) \left(\exp \left(\frac{1}{|B_{\mathbf{x}}|} \int_{B_{\mathbf{x}}} \ln \frac{1}{u_0}(\mathbf{y}) d\mathbf{y} \right) \right)^{\frac{q}{p}} d\mathbf{x} \right)^{\frac{1}{q}} < \infty,$$

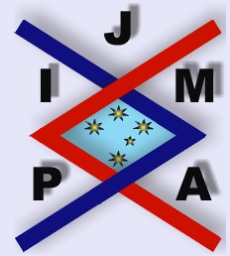
where

$$u_0(\mathbf{x}) := u(t^{-\frac{1}{\varepsilon}}\tau) \frac{1}{\varepsilon} t^{-N(1+\frac{1}{\varepsilon})}, \quad v_0(\mathbf{x}) := v(t^{-\frac{1}{\varepsilon}}\tau) \frac{1}{\varepsilon} t^{-N(1+\frac{1}{\varepsilon})}.$$

Moreover, the best constant C satisfies $\tilde{B} \leq C \leq e^{\frac{1}{p}} \tilde{B}$.

Remark 4.1. *Note that by choosing E as in Remark 3.3 we see that Corollary 4.2 in fact holds also when \mathbb{R}^N is replaced by \mathbb{R}_+^N or more general cones in \mathbb{R}^N .*

The corresponding result to Proposition 3.3 reads as follows and the proof is analogous.



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Proposition 4.3. Let $0 < p \leq q < \infty$, $\varepsilon > 0$, and $a, b \in \mathbb{R}$, and E, S_x be defined as in Theorem 2.1. Then the inequality

$$(4.3) \quad \left(\int_E \left(\exp \varepsilon |S_x|^\varepsilon \int_{E \setminus S_x} |S_y|^{-\varepsilon-1} \ln f(y) dy \right)^q |S_x|^a dx \right)^{\frac{1}{q}} \leq C \left(\int_E f^p(x) |S_x|^b dx \right)^{\frac{1}{p}}$$

holds for all $f > 0$ and some finite positive constant C if and only if

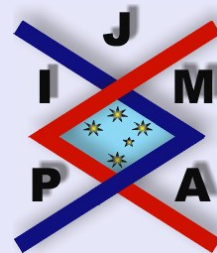
$$\frac{a+1}{q} = \frac{b+1}{p}$$

and the least constant C in (4.3) satisfies

$$\left(\frac{p}{q}\right)^{-\frac{1}{q}} \varepsilon^{\frac{1}{p}-\frac{1}{q}} e^{-\left(\frac{b+1}{\varepsilon p} + \frac{1}{p}\right)} \leq C \leq \left(\frac{p}{q}\right)^{-\frac{1}{q}} \varepsilon^{\frac{1}{p}-\frac{1}{q}} e^{-\frac{b+1}{\varepsilon p}}.$$

Remark 4.2 (Sharp Constant). Analogously to Proposition 3.3, in the above proposition we also find that if we take $p = q$, then $a = b$. In this situation (4.3) holds with the constant $C = e^{-(b+1)/\varepsilon p}$ and the constant is sharp. This can be shown by considering, for $\delta > 0$, the function

$$f_\delta(\mathbf{x}) = \begin{cases} e^{\frac{b+1}{\varepsilon p}} |S|^{-(b+1)} |\mathbf{x}|^{-\frac{N}{p}(b+1-\varepsilon\delta)}, & \mathbf{x} \in S \\ e^{\frac{b+1}{p}} |S|^{-(b+1)} |\mathbf{x}|^{-\frac{N}{p}(b+1+\varepsilon\delta)}, & \mathbf{x} \in E \setminus S. \end{cases}$$



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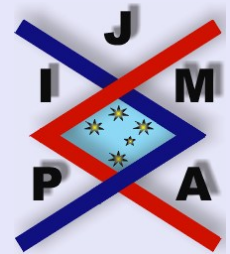
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Remark 4.3. *It is tempting to think that the results in this paper hold also in general star-shaped regions in \mathbb{R}^N (c.f. [22]) but this is not true in general as was pointed out to us by the referee. See also [22] and note that the results there also hold at least for cones in \mathbb{R}^N .*



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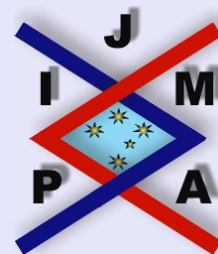
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References

- [1] A. ČIŽMEŠIJA, J. PEČARIĆ AND I. PERIĆ, Mixed means and inequalities of Hardy and Levin-Cochran-Lee type for multidimensional balls, *Proc. Amer. Math. Soc.*, **128**(9) (2000), 2543–2552.
- [2] J.A. COCHRAN AND C.S. LEE, Inequalities related to Hardy's and Heinig's, *Math. Proc. Cambridge Phil. Soc.*, **96** (1984), 1–7.
- [3] P. DRÁBEK, H.P. HEINIG AND A. KUFNER, Higher dimensional Hardy inequality, *Int. Ser. Num. Math.*, **123** (1997), 3–16.
- [4] G.H. HARDY, Notes on some points in the integral calculus, **LXIV** (1925), 150–156.
- [5] H.P. HEINIG, Weighted inequalities in Fourier analysis, *Nonlinear Analysis, Function Spaces and Applications*, Vol. 4, Teubner-Texte Math., band 119, Teubner, Leipzig, (1990), 42–85.
- [6] H.P. HEINIG, R. KERMAN AND M. KRBEC, Weighted exponential inequalities, *Georgian Math. J.*, (2001), 69–86.
- [7] P. JAIN AND A.P. SINGH, A characterization for the boundedness of geometric mean operator, *Applied Math. Letters* (Washington), **13**(8) (2000), 63–67.
- [8] P. JAIN, L.E. PERSSON AND A. WEDESTIG, From Hardy to Carleman and general mean-type inequalities, *Function Spaces and Applications*, CRC Press (New York)/Narosa Publishing House (New Delhi)/Alpha Science (Pangbourne) (2000), 117–130 .



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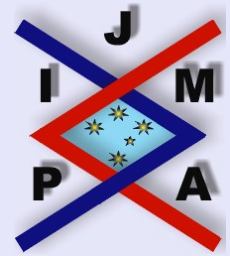
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- [9] P. JAIN, L.E. PERSSON AND A. WEDESTIG, Carleman-Knopp type inequalities via Hardy inequalities, *Math. Ineq. Appl.*, **4**(3) (2001), 343–355.
- [10] A.M. JARRAH AND A.P. SINGH, A limiting case of Hardy’s inequality, *Indian J. Math.*, **43**(1) (2001), 21–36.
- [11] S. KAIJSER, L.E. PERSSON AND A. ÖBERG, On Carleman’s and Knopp’s inequalities, *J. Approx. Theory*, to appear 2002.
- [12] K. KNOPP, Über Reihen mit positiven Gliedern, *J. London Math. Soc.*, **3** (1928), 205–211.
- [13] A. KUFNER AND L.E. PERSSON, *Weighted Inequalities of Hardy Type*, World Scientific, New Jersey/London/Singapore/Hong Kong, 2003.
- [14] E.R. LOVE, Inequalities related to those of Hardy and of Cochran and Lee, *Math. Proc. Camb. Phil. Soc.*, **99** (1986), 395–408.
- [15] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities Involving Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, 1991.
- [16] M. NASSYROVA, *Weighted inequalities involving Hardy-type and limiting geometric mean operators*, PhD Thesis, Department of Mathematics, Luleå University of Technology, 2002.
- [17] M. NASSYROVA, L.E. PERSSON AND V.D. STEPANOV, On weighted inequalities with geometric mean operator by the Hardy-type integral



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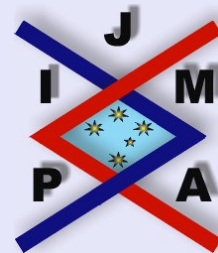
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transform, *J. Inequal. Pure Appl. Math.*, **3**(4) (2002), Art. 48. [ONLINE: http://jipam.vu.edu.au/v3n4/084_01.html]

- [18] B. OPIĆ AND P. GURKA, Weighted inequalities for geometric means, *Proc. Amer. Math. Soc.*, **3** (1994), 771–779.
- [19] V.D. STEPANOV, Weighted norm inequalities of Hardy type for a class of integral operators, *J. London Math. Soc.*, **50**(2) (1994), 105–120.
- [20] L.E. PERSSON AND V.D. STEPANOV, Weighted integral inequalities with the geometric mean operator, *J. Inequal. & Appl.*, **7**(5) (2002), 727–746 (an abbreviated version can also be found in *Russian Akad. Sci. Dokl. Math.*, **63** (2001), 201–202).
- [21] L. PICK AND B. OPIĆ, On the geometric mean operator, *J. Math. Anal. Appl.*, **183**(3) (1994), 652–662.
- [22] G. SINNAMON, One-dimensional Hardy-type inequalities in many dimensions, *Proc. Royal Soc. Edinburgh*, **128A** (1998), 833–848.



**Weighted Geometric Mean
Inequalities Over Cones in \mathbb{R}^N**

Babita Gupta, Pankaj Jain,
Lars-Erik Persson and
Anna Wedestig

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