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WEIGHTED GEOMETRIC MEAN INEQUALITIES OVER CONES IN $\mathbb{R}^N$

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Abstract

Let $0 < p \leq q < \infty$. Let A be a measurable subset of the unit sphere in $\mathbb{R}^N$, let $E = \left\{ \mathbf{x} \in \mathbb{R}^N : \mathbf{x} = s\sigma, 0 \leq s < \infty, \sigma \in A \right\}$ be a cone in $\mathbb{R}^N$ and let $S_{\mathbf{x}}$ be the part of E with 'radius' $\leq |\mathbf{x}|$. A characterization of the weights u and v on E is given such that the inequality

$$\left(\int_E \left(\exp \left(\frac{1}{|S_{\mathbf{x}}|} \int_{S_{\mathbf{x}}} \ln f(\mathbf{y}) d\mathbf{y} \right) \right)^q v(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \leq C \left(\int_E f^p(\mathbf{x}) u(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{p}}$$

holds for all $f \geq 0$ and some positive and finite constant C. The inequality is obtained as a limiting case of a corresponding new Hardy type inequality. Also the corresponding companion inequalities are proved and the sharpness of the constant C is discussed.

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Contents

1. Introduction

In their paper [2] J.A. Cochran and C.S. Lee proved the inequality

$$\int_0^\infty \left[\exp\left(\varepsilon x^{-\varepsilon}\int_0^x y^{\varepsilon-1}\ln f(y)dy\right)\right] x^a dx \le e^{\frac{a+1}{\varepsilon}}\int_0^\infty x^a f(x)dx, \tag{1.1}$$

where a, ε are real numbers with $\varepsilon > 0$, f is a positive function defined on $(0,\infty)$ and the constant $e^{\frac{a+1}{\varepsilon}}$ is the best possible. This inequality, in fact, is a generalization of what sometimes is referred to as Knopp's inequality [1] , which is obtained by taking $\varepsilon = 1$ and $a = 0$ in (1.1). Inequalities of the type (1.1) and its analogues have further been investigated and generalized by many authors e.g. see [1], [5] – [11], [14] and [16] – [21].

In particular, very recently A. Čižmešija, J. Pečarić and I. Perić [1, Th. 9, formula (23)] proved an $N-$ dimensional analogue of (1.1) by replacing the interval $(0,\infty)$ by $\mathbb{R}^N$ and the means are considered over the balls in $\mathbb{R}^N$ centered at the origin. Their inequality reads:

$$\int_{\mathbb{R}^N} \left[\exp\left(\varepsilon\left|B_{\mathbf{x}}\right|^{-\varepsilon}\int_{B_{\mathbf{x}}}\left|B_{\mathbf{y}}\right|^{\varepsilon-1}\ln f(\mathbf{y})d\mathbf{y}\right)\right]\left|B_{\mathbf{x}}\right|^a d\mathbf{x} \le e^{\frac{a+1}{\varepsilon}}\int_{\mathbb{R}^N} f(\mathbf{x})\left|B_{\mathbf{x}}\right|^a d\mathbf{x}, \tag{1.2}$$

where $a \in \mathbb{R}$, $\varepsilon > 0$, f is a positive function on $\mathbb{R}^N$, $B_{\mathbf{x}}$ is a ball in $\mathbb{R}^N$ with radius $|\mathbf{x}|$, $\mathbf{x} \in \mathbb{R}^N$, centered at the origin and $|B_{\mathbf{x}}|$ is its volume.

[1] See e.g. [15, p. 143–144] and [12]. Note however that according to G.H. Hardy [4, p 156] this inequality was pointed out to him already in 1925 by G. Polya.

In this paper we prove a more general result, namely we characterize the weights u and v on $\mathbb{R}^N$ such that for $0 < p \le q < \infty$

$$\left(\int_{\mathbb{R}^N} \left[\exp\left(\frac{1}{|B_{\mathbf{x}}|}\int_{B_{\mathbf{x}}} \ln f(\mathbf{y})d\mathbf{y}\right)\right]^q v(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^N} f^p(\mathbf{x})u(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}}$$

holds for some finite positive constant C (See Corollary 3.2). In the case when $v(\mathbf{x}) = |S_{\mathbf{x}}|^a$ and $u(\mathbf{x}) = |S_{\mathbf{x}}|^b$ we obtain a genuine generalization of (1.2) (see Proposition 3.3 and Remark 3.4).

In this paper we also generalize the results in another direction, namely when the geometric averages over spheres in $\mathbb{R}^N$ are replaced by such averages over spherical cones in $\mathbb{R}^N$ (see notation below). This means in particular that our inequalities above and later on also hold when $\mathbb{R}^N$ is replaced by $\mathbb{R}^N_+$ or even more general cones in $\mathbb{R}^N$.

The paper is organized in the following way. In Section 2 we collect some preliminaries and prove a new Hardy inequality that averages functions over the cones in $\mathbb{R}^N$ (see Theorem 2.1). In Section 3 we present and prove our main results concerning (the limiting) geometric mean operators (see Theorem 3.1 and Proposition 3.3). Finally, in Section 4 we present the corresponding companion inequalities (see Theorem 4.1, Corollary 4.2 and Proposition 4.3).

2. Preliminaries

Let Σ^{N-1} be the unit sphere in $\mathbb{R}^N$, that is, $\Sigma^{N-1} = \{\mathbf{x} \in \mathbb{R}^N : |\mathbf{x}| = 1\}$, where $|\mathbf{x}|$ denotes the Euclidean norm of the vector $\mathbf{x} \in \mathbb{R}^N$. Let A be a measurable subset of Σ^{N-1}, and let $E \subseteq \mathbb{R}^N$ be a spherical cone, i.e.,

$$E = \left\{\mathbf{x} \in \mathbb{R}^N : \mathbf{x} = s\sigma, 0 \leq s < \infty, \sigma \in A\right\}.$$

Let $S_{\mathbf{x}}$, $\mathbf{x} \in \mathbb{R}^N$ denote the part of E with 'radius' $\leq |\mathbf{x}|$, i.e.,

$$S_{\mathbf{x}} = \left\{\mathbf{y} \in \mathbb{R}^N : \mathbf{y} = s\sigma, 0 \leq s \leq |\mathbf{x}|, \sigma \in A\right\}.$$

For $0 < p < \infty$ and a non-negative measurable function w on E, by $L^p_w := L^p_w(E)$ we denote the weighted Lebesgue space with the weight function w, consisting of all measurable functions f on E such that

$$\|f\|_{L^p_w} = \left(\int_E |f(\mathbf{x})|^p w(\mathbf{x}) d\mathbf{x}\right)^{\frac{1}{p}} < \infty,$$

and make use of the abbreviations L^p and $\|f\|_{L^p}$ when $w(\mathbf{x}) \equiv 1$.

Let $S = S_{\mathbf{x}}, |\mathbf{x}| = 1$. The family of regions we shall average over is the collection of dilations of S. For $\mathbf{x} \in E \setminus \{\mathbf{0}\}$ denote by $|S_{\mathbf{x}}|$ the Lebesgue measure of $S_{\mathbf{x}}$. Using polar coordinates we obtain ($d\sigma$ denotes the usual surface measure on Σ^{N-1})

$$|S_{\mathbf{x}}| = \int_0^{|\mathbf{x}|} \int_A s^{N-1} d\sigma ds = \frac{|\mathbf{x}|^N}{N} |A|.$$

Moreover, we say that u is a weight function if it is a positive and measurable function on S. Throughout the paper, for any $p > 1$ we denote $p' = \frac{p}{p-1}$.

For later purposes but also of independent interest we now state and prove our announced Hardy inequality.

Theorem 2.1. *Let E be a cone in $\mathbb{R}^N$ and $S_{\mathbf{x}}$, A be defined as above. Suppose that $1 < p \le q < \infty$ and that u, v are weight functions on E. Then, the inequality*

$$\left(\int_E \left(\int_{S_{\mathbf{x}}} f(\mathbf{y})d\mathbf{y}\right)^q v(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{q}} \le C\left(\int_E f^p(\mathbf{x})u(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}} \tag{2.1}$$

holds for all $f \ge 0$ if and only if

$$D := \sup_{t>0} \left(\int_{tS} u^{1-p'}(\mathbf{x})d\mathbf{x}\right)^{-\frac{1}{p}} \times \left(\int_{tS} v(\mathbf{x}) \left(\int_{S_{\mathbf{x}}} u^{1-p'}(\mathbf{y})d\mathbf{y}\right)^q d\mathbf{x}\right)^{\frac{1}{q}} < \infty. \tag{2.2}$$

Moreover, the best constant C in (2.1) can be estimated as follows:

$$D \le C \le p'D.$$

Remark 2.1. *Another weight characterization of (2.1) over balls in $\mathbb{R}^N$ was proved by P. Drábek, H.P. Heinig and A. Kufner [3] . This result may be regarded as a generalization of the usual (Muckenhaupt type) characterization in 1-dimension (see e.g. [13]) while our result may be seen as a higher dimensional version of another characterization by V.D. Stepanov and L.E. Persson (see [19] , [20]).*

Proof. By the duality principle (see e.g. [13]), it can be shown that the inequality (2.1) is equivalent to that the inequality

$$\left(\int_E \left(\int_{E\setminus S_{\mathbf{x}}} g(\mathbf{y})dy\right)^{p'} u^{1-p'}(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p'}} \leq C\left(\int_E g^{q'}(\mathbf{x})v^{1-q'}(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{q'}} \tag{2.3}$$

holds for all $g \geq 0$ and with the same best constant C. First assume that (2.2) holds. Using polar coordinates and putting

$$\widetilde{g}(t) = \int_A g(t\sigma)t^{N-1}d\sigma, \qquad t \in (0,\infty) \tag{2.4}$$

and

$$\widetilde{u}(t) = \left(\int_A u^{1-p'}(t\tau)t^{N-1}d\tau\right)^{1-p}, \qquad t \in (0,\infty) \tag{2.5}$$

we have

$$\begin{aligned}\int_E \left(\int_{E\setminus S_x} g(\mathbf{y})d\mathbf{y}\right)^{p'} u^{1-p'}(\mathbf{x})d\mathbf{x} \\ &= \int_0^\infty \int_A \left(\int_t^\infty \int_A g(s\sigma)s^{N-1}d\sigma ds\right)^{p'} u^{1-p'}(t\tau)t^{N-1}d\tau dt \\ &= \int_0^\infty \left(\int_t^\infty \widetilde{g}(s)ds\right)^{p'} \widetilde{u}^{1-p'}(t)dt.\end{aligned}$$

Thus, using this, changing the order of integration and finally using Hölder's inequality, we get

$$I := \int_E \left(\int_{E \setminus S_{\mathbf{x}}} g(\mathbf{y}) d\mathbf{y} \right)^{p'} u^{1-p'}(\mathbf{x}) d\mathbf{x} \tag{2.6}$$
$$= \int_0^\infty \left(\int_t^\infty \widetilde{g}(s) ds \right)^{p'} \widetilde{u}^{1-p'}(t) dt$$
$$= \int_0^\infty \left(\int_z^\infty -\frac{d}{dt} \left(\int_t^\infty \widetilde{g}(s) ds \right)^{p'} dt \right) \widetilde{u}^{1-p'}(z) dz$$
$$= p' \int_0^\infty \left(\int_z^\infty \left(\int_t^\infty \widetilde{g}(s) ds \right)^{p'-1} \widetilde{g}(t) dt \right) \widetilde{u}^{1-p'}(z) dz$$
$$= p' \int_0^\infty \left(\int_t^\infty \widetilde{g}(s) ds \right)^{p'-1} \widetilde{g}(t) \left(\int_0^t \widetilde{u}^{1-p'}(z) dz \right) dt$$
$$= p' \int_0^\infty \int_A \left(\int_t^\infty \widetilde{g}(s) ds \right)^{p'-1} \left(\int_0^t \widetilde{u}^{1-p'}(s) ds \right) g(t\tau) t^{N-1} d\tau dt$$
$$\leq p' \left(\int_0^\infty \int_A g^{q'}(t\tau) v^{1-q'}(t\tau) t^{N-1} d\tau dt \right)^{\frac{1}{q'}}$$
$$\times \left(\int_0^\infty \int_A \left(\int_t^\infty \widetilde{g}(s) ds \right)^{(p'-1)q}\right.$$
$$\left. \times \left(\int_0^t \widetilde{u}^{1-p'}(s) ds \right)^q v(t\tau) t^{N-1} d\tau dt \right)^{\frac{1}{q}}$$
$$= p' \left(\int_E g^{q'}(\mathbf{x}) v^{1-q'}(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q'}} J^{\frac{1}{q}},$$

where

$$J=\int_0^\infty\left(\int_t^\infty \widetilde{g}(s)ds\right)^{(p'-1)q}\left(\int_0^t \widetilde{u}^{1-p'}(s)ds\right)^q \widetilde{v}(t)dt$$

with

$$\widetilde{v}(t)=\int_A v(t\tau)t^{N-1}d\tau. \tag{2.7}$$

Using Fubini's theorem, (2.2), (2.5) and (2.7), we get

$$\begin{aligned}
J&=\int_0^\infty\int_t^\infty \frac{d}{dz}\left(-\left(\int_z^\infty \widetilde{g}(s)ds\right)^{(p'-1)q}\right)dz\left(\int_0^t \widetilde{u}^{1-p'}(s)ds\right)^q \widetilde{v}(t)dt\\
&=\int_0^\infty\left[\frac{d}{dz}\left(-\left(\int_z^\infty \widetilde{g}(s)ds\right)^{(p'-1)q}\right)\right]\int_0^z\left(\int_0^t \widetilde{u}^{1-p'}(s)ds\right)^q \widetilde{v}(t)dtdz\\
&=\int_0^\infty\left[\frac{d}{dz}\left(-\left(\int_z^\infty \widetilde{g}(s)ds\right)^{(p'-1)q}\right)\right]\\
&\qquad\times\left(\int_0^z\int_A\left(\int_0^t\int_A u^{1-p'}(s\sigma)s^{N-1}d\sigma ds\right)^q v(t\tau)t^{N-1}d\tau dt\right)dz\\
&=\int_0^\infty\left[\frac{d}{dz}\left(-\left(\int_z^\infty \widetilde{g}(s)ds\right)^{(p'-1)q}\right)\right]\\
&\qquad\times\left(\int_{zS}\left(\int_{S_{\mathbf{x}}} u^{1-p'}(\mathbf{y})d\mathbf{y}\right)^q v(\mathbf{x})d\mathbf{x}\right)dz
\end{aligned}$$

$$\leq D^q \int_0^\infty \left[\frac{d}{dz}\left(-\left(\int_z^\infty \widetilde{g}(s)ds\right)^{(p'-1)q}\right)\right]\left(\int_{zS} u^{1-p'}(\mathbf{x})d\mathbf{x}\right)^{\frac{q}{p}} dz$$
$$= D^q \int_0^\infty \left[\frac{d}{dz}\left(-\left(\int_z^\infty \widetilde{g}(s)ds\right)^{(p'-1)q}\right)\right]\left(\int_0^z \widetilde{u}^{1-p'}(t)dt\right)^{\frac{q}{p}} dz.$$

Thus, using Minkowski's integral inequality, (2.4) and (2.5) we have

$$J \leq D^q \left(\int_0^\infty \left(\int_t^\infty \left[\frac{d}{dz}\left(-\left(\int_z^\infty \widetilde{g}(s)ds\right)^{(p'-1)q}\right)\right] dz\right)^{\frac{p}{q}} \widetilde{u}^{1-p'}(t)dt\right)^{\frac{q}{p}}$$
$$= D^q \left(\int_0^\infty \left(\int_t^\infty \widetilde{g}(s)ds\right)^{p'} \widetilde{u}^{1-p'}(t)dt\right)^{\frac{q}{p}}$$
$$= D^q \left(\int_E \left(\int_{E\backslash S_{\mathbf{x}}} g(\mathbf{y})d\mathbf{y}\right)^{p'} u^{1-p'}(\mathbf{x})d\mathbf{x}\right)^{\frac{q}{p}}.$$

Assume first that in (2.6) $I < \infty$. Then

$$\left(\int_E \left(\int_{E\backslash S_{\mathbf{x}}} g(\mathbf{y})d\mathbf{y}\right)^{p'} u^{1-p'}(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p'}} \leq p'D\left(\int_E g^{q'}(\mathbf{x})v^{1-q'}(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{q'}}$$

i.e., (2.3) holds for all $g \geq 0$ and also the constant C in (2.3) satisfies $C \leq p'D$. For the case $I = \infty$ replace $g(\mathbf{y})$ by an approximating sequence $g_n(\mathbf{y}) \leq g(\mathbf{y})$ (such that the corresponding $I_n < \infty$) and use the Monotone Convergence Theorem to obtain the result.

Conversely, suppose that (2.1) holds for all $f \geq 0$. In this inequality, taking for any fixed $t > 0$ the function $f_t = \chi_{tS} u^{1-p'}$, we find that

$$C \geq \left(\int_E \left(\int_{S_{\mathbf{x}}} f_t(\mathbf{y}) d\mathbf{y} \right)^q v(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} \left(\int_E f_t^p(\mathbf{x}) u(\mathbf{x}) d\mathbf{x} \right)^{-\frac{1}{p}}$$

$$\geq \left(\int_{tS} \left(\int_{S_{\mathbf{x}}} u^{1-p'}(\mathbf{y}) d\mathbf{y} \right)^q v(\mathbf{x}) dx \right)^{\frac{1}{q}} \left(\int_{tS} u^{1-p'}(\mathbf{x}) d\mathbf{x} \right)^{-\frac{1}{p}}.$$

By taking the supremum we find that (2.2) holds and, moreover, $D \leq C$. The proof is complete. □

3. Geometric Mean Inequalities

Here we prove our main geometric mean inequality by making a limit procedure in Theorem 2.1.

Theorem 3.1. *Let $0 < p \le q < \infty$ and suppose that all other assumptions of Theorem 2.1 are satisfied. Then the inequality*

$$\left(\int_E \left(\exp\left(\frac{1}{|S_{\mathbf{x}}|}\int_{S_x} \ln f(\mathbf{y})d\mathbf{y}\right)\right)^q v(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{q}} \le C\left(\int_E f^p(\mathbf{x})u(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}} \tag{3.1}$$

holds for all $f > 0$ if and only if

$$D_1 := \sup_{t>0} |tS|^{-\frac{1}{p}}\left(\int_{tS} w(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{q}} < \infty,$$

where

$$w(\mathbf{t}) := v(\mathbf{x})\left(\exp\left(\frac{1}{|S_{\mathbf{x}}|}\int_{S_{\mathbf{x}}} \ln\frac{1}{u(\mathbf{y})}d\mathbf{y}\right)\right)^{\frac{q}{p}} < \infty. \tag{3.2}$$

Moreover, the best constant C satisfies $D_1 \le C \le e^{\frac{1}{p}}D_1$.

Proof. It is easy to see that (3.1) is equivalent to

$$\left(\int_E \left(\exp\left(\frac{1}{|S_{\mathbf{x}}|}\int_{S_x} \ln f(\mathbf{y})d\mathbf{y}\right)\right)^q w(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{q}} \le C\left(\int_E f^p(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}}$$

with $w(\mathbf{x})$ defined by (3.2). Let $v(\mathbf{x}) = w(\mathbf{x})\,|S_{\mathbf{x}}|^{-q}$ and $u(\mathbf{x}) = 1$ in Theorem 2.1 and choose an α such that $0 < \alpha < p \leq q < \infty$. Then $1 < \frac{p}{\alpha} \leq \frac{q}{\alpha} < \infty$. Now, replacing f, p, q and $v(\mathbf{x})$ by $f^{\alpha}, \frac{p}{\alpha}, \frac{q}{\alpha}$ in Theorem 2.1, we find that the inequality

$$\left(\int_E \left(\frac{1}{|S_{\mathbf{x}}|} \int_{S_{\mathbf{x}}} f^{\alpha}(\mathbf{y})d\mathbf{y} \right)^{\frac{q}{\alpha}} w(\mathbf{x})d\mathbf{x} \right)^{\frac{1}{q}} \leq C_{\alpha} \left(\int_E f^{p}(\mathbf{x})d\mathbf{x} \right)^{\frac{1}{p}} \tag{3.3}$$

holds for all functions $f > 0$ if and only if D_1 holds. Moreover, it is easy to see that (c.f. [20])

$$D_1 \leq C_{\alpha} \leq \left(\frac{p}{p-\alpha} \right)^{\frac{1}{\alpha}} D_1. \tag{3.4}$$

By letting $\alpha \to 0^{+}$ in (3.3) and (3.4) we find that $\left(\frac{p}{p-\alpha} \right)^{\frac{1}{\alpha}} \to e^{\frac{1}{p}}$ and

$$\left(\frac{1}{|S_{\mathbf{x}}|} \int_{S_{\mathbf{x}}} f^{\alpha}(\mathbf{y})d\mathbf{y} \right)^{\frac{1}{\alpha}} \to \exp \left(\frac{1}{|S_{\mathbf{x}}|} \int_{S_{\mathbf{x}}} \ln f(\mathbf{y})d\mathbf{y} \right),$$

i.e. the scale of power means converge to the geometric mean, and the proof follows. □

Remark 3.1. *Our proof above shows that (3.1) in Theorem 3.1 may be regarded as a natural limiting case of Hardy's inequality (2.1) as it is in the classical one-dimensional situation. This fact indicates that our formulation of Hardy's inequality in Theorem 2.1 is very natural from this point of view.*

As a special case, if we take $E = \mathbb{R}^N$ and $S_{\mathbf{x}} = B_{\mathbf{x}}$ the ball centered at the origin and with radius $|\mathbf{x}|$, and $|B_x|$ its volume, then we immediately obtain the following corollary to Theorem 3.1 that averages functions over balls in $\mathbb{R}^N$:

Corollary 3.2. *Let $0 < p \le q < \infty$ and u, v be weight functions in $\mathbb{R}^N$. Then the inequality*

$$\left(\int_{\mathbb{R}^N}\left(\exp\left(\frac{1}{|B_{\mathbf{x}}|}\int_{B_{\mathbf{x}}}\ln f(\mathbf{y})d\mathbf{y}\right)\right)^q v(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^N} f^p(\mathbf{x})u(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}}$$

holds for all $f > 0$ if and only if

$$D_2 := \sup_{\mathbf{z}\in\mathbb{R}^N\setminus\{0\}} |B_{\mathbf{z}}|^{-\frac{1}{p}}\left(\int_{B_{\mathbf{z}}} v(\mathbf{x})\left(\exp\left(\frac{1}{|B_{\mathbf{x}}|}\int_{B_{\mathbf{x}}}\ln\frac{1}{u(\mathbf{y})}d\mathbf{y}\right)\right)^{\frac{q}{p}} d\mathbf{x}\right)^{\frac{1}{q}} < \infty.$$

Moreover, the best constant C satisfies $D_2 \le C \le e^{\frac{1}{p}} D_2$.

Remark 3.2. *Corollary 3.2 extends a result of P. Drábek, H.P. Heinig and A. Kufner [3, Theorem 4.1], who obtained it for the case $p = q = 1$ and with a completely different proof.*

Remark 3.3. *Setting $E = \mathbb{R}^N_+ = \{(x_1, \dots, x_N) \in \mathbb{R}^N,\ x_1 \ge 0, \dots, x_N \ge 0\}$ in Theorem 3.1 we obtain that Corollary 3.2 holds also for $\mathbb{R}^N_+$ instead of $\mathbb{R}^N$ and $B_{\mathbf{x}} \cap \mathbb{R}^N_+$ instead of $B_{\mathbf{x}}$.*

We shall now consider the special weights discussed in our introduction and in [1].

Proposition 3.3. *Let $0 < p \leq q < \infty$, $a, b \in \mathbb{R}$, $\varepsilon \in \mathbb{R}_+$, and $E, S_{\mathbf{x}}$ be defined as in Theorem 2.1. Then*

$$\left(\int_E \left[\exp\left(\varepsilon \left|S_{\mathbf{x}}\right|^{-\varepsilon} \int_{S_{\mathbf{x}}} \left|S_{\mathbf{y}}\right|^{\varepsilon-1} \ln f(\mathbf{y}) d\mathbf{y}\right)\right]^q \left|S_{\mathbf{x}}\right|^a d\mathbf{x}\right)^{\frac{1}{q}} \leq C \left(\int_E f^p(\mathbf{x}) \left|S_{\mathbf{x}}\right|^b d\mathbf{x}\right)^{\frac{1}{p}} \tag{3.5}$$

holds for all positive functions f for some finite constant C if and only if

$$\frac{a+1}{q} = \frac{b+1}{p} \tag{3.6}$$

and the least constant C in (3.5) satisfies

$$\left(\frac{p}{q}\right)^{\frac{1}{q}} \varepsilon^{\frac{1}{p}-\frac{1}{q}} e^{\frac{b+1}{\varepsilon p}-\frac{1}{p}} \leq C \leq \left(\frac{p}{q}\right)^{\frac{1}{q}} \varepsilon^{\frac{1}{p}-\frac{1}{q}} e^{\frac{b+1}{\varepsilon p}}.$$

Proof. By writing (3.5) in polar coordinates we find that

$$\left(\int_0^\infty \int_A \left[\exp \frac{\varepsilon N^\varepsilon}{t^{N\varepsilon} \left|A\right|^\varepsilon} \times \int_0^t \int_A \left(\frac{|A|}{N}\right)^{\varepsilon-1} s^{N\varepsilon-1} \ln f(s\sigma) d\sigma ds\right]^q t^{Na+N-1} \left(\frac{|A|}{N}\right)^a d\tau dt\right)^{\frac{1}{q}}$$
$$\leq \left(\int_0^\infty \int_A f^p(t\tau) \left(\frac{|A|}{N}\right)^b t^{Nb+N-1} d\tau dt\right)^{\frac{1}{p}}.$$

Exchanging variables, $s = r^{\frac{1}{\varepsilon}}$ and $t = z^{\frac{1}{\varepsilon}}$ we find that this inequality can be rewritten as

$$\left(\int_0^\infty \int_A \left(\exp\left(\frac{N}{|A|\, z^N} \int_0^z \int_A \ln f\left(r^{\frac{1}{\varepsilon}}\sigma\right) r^{N-1} d\sigma dr\right)\right)^q \right.$$
$$\left. \times \left(\frac{|A|}{N}\right)^a z^{N\left(\frac{a+1}{\varepsilon}-1\right)} z^{N-1} \frac{1}{\varepsilon} d\tau dz\right)^{\frac{1}{q}}$$
$$\leq C \left(\int_0^\infty \int_A f^p\left(z^{\frac{1}{\varepsilon}}\tau\right) \left(\frac{|A|}{N}\right)^b z^{N\left(\frac{b+1}{\varepsilon}-1\right)} z^{N-1} \frac{1}{\varepsilon} d\tau dz\right)^{\frac{1}{p}},$$

that is,

$$\left(\int_E \left(\exp\left(\frac{1}{|S_{\mathbf{x}}|} \int_{S_{\mathbf{x}}} \ln f_1(\mathbf{y}) d\mathbf{y}\right)\right)^q |S_{\mathbf{x}}|^{\frac{a+1}{\varepsilon}-1} d\mathbf{x}\right)^{\frac{1}{q}}$$
$$\leq C \left(\frac{|A|}{N}\right)^{\left(\frac{b+1}{p}-\frac{a+1}{q}\right)\left(1-\frac{1}{\varepsilon}\right)} \varepsilon^{\frac{1}{q}-\frac{1}{p}} \left(\int_E f_1^p(\mathbf{x})\, |S_{\mathbf{x}}|^{\frac{b+1}{\varepsilon}-1} d\mathbf{x}\right)^{\frac{1}{p}}, \tag{3.7}$$

where $f_1(r\sigma) = f(r^{\frac{1}{\varepsilon}}\sigma)$. This means that (3.5) is equivalent to (3.7) i.e., (3.1) holds with the weights $v(x) = |S_x|^{\frac{a+1}{\varepsilon}-1}$ and $u(x) = |S_x|^{\frac{b+1}{\varepsilon}-1}$. We note that for these weights we find after a direct calculation that the constant D_1 from Theorem 3.1 is

$$D_1 = \sup_{t>0} \frac{|tS|^{\frac{a+1}{\varepsilon q}-\frac{b+1}{\varepsilon p}} e^{\frac{1}{p}\left(\frac{b+1}{\varepsilon}-1\right)}}{\left(\frac{a+1}{\varepsilon} - \frac{q}{p}\left(\frac{b+1}{\varepsilon}-1\right)\right)^{\frac{1}{q}}}$$

so we conclude that (3.6) must hold and then

$$D_1 = e^{\frac{1}{p}\left(\frac{b+1}{\varepsilon}-1\right)} \left(\frac{p}{q}\right)^{\frac{1}{q}}.$$

Thus, the proof follows from Theorem 3.1. □

Remark 3.4. *Setting $p = q = 1$, $a = b$, we have that (3.5) implies the estimate (1.2).*

Remark 3.5 (Sharp Constant). *In the above proposition, if we take $p = q$, then $a = b$. In this situation (3.5) holds with the constant $C = e^{(b+1)/p}$. Indeed, this constant is sharp. In order to show this for $\delta > 0$, we consider the function*

$$f_\delta(x) = \begin{cases} e^{-\frac{b+1}{\varepsilon p}} |S|^{-(b+1)} |\mathbf{x}|^{-\frac{N}{p}(b+1-\varepsilon\delta)}, & x \in S, \\ e^{-\frac{b+1}{\varepsilon p}} |S|^{-(b+1)} |\mathbf{x}|^{-\frac{N}{p}(b+1+\varepsilon\delta)}, & x \in E\backslash S. \end{cases}$$

By using this function in (3.5), we find that

$$1 \le \frac{RHS}{LHS} \le e^{\frac{\delta}{p}} \to 1 \qquad \textit{as} \qquad \delta \to 0$$

and consequently the constant is sharp. Note that the sharpness of the constant for $p = q$, in Proposition 3.3 has been proved in the more general setting than that in [1].

4. The Companion Inequalities

We present the following result which is a companion of Theorem 3.1:

Theorem 4.1. *Let $0 < p \leq q < \infty$, $\varepsilon > 0$, and suppose that all other hypotheses of Theorem 3.1 are satisfied. Then the inequality*

$$\left(\int_E \left(\exp\left(\varepsilon\,|S_{\mathbf{x}}|^{\varepsilon}\int_{E\setminus S_{\mathbf{x}}}|S_{\mathbf{y}}|^{-\varepsilon-1}\ln f(\mathbf{y})d\mathbf{y}\right)\right)^q v(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{q}} \leq C\left(\int_E f^p(\mathbf{x})u(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}} \tag{4.1}$$

holds for all $f > 0$ if and only if

$$D_3 := \sup_{t>0}|tS|^{-\frac{1}{p}}\left(\int_{tS} v_*(\mathbf{x})\left(\exp\left(\frac{1}{|S_{\mathbf{x}}|}\int_{S_{\mathbf{x}}}\ln\frac{1}{u_*(\mathbf{y})}d\mathbf{y}\right)\right)^{\frac{q}{p}}d\mathbf{x}\right)^{\frac{1}{q}} < \infty,$$

where

$$u_*(\mathbf{y}) := u(s^{-\frac{1}{\varepsilon}}\sigma)\frac{1}{\varepsilon}s^{-N\left(1+\frac{1}{\varepsilon}\right)}, \quad v_*(\mathbf{y}) := v(s^{-\frac{1}{\varepsilon}}\sigma)\frac{1}{\varepsilon}s^{-N\left(1+\frac{1}{\varepsilon}\right)}.$$

Moreover, the constant C satisfies $D_3 \leq C \leq e^{\frac{1}{p}}D_3$.

Proof. Note that for $x \in \mathbb{R}^N$

$$|S_{\mathbf{x}}| = \int_0^{|\mathbf{x}|}\int_A t^{N-1}d\tau dt = \frac{|\mathbf{x}|^N}{N}|A|.$$

Now, using polar coordinates, (4.1) can be written as

$$\left(\int_0^\infty \int_A \left(\exp \frac{\varepsilon |A|^\varepsilon t^{N\varepsilon}}{N} \times \int_t^\infty \int_A \left(\frac{|A|}{N}\right)^{-\varepsilon-1} s^{-N\varepsilon-1} \ln f(s\sigma) d\sigma ds\right)^q v(t\tau) t^{N-1} d\tau dt\right)^{\frac{1}{q}} \leq C \left(\int_0^\infty \int_A f^p(t\tau) u(t\tau) t^{N-1} d\tau dt\right)^{\frac{1}{p}}.$$

Using the exchange of variables $s = r^{-1/\varepsilon}$ and $t = z^{-1/\varepsilon}$ we obtain

$$\left(\int_0^\infty \int_A \left[\exp\left(\frac{N}{|A|\, z^N} \int_A \int_0^z \ln f(r^{-\frac{1}{\varepsilon}}\sigma) r^{N-1} d\sigma dr\right)\right]^q \times v(z^{-\frac{1}{\varepsilon}}\tau) z^{-N(1+\frac{1}{\varepsilon})} \frac{1}{\varepsilon} z^{N-1} d\tau dz\right)^{\frac{1}{q}} \leq C \left(\int_0^\infty \int_A f^p(z^{-\frac{1}{\varepsilon}}\tau) u(z^{-\frac{1}{\varepsilon}}\tau) z^{-N(1+\frac{1}{\varepsilon})} \frac{1}{\varepsilon} z^{N-1} d\tau dz\right)^{\frac{1}{p}}$$

and put $f_*(t\tau) = f(t^{-\frac{1}{\varepsilon}}\tau)$. (4.1) can be equivalently rewritten as

$$\left(\int_E \left(\exp\left(\frac{1}{|S_\mathbf{x}|} \int_{S_\mathbf{x}} \ln f_*(\mathbf{y}) d\mathbf{y}\right)\right)^q v_*(\mathbf{x}) d\mathbf{x}\right)^{\frac{1}{q}} \leq C \left(\int_E f_*^p(\mathbf{x}) u_*(\mathbf{x}) d\mathbf{x}\right)^{\frac{1}{p}}.$$

Now, the result is obtained by using Theorem 3.1. □

Analogously to Corollary 3.2, we can immediately obtain a special case of Theorem 4.1 that averages functions over balls in $\mathbb{R}^N$ centered at origin.

Corollary 4.2. *Let* $0 < p \le q < \infty$, $\varepsilon > 0$, *and* u, v *be weight functions in* $\mathbb{R}^N$. *Then the inequality*

$$\left(\int_{\mathbb{R}^N}\left(\exp\left(\varepsilon\,|B_{\mathbf{x}}|^{\varepsilon}\int_{\mathbb{R}^N\setminus B_{\mathbf{x}}}|B_{\mathbf{y}}|^{-\varepsilon-1}\ln f(\mathbf{y})d\mathbf{y}\right)\right)^{q} v(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^N} f^p(\mathbf{x})u(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}} \tag{4.2}$$

holds for all $f > 0$ *if and only if*

$$\widetilde{B} := \sup_{z\in\mathbb{R}^N}|B_{\mathbf{z}}|^{-\frac{1}{p}}\left(\int_{B_{\mathbf{z}}} v_0(\mathbf{x})\left(\exp\left(\frac{1}{|B_{\mathbf{x}}|}\int_{B_{\mathbf{x}}}\ln\frac{1}{u_0}(\mathbf{y})d\mathbf{y}\right)\right)^{\frac{q}{p}} d\mathbf{x}\right)^{\frac{1}{q}} < \infty,$$

where

$$u_0(\mathbf{x}) := u(t^{-\frac{1}{\varepsilon}}\tau)\frac{1}{\varepsilon}t^{-N(1+\frac{1}{\varepsilon})}, \qquad v_0(\mathbf{x}) := v(t^{-\frac{1}{\varepsilon}}\tau)\frac{1}{\varepsilon}t^{-N(1+\frac{1}{\varepsilon})}.$$

Moreover, the best constant C *satisfies* $\widetilde{B} \le C \le e^{\frac{1}{p}}\widetilde{B}$.

Remark 4.1. *Note that by choosing* E *as in Remark 3.3 we see that Corollary 4.2 in fact holds also when* $\mathbb{R}^N$ *is replaced by* $\mathbb{R}^N_+$ *or more general cones in* $\mathbb{R}^N$.

The corresponding result to Proposition 3.3 reads as follows and the proof is analogous.

Proposition 4.3. *Let $0 < p \le q < \infty$, $\varepsilon > 0$, and $a, b \in \mathbb{R}$, and E, S_x be defined as in Theorem 2.1. Then the inequality*

$$\left(\int_E \left(\exp \varepsilon \left|S_{\mathbf{x}}\right|^{\varepsilon} \int_{E \setminus S_{\mathbf{x}}} \left|S_{\mathbf{y}}\right|^{-\varepsilon-1} \ln f(\mathbf{y}) d\mathbf{y}\right)^q \left|S_{\mathbf{x}}\right|^a d\mathbf{x}\right)^{\frac{1}{q}} \le C \left(\int_E f^p(\mathbf{x}) \left|S_{\mathbf{x}}\right|^b d\mathbf{x}\right)^{\frac{1}{p}} \tag{4.3}$$

holds for all $f > 0$ and some finite positive constant C if and only if

$$\frac{a+1}{q} = \frac{b+1}{p}$$

and the least constant C in (4.3) satisfies

$$\left(\frac{p}{q}\right)^{-\frac{1}{q}} \varepsilon^{\frac{1}{p}-\frac{1}{q}} e^{-\left(\frac{b+1}{\varepsilon p}+\frac{1}{p}\right)} \le C \le \left(\frac{p}{q}\right)^{-\frac{1}{q}} \varepsilon^{\frac{1}{p}-\frac{1}{q}} e^{-\frac{b+1}{\varepsilon p}}.$$

Remark 4.2 (Sharp Constant). *Analogously to Proposition 3.3, in the above proposition we also find that if we take $p = q$, then $a = b$. In this situation (4.3) holds with the constant $C = e^{-(b+1)/\varepsilon p}$ and the constant is sharp. This can be shown by considering, for $\delta > 0$, the function*

$$f_\delta(\mathbf{x}) = \begin{cases} e^{\frac{b+1}{\varepsilon p}} \left|S\right|^{-(b+1)} \left|\mathbf{x}\right|^{-\frac{N}{p}(b+1-\varepsilon\delta)}, & \mathbf{x} \in S \\ e^{\frac{b+1}{p}} \left|S\right|^{-(b+1)} \left|\mathbf{x}\right|^{-\frac{N}{p}(b+1+\varepsilon\delta)}, & \mathbf{x} \in E \setminus S. \end{cases}$$

Remark 4.3. *It is tempting to think that the results in this paper hold also in general star-shaped regions in* $\mathbb{R}^N$ *(c.f. [22]) but this is not true in general as was pointed out to us by the referee. See also [22] and note that the results there also hold at least for cones in* $\mathbb{R}^N$*.*

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