Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 2, Article 46, 2005

REVERSES OF THE CONTINUOUS TRIANGLE INEQUALITY FOR BOCHNER INTEGRAL OF VECTOR-VALUED FUNCTIONS IN HILBERT SPACES

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Received 07 June, 2004; accepted 15 April, 2005
Communicated by R.N. Mohapatra

ABSTRACT. Some reverses of the continuous triangle inequality for Bochner integral of vector-valued functions in Hilbert spaces are given. Applications for complex-valued functions are provided as well.

Key words and phrases: Triangle inequality, Reverse inequality, Hilbert spaces, Bochner integral.

2000 *Mathematics Subject Classification.* 46C05, 26D15, 26D10.

1. INTRODUCTION

Let $f : [a, b] \to \mathbb{K}$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ be a Lebesgue integrable function. The following inequality, which is the continuous version of the *triangle inequality*,

$$\left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx, \tag{1.1}$$

plays a fundamental role in Mathematical Analysis and its applications.

It appears, see [4, p. 492], that the first reverse inequality for (1.1) was obtained by J. Karamata in his book from 1949, [2]. It can be stated as

$$\cos\theta \int_a^b |f(x)|\, dx \le \left| \int_a^b f(x)\, dx \right|, \tag{1.2}$$

provided

$$-\theta \le \arg f(x) \le \theta, \quad x \in [a, b]$$

for given $\theta \in \left(0, \frac{\pi}{2}\right)$.

ISSN (electronic): 1443-5756

115-04

This integral inequality is the continuous version of a reverse inequality for the generalised triangle inequality

$$\cos\theta\sum_{i=1}^{n}|z_i|\leq\left|\sum_{i=1}^{n}z_i\right|, \tag{1.3}$$

provided

$$a-\theta\leq\arg(z_i)\leq a+\theta,\quad\text{for }\ i\in\{1,\dots,n\},$$

where $a\in\mathbb{R}$ and $\theta\in\left(0,\frac{\pi}{2}\right)$, which, as pointed out in [4, p. 492], was first discovered by M. Petrovich in 1917, [5], and, subsequently rediscovered by other authors, including J. Karamata [2, p. 300 – 301], H.S. Wilf [6], and in an equivalent form, by M. Marden [3].

The first to consider the problem in the more general case of Hilbert and Banach spaces, were J.B. Diaz and F.T. Metcalf [1] who showed that, in an inner product space H over the real or complex number field, the following reverse of the triangle inequality holds

$$r\sum_{i=1}^{n}\|x_i\|\leq\left\|\sum_{i=1}^{n}x_i\right\|, \tag{1.4}$$

provided

$$0\leq r\leq\frac{\operatorname{Re}\langle x_i,a\rangle}{\|x_i\|},\qquad i\in\{1,\dots,n\},$$

and $a\in H$ is a unit vector, i.e., $\|a\|=1$. The case of equality holds in (1.4) if and only if

$$\sum_{i=1}^{n}x_i=r\left(\sum_{i=1}^{n}\|x_i\|\right)a. \tag{1.5}$$

The main aim of this paper is to point out some reverses of the triangle inequality for Bochner integrable functions f with values in Hilbert spaces and defined on a compact interval $[a,b]\subset\mathbb{R}$. Applications for Lebesgue integrable complex-valued functions are provided as well.

2. Reverses for a Unit Vector

We recall that $f\in L([a,b];H)$, the space of Bochner integrable functions with values in a Hilbert space H, if and only if $f:[a,b]\to H$ is Bochner measurable on $[a,b]$ and the Lebesgue integral $\int_a^b\|f(t)\|\,dt$ is finite.

The following result holds:

Theorem 2.1. *If $f\in L([a,b];H)$ is such that there exists a constant $K\geq1$ and a vector $e\in H$, $\|e\|=1$ with*

$$\|f(t)\|\leq K\operatorname{Re}\langle f(t),e\rangle\quad\textit{for a.e. }t\in[a,b], \tag{2.1}$$

then we have the inequality:

$$\int_a^b\|f(t)\|\,dt\leq K\left\|\int_a^b f(t)\,dt\right\|. \tag{2.2}$$

The case of equality holds in (2.2) if and only if

$$\int_a^b f(t)\,dt=\frac{1}{K}\left(\int_a^b\|f(t)\|\,dt\right)e. \tag{2.3}$$

Proof. By the Schwarz inequality in inner product spaces, we have

$$\begin{aligned}(2.4)\qquad \left\|\int_a^b f(t)\,dt\right\| &= \left\|\int_a^b f(t)\,dt\right\|\|e\| \\ &\geq \left|\left\langle \int_a^b f(t)\,dt, e\right\rangle\right| \\ &\geq \left|\operatorname{Re}\left\langle \int_a^b f(t)\,dt, e\right\rangle\right| \\ &\geq \operatorname{Re}\left\langle \int_a^b f(t)\,dt, e\right\rangle = \int_a^b \operatorname{Re}\langle f(t), e\rangle\,dt.\end{aligned}$$

From the condition (2.1), on integrating over $[a,b]$, we deduce

$$(2.5)\qquad \int_a^b \operatorname{Re}\langle f(t), e\rangle\,dt \geq \frac{1}{K}\int_a^b \|f(t)\|\,dt,$$

and thus, on making use of (2.4) and (2.5), we obtain the desired inequality (2.2).

If (2.3) holds true, then, obviously

$$K\left\|\int_a^b f(t)\,dt\right\| = \|e\|\int_a^b \|f(t)\|\,dt = \int_a^b \|f(t)\|\,dt,$$

showing that (2.2) holds with equality.

If we assume that the equality holds in (2.2), then by the argument provided at the beginning of our proof, we must have equality in each of the inequalities from (2.4) and (2.5).

Observe that in Schwarz's inequality $\|x\|\|y\| \geq \operatorname{Re}\langle x,y\rangle$, $x,y \in H$, the case of equality holds if and only if there exists a positive scalar μ such that $x = \mu e$. Therefore, equality holds in the first inequality in (2.4) iff $\int_a^b f(t)\,dt = \lambda e$, with $\lambda \geq 0$.

If we assume that a strict inequality holds in (2.1) on a subset of nonzero Lebesgue measures, then $\int_a^b \|f(t)\|\,dt < K\int_a^b \operatorname{Re}\langle f(t), e\rangle\,dt$, and by (2.4) we deduce a strict inequality in (2.2), which contradicts the assumption. Thus, we must have $\|f(t)\| = K\operatorname{Re}\langle f(t), e\rangle$ for a.e. $t \in [a,b]$.

If we integrate this equality, we deduce

$$\begin{aligned}\int_a^b \|f(t)\|\,dt &= K\int_a^b \operatorname{Re}\langle f(t), e\rangle\,dt \\ &= K\operatorname{Re}\left\langle \int_a^b f(t)\,dt, e\right\rangle \\ &= K\operatorname{Re}\langle \lambda e, e\rangle = \lambda K,\end{aligned}$$

giving

$$\lambda = \frac{1}{K}\int_a^b \|f(t)\|\,dt,$$

and thus the equality (2.3) is necessary.

This completes the proof. □

A more appropriate result from an applications point of view is perhaps the following result.

Corollary 2.2. *Let e be a unit vector in the Hilbert space $(H;\langle\cdot,\cdot\rangle)$, $\rho \in (0,1)$ and $f \in L([a,b];H)$ so that*

$$(2.6)\qquad \|f(t) - e\| \leq \rho \quad \text{for a.e. } t \in [a,b].$$

Then we have the inequality

$$\sqrt{1-\rho^{2}}\int_{a}^{b}\left\Vert f\left(t\right)\right\Vert dt\leq\left\Vert \int_{a}^{b}f\left(t\right)dt\right\Vert , \tag{2.7}$$

with equality if and only if

$$\int_{a}^{b}f\left(t\right)dt=\sqrt{1-\rho^{2}}\left(\int_{a}^{b}\left\Vert f\left(t\right)\right\Vert dt\right)\cdot e. \tag{2.8}$$

Proof. From (2.6), we have

$$\left\Vert f\left(t\right)\right\Vert ^{2}-2\operatorname{Re}\left\langle f\left(t\right),e\right\rangle +1\leq\rho^{2},$$

giving

$$\left\Vert f\left(t\right)\right\Vert ^{2}+1-\rho^{2}\leq2\operatorname{Re}\left\langle f\left(t\right),e\right\rangle$$

for a.e. $t\in\left[a,b\right]$.

Dividing by $\sqrt{1-\rho^{2}}>0$, we deduce

$$\frac{\left\Vert f\left(t\right)\right\Vert ^{2}}{\sqrt{1-\rho^{2}}}+\sqrt{1-\rho^{2}}\leq\frac{2\operatorname{Re}\left\langle f\left(t\right),e\right\rangle }{\sqrt{1-\rho^{2}}} \tag{2.9}$$

for a.e. $t\in\left[a,b\right]$.

On the other hand, by the elementary inequality

$$\frac{p}{\alpha}+q\alpha\geq2\sqrt{pq},\quad p,q\geq0,\ \alpha>0$$

we have

$$2\left\Vert f\left(t\right)\right\Vert \leq\frac{\left\Vert f\left(t\right)\right\Vert ^{2}}{\sqrt{1-\rho^{2}}}+\sqrt{1-\rho^{2}} \tag{2.10}$$

for each $t\in\left[a,b\right]$.

Making use of (2.9) and (2.10), we deduce

$$\left\Vert f\left(t\right)\right\Vert \leq\frac{1}{\sqrt{1-\rho^{2}}}\operatorname{Re}\left\langle f\left(t\right),e\right\rangle$$

for a.e. $t\in\left[a,b\right]$.

Applying Theorem 2.1 for $K=\frac{1}{\sqrt{1-\rho^{2}}}$, we deduce the desired inequality (2.7). □

In the same spirit, we also have the following corollary.

Corollary 2.3. *Let e be a unit vector in H and $M\geq m>0$. If $f\in L\left(\left[a,b\right];H\right)$ is such that*

$$\operatorname{Re}\left\langle Me-f\left(t\right),f\left(t\right)-me\right\rangle \geq0\quad\text{for a.e. }t\in\left[a,b\right], \tag{2.11}$$

or, equivalently,

$$\left\Vert f\left(t\right)-\frac{M+m}{2}e\right\Vert \leq\frac{1}{2}\left(M-m\right)\quad\text{for a.e. }t\in\left[a,b\right], \tag{2.12}$$

then we have the inequality

$$\frac{2\sqrt{mM}}{M+m}\int_{a}^{b}\left\Vert f\left(t\right)\right\Vert dt\leq\left\Vert \int_{a}^{b}f\left(t\right)dt\right\Vert , \tag{2.13}$$

or, equivalently,

$$\left(0\leq\right)\int_{a}^{b}\left\Vert f\left(t\right)\right\Vert dt-\left\Vert \int_{a}^{b}f\left(t\right)dt\right\Vert \leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^{2}}{M+m}\left\Vert \int_{a}^{b}f\left(t\right)dt\right\Vert . \tag{2.14}$$

The equality holds in (2.13) (or in the second part of (2.14)) if and only if

$$\int_a^b f(t)\,dt = \frac{2\sqrt{mM}}{M+m}\left(\int_a^b \|f(t)\|\,dt\right) e. \tag{2.15}$$

Proof. Firstly, we remark that if $x, z, Z \in H$, then the following statements are equivalent

(i) $\operatorname{Re}\langle Z - x, x - z\rangle \geq 0$
and
(ii) $\left\|x - \frac{Z+z}{2}\right\| \leq \frac{1}{2}\|Z - z\|$.

Using this fact, we may simply realise that (2.9) and (2.10) are equivalent.

Now, from (2.9), we obtain

$$\|f(t)\|^2 + mM \leq (M+m)\operatorname{Re}\langle f(t), e\rangle$$

for a.e. $t \in [a,b]$. Dividing this inequality with $\sqrt{mM} > 0$, we deduce the following inequality that will be used in the sequel

$$\frac{\|f(t)\|^2}{\sqrt{mM}} + \sqrt{mM} \leq \frac{M+m}{\sqrt{mM}}\operatorname{Re}\langle f(t), e\rangle \tag{2.16}$$

for a.e. $t \in [a,b]$.

On the other hand

$$2\|f(t)\| \leq \frac{\|f(t)\|^2}{\sqrt{mM}} + \sqrt{mM}, \tag{2.17}$$

for any $t \in [a,b]$.

Utilising (2.16) and (2.17), we may conclude with the following inequality

$$\|f(t)\| \leq \frac{M+m}{2\sqrt{mM}}\operatorname{Re}\langle f(t), e\rangle,$$

for a.e. $t \in [a,b]$.

Applying Theorem 2.1 for the constant $K := \frac{m+M}{2\sqrt{mM}} \geq 1$, we deduce the desired result. □

3. REVERSES FOR ORTHONORMAL FAMILIES OF VECTORS

The following result for orthonormal vectors in H holds.

Theorem 3.1. *Let $\{e_1, \ldots, e_n\}$ be a family of orthonormal vectors in H, $k_i \geq 0$, $i \in \{1, \ldots, n\}$ and $f \in L([a,b]; H)$ such that*

$$k_i\|f(t)\| \leq \operatorname{Re}\langle f(t), e_i\rangle \tag{3.1}$$

for each $i \in \{1, \ldots, n\}$ and for a.e. $t \in [a,b]$.

Then

$$\left(\sum_{i=1}^n k_i^2\right)^{\frac{1}{2}} \int_a^b \|f(t)\|\,dt \leq \left\|\int_a^b f(t)\,dt\right\|, \tag{3.2}$$

where equality holds if and only if

$$\int_a^b f(t)\,dt = \left(\int_a^b \|f(t)\|\,dt\right)\sum_{i=1}^n k_i e_i. \tag{3.3}$$

Proof. By Bessel's inequality applied for $\int_a^b f(t)\,dt$ and the orthonormal vectors $\{e_1, \dots, e_n\}$, we have

$$\left\|\int_a^b f(t)\,dt\right\|^2 \geq \sum_{i=1}^n \left|\left\langle \int_a^b f(t)\,dt, e_i\right\rangle\right|^2 \geq \sum_{i=1}^n \left[\operatorname{Re}\left\langle \int_a^b f(t)\,dt, e_i\right\rangle\right]^2 = \sum_{i=1}^n \left[\int_a^b \operatorname{Re}\langle f(t), e_i\rangle\,dt\right]^2. \tag{3.4}$$

Integrating (3.1), we get for each $i \in \{1, \dots, n\}$

$$0 \leq k_i \int_a^b \|f(t)\|\,dt \leq \int_a^b \operatorname{Re}\langle f(t), e_i\rangle\,dt,$$

implying

$$\sum_{i=1}^n \left[\int_a^b \operatorname{Re}\langle f(t), e_i\rangle\,dt\right]^2 \geq \sum_{i=1}^n k_i^2 \left(\int_a^b \|f(t)\|\,dt\right)^2. \tag{3.5}$$

On making use of (3.4) and (3.5), we deduce

$$\left\|\int_a^b f(t)\,dt\right\|^2 \geq \sum_{i=1}^n k_i^2 \left(\int_a^b \|f(t)\|\,dt\right)^2,$$

which is clearly equivalent to (3.2).

If (3.3) holds true, then

$$\left\|\int_a^b f(t)\,dt\right\| = \left(\int_a^b \|f(t)\|\,dt\right)\left\|\sum_{i=1}^n k_i e_i\right\| = \left(\int_a^b \|f(t)\|\,dt\right)\left[\sum_{i=1}^n k_i^2 \|e_i\|^2\right]^{\frac{1}{2}} = \left(\sum_{i=1}^n k_i^2\right)^{\frac{1}{2}} \int_a^b \|f(t)\|\,dt,$$

showing that (3.2) holds with equality.

Now, suppose that there is an $i_0 \in \{1, \dots, n\}$ for which

$$k_{i_0}\|f(t)\| < \operatorname{Re}\langle f(t), e_{i_0}\rangle$$

on a subset of nonzero Lebesgue measures enclosed in $[a, b]$. Then obviously

$$k_{i_0}\int_a^b \|f(t)\|\,dt < \int_a^b \operatorname{Re}\langle f(t), e_{i_0}\rangle\,dt,$$

and using the argument given above, we deduce

$$\left(\sum_{i=1}^n k_i^2\right)^{\frac{1}{2}} \int_a^b \|f(t)\|\,dt < \left\|\int_a^b f(t)\,dt\right\|.$$

Therefore, if the equality holds in (3.2), we must have

$$k_i\|f(t)\| = \operatorname{Re}\langle f(t), e_i\rangle \tag{3.6}$$

for each $i \in \{1, \ldots, n\}$ and a.e. $t \in [a, b]$.

Also, if the equality holds in (3.2), then we must have equality in all inequalities (3.4), this means that

$$\int_a^b f(t)\,dt = \sum_{i=1}^n \left\langle \int_a^b f(t)\,dt, e_i \right\rangle e_i \tag{3.7}$$

and

$$\operatorname{Im} \left\langle \int_a^b f(t)\,dt, e_i \right\rangle = 0 \text{ for each } i \in \{1, \ldots, n\}. \tag{3.8}$$

Using (3.6) and (3.8) in (3.7), we deduce

$$\begin{aligned} \int_a^b f(t)\,dt &= \sum_{i=1}^n \operatorname{Re} \left\langle \int_a^b f(t)\,dt, e_i \right\rangle e_i \\ &= \sum_{i=1}^n \int_a^b \operatorname{Re} \langle f(t), e_i \rangle e_i dt \\ &= \sum_{i=1}^n \left(\int_a^b \|f(t)\|\,dt \right) k_i e_i \\ &= \int_a^b \|f(t)\|\,dt \sum_{i=1}^n k_i e_i, \end{aligned}$$

and the condition (3.3) is necessary.

This completes the proof. □

The following two corollaries are of interest.

Corollary 3.2. *Let $\{e_1, \ldots, e_n\}$ be a family of orthonormal vectors in H, $\rho_i \in (0,1)$, $i \in \{1, \ldots, n\}$ and $f \in L([a,b]; H)$ such that:*

$$\|f(t) - e_i\| \le \rho_i \text{ for } i \in \{1, \ldots, n\} \text{ and a.e. } t \in [a,b]. \tag{3.9}$$

Then we have the inequality

$$\left(n - \sum_{i=1}^n \rho_i^2 \right)^{\frac{1}{2}} \int_a^b \|f(t)\|\,dt \le \left\| \int_a^b f(t)\,dt \right\|,$$

with equality if and only if

$$\int_a^b f(t)\,dt = \int_a^b \|f(t)\|\,dt \left(\sum_{i=1}^n \left(1 - \rho_i^2\right)^{\frac{1}{2}} e_i \right).$$

Proof. From the proof of Theorem 2.1, we know that (3.3) implies the inequality

$$\sqrt{1 - \rho_i^2}\, \|f(t)\| \le \operatorname{Re} \langle f(t), e_i \rangle, \quad i \in \{1, \ldots, n\}, \text{ for a.e. } t \in [a,b].$$

Now, applying Theorem 3.1 for $k_i := \sqrt{1 - \rho_i^2}$, $i \in \{1, \ldots, n\}$, we deduce the desired result. □

Corollary 3.3. *Let $\{e_1, \ldots, e_n\}$ be a family of orthonormal vectors in H, $M_i \ge m_i > 0$, $i \in \{1, \ldots, n\}$ and $f \in L([a,b]; H)$ such that*

$$\operatorname{Re} \langle M_i e_i - f(t), f(t) - m_i e_i \rangle \ge 0 \tag{3.10}$$

or, equivalently,

$$\left\| f(t) - \frac{M_i + m_i}{2} e_i \right\| \le \frac{1}{2}(M_i - m_i)$$

for $i \in \{1, \dots, n\}$ *and a.e.* $t \in [a, b]$. *Then we have the reverse of the generalised triangle inequality*

$$\left[\sum_{i=1}^{n} \frac{4 m_i M_i}{(m_i + M_i)^2} \right]^{\frac{1}{2}} \int_a^b \| f(t) \| \, dt \le \left\| \int_a^b f(t) \, dt \right\|,$$

with equality if and only if

$$\int_a^b f(t) \, dt = \int_a^b \| f(t) \| \, dt \left(\sum_{i=1}^{n} \frac{2\sqrt{m_i M_i}}{m_i + M_i} e_i \right).$$

Proof. From the proof of Corollary 2.3, we know (3.10) implies that

$$\frac{2\sqrt{m_i M_i}}{m_i + M_i} \| f(t) \| \le \operatorname{Re} \langle f(t), e_i \rangle, \quad i \in \{1, \dots, n\} \text{ and a.e. } t \in [a, b].$$

Now, applying Theorem 3.1 for $k_i := \frac{2\sqrt{m_i M_i}}{m_i + M_i}$, $i \in \{1, \dots, n\}$, we deduce the desired result. □

4. Applications for Complex-Valued Functions

Let $e = \alpha + i\beta$ $(\alpha, \beta \in \mathbb{R})$ be a complex number with the property that $|e| = 1$, i.e., $\alpha^2 + \beta^2 = 1$.

The following proposition holds.

Proposition 4.1. *If* $f : [a, b] \to \mathbb{C}$ *is a Lebesgue integrable function with the property that there exists a constant* $K \ge 1$ *such that*

$$|f(t)| \le K [\alpha \operatorname{Re} f(t) + \beta \operatorname{Im} f(t)] \tag{4.1}$$

for a.e. $t \in [a, b]$, *where* $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 = 1$ *are given, then we have the following reverse of the continuous triangle inequality:*

$$\int_a^b |f(t)| \, dt \le K \left| \int_a^b f(t) \, dt \right|. \tag{4.2}$$

The case of equality holds in (2.2) if and only if

$$\int_a^b f(t) \, dt = \frac{1}{K} (\alpha + i\beta) \int_a^b |f(t)| \, dt.$$

The proof is obvious by Theorem 2.1, and we omit the details.

Remark 4.2. If in the above Proposition 4.1 we choose $\alpha = 1$, $\beta = 0$, then the condition (4.1) for $\operatorname{Re} f(t) > 0$ is equivalent to

$$[\operatorname{Re} f(t)]^2 + [\operatorname{Im} f(t)]^2 \le K^2 [\operatorname{Re} f(t)]^2$$

or with the inequality:

$$\frac{|\operatorname{Im} f(t)|}{\operatorname{Re} f(t)} \le \sqrt{K^2 - 1}.$$

Now, if we assume that

$$|\arg f(t)| \le \theta, \qquad \theta \in \left(0, \frac{\pi}{2}\right), \tag{4.3}$$

then, for $\operatorname{Re} f(t) > 0$,

$$|\tan[\arg f(t)]| = \frac{|\operatorname{Im} f(t)|}{\operatorname{Re} f(t)} \le \tan\theta,$$

and if we choose $K = \frac{1}{\cos\theta} > 1$, then

$$\sqrt{K^2 - 1} = \tan\theta,$$

and by Proposition 4.1, we deduce

$$\cos\theta \int_a^b |f(t)|\,dt \le \left|\int_a^b f(t)\,dt\right|, \tag{4.4}$$

which is exactly the Karamata inequality (1.2) from the Introduction.

Obviously, the result from Proposition 4.1 is more comprehensive since for other values of $(\alpha, \beta) \in \mathbb{R}^2$ with $\alpha^2 + \beta^2 = 1$ we can get different sufficient conditions for the function f such that the inequality (4.2) holds true.

A different sufficient condition in terms of complex disks is incorporated in the following proposition.

Proposition 4.3. *Let $e = \alpha + i\beta$ with $\alpha^2 + \beta^2 = 1$, $r \in (0, 1)$ and $f : [a, b] \to \mathbb{C}$ be a Lebesgue integrable function such that*

$$f(t) \in \bar{D}(e, r) := \{z \in \mathbb{C} |\ |z - e| \le r\} \quad \text{for a.e. } t \in [a, b]. \tag{4.5}$$

Then we have the inequality

$$\sqrt{1 - r^2} \int_a^b |f(t)|\,dt \le \left|\int_a^b f(t)\,dt\right|. \tag{4.6}$$

The case of equality holds in (4.6) if and only if

$$\int_a^b f(t)\,dt = \sqrt{1 - r^2}\,(\alpha + i\beta) \int_a^b |f(t)|\,dt.$$

The proof follows by Corollary 2.2 and we omit the details.

Finally, we may state the following proposition as well.

Proposition 4.4. *Let $e = \alpha + i\beta$ with $\alpha^2 + \beta^2 = 1$ and $M \ge m > 0$. If $f : [a, b] \to \mathbb{C}$ is such that*

$$\operatorname{Re}\left[(Me - f(t))\left(\overline{f(t)} - m\bar{e}\right)\right] \ge 0 \quad \text{for a.e. } t \in [a, b], \tag{4.7}$$

or, equivalently,

$$\left|f(t) - \frac{M + m}{2} e\right| \le \frac{1}{2}(M - m) \quad \text{for a.e. } t \in [a, b], \tag{4.8}$$

then we have the inequality

$$\frac{2\sqrt{mM}}{M + m} \int_a^b |f(t)|\,dt \le \left|\int_a^b f(t)\,dt\right|, \tag{4.9}$$

or, equivalently,

$$(0 \le) \int_a^b |f(t)|\,dt - \left|\int_a^b f(t)\,dt\right| \le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{M + m} \left|\int_a^b f(t)\,dt\right|. \tag{4.10}$$

The equality holds in (4.9) (or in the second part of (4.10)) if and only if

$$\int_a^b f(t)\,dt = \frac{2\sqrt{mM}}{M+m}(\alpha + i\beta)\int_a^b |f(t)|\,dt.$$

The proof follows by Corollary 2.3 and we omit the details.

Remark 4.5. Since

$$Me - f(t) = M\alpha - \operatorname{Re} f(t) + i\left[M\beta - \operatorname{Im} f(t)\right],$$
$$\overline{f(t)} - m\bar{e} = \operatorname{Re} f(t) - m\alpha - i\left[\operatorname{Im} f(t) - m\beta\right]$$

hence

$$\operatorname{Re}\left[(Me - f(t))\left(\overline{f(t)} - m\bar{e}\right)\right] = \left[M\alpha - \operatorname{Re} f(t)\right]\left[\operatorname{Re} f(t) - m\alpha\right] + \left[M\beta - \operatorname{Im} f(t)\right]\left[\operatorname{Im} f(t) - m\beta\right]. \tag{4.11}$$

It is obvious that, if

$$m\alpha \le \operatorname{Re} f(t) \le M\alpha \quad \text{for a.e. } t \in [a,b], \tag{4.12}$$

and

$$m\beta \le \operatorname{Im} f(t) \le M\beta \quad \text{for a.e. } t \in [a,b], \tag{4.13}$$

then, by (4.11),

$$\operatorname{Re}\left[(Me - f(t))\left(\overline{f(t)} - m\bar{e}\right)\right] \ge 0 \quad \text{for a.e. } t \in [a,b],$$

and then either (4.9) or (4.12) hold true.

We observe that the conditions (4.12) and (4.13) are very easy to verify in practice and may be useful in various applications where reverses of the continuous triangle inequality are required.

Remark 4.6. Similar results may be stated for functions $f : [a,b] \to \mathbb{R}^n$ or $f : [a,b] \to H$, with H particular instances of Hilbert spaces of significance in applications, but we leave them to the interested reader.

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