Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 5, Article 94, 2003

ON HARDY-HILBERT'S INTEGRAL INEQUALITY WITH PARAMETERS

LEPING HE, MINGZHE GAO, AND WEIJIAN JIA

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE,
NORMAL COLLEGE, JISHOU UNIVERSITY,
JISHOU HUNAN, 416000
PEOPLE'S REPUBLIC OF CHINA.
lianheping@163.com

mingzhegao@163.com

jwj1959@163.com

Received 01 August, 2003; accepted 28 August, 2003
Communicated by L. Debnath

ABSTRACT. In this paper, by means of a sharpening of Hölder's inequality, Hardy-Hilbert's integral inequality with parameters is improved. Some new inequalities are established.

Key words and phrases: Hardy-Hilbert integral inequality, Hölder's inequality, Weight function, Beta function.

2000 *Mathematics Subject Classification.* 26D15, 46C99.

1. INTRODUCTION

Let $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $f, g > 0$. If $0 < \int_0^\infty f^p(t)dt < +\infty$, $0 < \int_0^\infty g^q(t)dt < +\infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(t)\,dt \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(t)\,dt \right)^{\frac{1}{q}}, \tag{1.1}$$

where the constant $\frac{\pi}{\sin(\pi/p)}$ is best possible. The inequality (1.1) is well known as Hardy-Hilbert's integral inequality. In recent years, some improvements and extensions of Hilbert's inequality and Hardy-Hilbert's inequality have been given in [2] – [6], Yang [2] gave a generalization of (1.1) as follows:

ISSN (electronic): 1443-5756

The authors thank the referee for his help and patience in improving the paper.
108-03

If $\lambda > 2 - \min\{p, q\}, \alpha < T \leqslant \infty$ then

$$\int_{\alpha}^{T}\int_{\alpha}^{T}\frac{f(x)g(y)}{(x+y-2\alpha)^{\lambda}}dxdy < \left\{\int_{\alpha}^{T}\left[k_{\lambda}(p)-\theta_{\lambda}(p)\left(\frac{t-\alpha}{T-\alpha}\right)^{\frac{p+\lambda-2}{p}}\right](t-\alpha)^{1-\lambda}f^{p}(t)dt\right\}^{\frac{1}{p}} \times\left\{\int_{\alpha}^{T}\left[k_{\lambda}(p)-\theta_{\lambda}(q)\left(\frac{t-\alpha}{T-\alpha}\right)^{\frac{q+\lambda-2}{q}}\right](t-\alpha)^{1-\lambda}g^{q}(t)dt\right\}^{\frac{1}{q}} \quad (T<\infty) \tag{1.2}$$

and

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)g(y)}{(x+y-2\alpha)^{\lambda}}dxdy < k_{\lambda}(p)\left\{\int_{\alpha}^{\infty}(t-\alpha)^{1-\lambda}f^{p}(t)dt\right\}^{\frac{1}{p}}\left\{\int_{\alpha}^{\infty}(t-\alpha)^{1-\lambda}g^{q}(t)dt\right\}^{\frac{1}{q}}, \tag{1.3}$$

where

$$k_{\lambda}(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right),$$

$$\theta_{\lambda}(r) = \int_{0}^{1}\frac{1}{(1+u)^{\lambda}}\left(\frac{1}{u}\right)^{(2-\lambda)/r}du \qquad (r = p, q).$$

The main purpose of this paper is to build a few new inequalities which include improvements of the inequalities (1.2) and (1.3), and extensions of corresponding results in [3] – [5].

2. LEMMAS AND THEIR PROOFS

For convenience, we firstly introduce some notations:

$$(f^{r}, g^{s}) = \int_{\alpha}^{T} f^{r}(x)g^{s}(x)\,dx, \quad \|f\|_{p} = \left(\int_{\alpha}^{T} f^{p}(x)\,dx\right)^{\frac{1}{p}}, \quad \|f\|_{2} = \|f\|.$$

We next introduce a function defined by

$$S_{r}(H, x) = \left(H^{r/2}, x\right)\|H\|_{r}^{-r/2},$$

where x is a parametric variable vector which is a variable unit vector. Under the general case, it is properly chosen such that the specific problems discussed are simplified.

Clearly, $S_r(H, x) = 0$ when the vector x selected is orthogonal to $H^{p/2}$. Throughout this paper, the exponent m indicates $m = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}, \alpha < T \leqslant \infty$.

In order to verify our assertions, we need to build the following lemmas.

Lemma 2.1. *Let* $f(x), g(x) > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p > 1$. *If* $0 < \|f\|_p < +\infty$ *and* $0 < \|g\|_q < +\infty$, *then*

$$(f, g) < \|f\|_{p}\|g\|_{q}(1-R)^{m}, \tag{2.1}$$

where $R = (S_p(f, h) - S_q(g, h))^2$, $\|h\| = 1$, $f^{p/2}(x)$, $g^{q/2}(x)$*and* $h(x)$ *are linearly independent.*

Proof. First of all, we discuss the case of $p \neq q$. Without loss of generality, suppose that $p > q > 1$, since $\frac{1}{p} + \frac{1}{q} = 1$, we have $p > 2$. Let $R = \frac{p}{2}$, $Q = \frac{p}{p-2}$. Then $\frac{1}{R} + \frac{1}{Q} = 1$. By Hölder's inequality we obtain,

$$(f,g) = \int_a^T f(x) g(x)\, dx = \int_a^T \left(f \cdot g^{q/p}\right) g^{1-(q/p)} dx \leqslant \left(\int_a^T \left(f \cdot g^{q/p}\right)^R dx\right)^{\frac{1}{R}} \left(\int_a^T \left(g^{1-(q/p)}\right)^Q dx\right)^{\frac{1}{Q}} = \left(f^{p/2}, g^{q/2}\right)^{\frac{2}{p}} \|g\|_q^{q\left(1-\frac{2}{p}\right)}. \tag{2.2}$$

And the equality in (2.2) holds if and only if $f^{p/2}$ and $g^{q/2}$ are linearly dependent. In fact, the equality in (2.2) holds if and only if, there exists a c_1 such that $\left(f \cdot g^{q/p}\right)^R = c_1 \left(g^{1-(q/p)}\right)^Q$. It is easy to deduce that $f^{p/2} = c_1 g^{q/2}$.

In our previous paper [3], with the help of the positive definiteness of the Gram matrix, we established an important inequality of the form

$$(\alpha,\beta)^2 \leqslant \|\alpha\|^2 \|\beta\|^2 - (\|\alpha\| x - \|\beta\| y)^2 = \|\alpha\|^2 \|\beta\|^2 (1 - \overline{\gamma}) \tag{2.3}$$

where $\overline{\gamma} = \left(\frac{y}{\|\alpha\|} - \frac{x}{\|\beta\|}\right)^2$, $x = (\beta,\gamma)$, $y = (\alpha,\gamma)$ with $\|\gamma\| = 1$ and $xy \geqslant 0$. The equality in (2.3) holds if and only if α and β are linearly dependent; or the vector γ is a linear combination of α and β, and $xy = 0$ but $x \neq y$. If $\alpha,\ \beta$ and γ in (2.3) are replaced by $f^{p/2}$, $g^{q/2}$ and h respectively, then we get

$$\left(f^{p/2}, g^{q/2}\right)^2 \leqslant \|f\|_p^p \|g\|_q^q (1 - R), \tag{2.4}$$

where $R = (S_p(f,h) - S_q(g,h))^2$ with $||h|| = 1$. The equality in (2.4) holds if and only if $f^{p/2}$and $g^{q/2}$ are linearly dependent, or h is a linear combination of $f^{p/2}$ and $g^{q/2}$, and $\left(f^{p/2}, h\right)\left(g^{q/2}, h\right) = 0$, but $\left(f^{p/2}, h\right) \neq \left(g^{q/2}, h\right)$. Since $f^{p/2}$ and $g^{q/2}$ are linearly independent, it is impossible to have equality in (2.4). Substituting (2.4) into (2.2), we obtain after simplifications

$$(f,g) < \|f\|_p \|g\|_q (1 - R)^{\frac{1}{p}}. \tag{2.5}$$

Provided that $h(x)$ is properly chosen, then $R \neq 0$ is achieved. (The choice of $h(x)$ is quite flexible, as long as condition $\|h\| = 1$ is satisfied, on which we can refer to [3, 4], etc.). Noticing the symmetry of p and q, the inequality (2.1) follows from (2.5).

Next, we discuss the case of $p = q$. According to the hypothesis: when f, g and h are linearly independent, we immediately obtain from (2.3) the following result:

$$(f,g) < \|f\| \|g\| (1 - \bar{r})^{\frac{1}{2}},$$

where $\bar{r} = \left(\frac{(f,h)}{\|f\|} - \frac{(g,h)}{\|g\|}\right)^2$, and $\|h\| = 1$. Thus the lemma is proved. □

Lemma 2.2. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p,q\}$, $\alpha < T < \infty$. *Define the weight function* ω_λ *as follows:*

$$\omega_\lambda(\alpha, T, r, x) = \int_\alpha^T \frac{1}{(x+y-2\alpha)^\lambda} \left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{r}} dy \quad x \in (\alpha, T]. \tag{2.6}$$

Setting $\omega_\lambda(\alpha,\infty,r,x)=\lim_{T\to\infty}\omega_\lambda(\alpha,T,r,x)$ *and* $k_\lambda(p)=B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$*,*

$$\overline{\theta}_\lambda(r)=\int_0^1\frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{(2-\lambda)(1-1/r)}du,\qquad(r=p,q),\tag{2.7}$$

then we have

$$\omega_\lambda(\alpha,\infty,r,x)=k_\lambda(p)(x-\alpha)^{1-\lambda},\quad x\in(\alpha,\infty)\tag{2.8}$$

and

$$\omega_\lambda(\alpha,T,r,x)<\left(k_\lambda(p)-\overline{\theta}(r)\left(\frac{x-\alpha}{T-\alpha}\right)^{1+(\lambda-2)(1-1/r)}\right)(x-\alpha)^{1-\lambda},\quad x\in(\alpha,T),\tag{2.9}$$

where $B(m,n)$ *is the beta function.*

The proof of this lemma is given in the paper [2]; it is omitted here.

3. MAIN RESULTS

In order to state it conveniently, we need again to define the functions and introduce some notations

$$F=\frac{f(x)}{(x+y-2\alpha)^{\lambda/p}}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{2-\lambda}{pq}},\quad G=\frac{g(y)}{(x+y-2\alpha)^{\lambda/q}}\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{2-\lambda}{pq}},$$

$$S_p(F,h_T)=\left\{\int_\alpha^T\int_\alpha^T F^{p/2}h_T dxdy\right\}\left\{\int_\alpha^T\int_\alpha^T F^p dxdy\right\}^{-\frac{1}{2}},$$
$$S_q(G,h_T)=\left\{\int_\alpha^T\int_\alpha^T G^{q/2}h_T dxdy\right\}\left\{\int_\alpha^T\int_\alpha^T G^q dxdy\right\}^{-\frac{1}{2}},$$

where $h_T=h_T(x,y)$ is a unit vector with two variants, namely

$$||h_T||=\left\{\int_\alpha^T\int_\alpha^T h_T^2 dxdy\right\}^{\frac{1}{2}}=1,\quad \alpha<T\leqslant\infty,$$

and $F^{p/2}$, $G^{q/2}$, h_T are linearly independent.

Theorem 3.1. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda>2-\min\{p,q\}$, $\alpha<T\leqslant\infty$, $f(t),g(t)>0$. *If*

$$0<\int_\alpha^\infty(t-\alpha)^{1-\lambda}f^p(t)\,dt<+\infty\quad\textit{and}$$
$$0<\int_\alpha^\infty(t-\alpha)^{1-\lambda}g^q(t)\,dt<+\infty,$$

then

(i) *For* $T<\infty$*, we have*

$$\int_\alpha^T\int_\alpha^T\frac{f(x)g(y)}{(x+y-2\alpha)^\lambda}dxdy\tag{3.1}$$
$$<\left\{\int_\alpha^T\left(k_\lambda(p)-\theta_\lambda(p)\left(\frac{t-\alpha}{T-\alpha}\right)^{(p+\lambda-2)/p}\right)(t-\alpha)^{1-\lambda}f^p(t)dt\right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_\alpha^T \left(k_\lambda(p) - \theta_\lambda(q) \left(\frac{t-\alpha}{T-\alpha} \right)^{(q+\lambda+2)/q} \right) (t-\alpha)^{1-\lambda} g^q(t) dt \right\}^{\frac{1}{q}} (1-R_T)^m,$$

where

$$k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right),$$
$$\theta_\lambda(r) = \int_0^1 \frac{1}{(1+u)^\lambda} \left(\frac{1}{u} \right)^{\frac{2-\lambda}{r}} du \quad (r = p, q).$$

(ii) *For $T = \infty$, we have*

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda} dxdy < k_\lambda(p) \left(\int_\alpha^\infty (t-\alpha)^{1-\lambda} f^p(t) dt \right)^{\frac{1}{p}} \left(\int_\alpha^\infty (t-\alpha)^{1-\lambda} g^q(t) dt \right)^{\frac{1}{q}} (1-R_\infty)^m, \tag{3.2}$$

where $R_T = (S_p(F, h_T) - S_q(G, h_T))^2$,

$$h_T(x,y) = \begin{cases} \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \frac{e^{a-x}}{(x+y-2a)^{\frac{1}{2}}} \left(\frac{x-a}{y-a} \right)^{\frac{1}{4}}, & T = \infty; \\ \frac{T-\alpha}{(x-\alpha)(y-\alpha)} e^{\left(1 - \frac{T-\alpha}{2(x-\alpha)} - \frac{T-\alpha}{2(y-\alpha)}\right)}, & T < \infty. \end{cases} \tag{3.3}$$

Proof. By Lemma 2.1, we get

$$\begin{aligned} \int_\alpha^T \int_\alpha^T \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda} dxdy &= \int_\alpha^T \int_\alpha^T FGdxdy \\ &\leqslant \left\{ \int_\alpha^T \int_\alpha^T F^p dxdy \right\}^{\frac{1}{p}} \left\{ \int_\alpha^T \int_\alpha^T G^q dxdy \right\}^{\frac{1}{q}} (1-R_T)^m \\ &= \left(\int_\alpha^T \omega_\lambda(\alpha,\beta,q,t) f^p(t)\, dt \right)^{\frac{1}{p}} \left(\int_\alpha^T \omega_\lambda(\alpha,\beta,p,t) g^q(t)\, dt \right)^{\frac{1}{q}} (1-R_T)^m, \end{aligned} \tag{3.4}$$

where $\omega_\lambda(\alpha, T, r, t) \quad (r = p, q)$ is the function defined by (2.6) .

Now notice that $\theta_\lambda(p) = \overline{\theta}_\lambda(q)$, $\theta_\lambda(q) = \overline{\theta}_\lambda(p)$ and substituting (2.9) and (2.8) into (3.4) respectively, the inequalities (3.1) and (3.2) follow.

It remains to discuss the expression of R_T. We may choose the function h_T indicated by (3.3).

When $T = \infty$, setting $s = x - \alpha$, $t = y - \alpha$, then

$$||h_\infty|| = \left(\int_\alpha^\infty \int_\alpha^\infty h_\infty^2(x,y) dxdy \right)^{\frac{1}{2}} = \left\{ \frac{2}{\pi} \int_0^\infty e^{-2s} ds \int_0^\infty \frac{1}{s+t} \left(\frac{s}{t} \right)^{\frac{1}{2}} dt \right\}^{\frac{1}{2}} = 1.$$

When $T<\infty$, setting $\xi=\frac{T-\alpha}{x-\alpha}$, $\eta=\frac{T-\alpha}{y-\alpha}$, then we have

$$\begin{aligned}\|h_T\| &= \left(\int_\alpha^T\int_\alpha^T h_T^2 dxdy\right)^{\frac{1}{2}}\\ &= \left\{\int_\alpha^T \frac{T-\alpha}{(x-\alpha)^2}e^{\left(1-\frac{T-\alpha}{x-\alpha}\right)}dx\cdot\int_\alpha^T \frac{T-\alpha}{(y-\alpha)^2}e^{\left(1-\frac{T-\alpha}{y-\alpha}\right)}dy\right\}^{\frac{1}{2}}\\ &= \left\{\int_1^\infty e^{1-\xi}d\xi\cdot\int_1^\infty e^{1-\eta}d\eta\right\}^{\frac{1}{2}}=1.\end{aligned}$$

According to Lemma 2.1 and the given h_T, we have $R_T=\left(S_p\left(F,h_T\right)-S_q\left(G,h_T\right)\right)^2$. It is obvious that $F^{p/2}$, $G^{q/2}$ and h_T are linearly independent, so it is impossible for equality to hold in (3.4). Thus the proof of theorem is completed. □

Remark 3.2. Clearly, the inequalities (3.1) and (3.2) are the improvements of (1.2) and (1.3) respectively .

Owing to $p,q>1$, when $\lambda=1,2,3$, the condition $\lambda>2-\min(p,q)$ is satisfied, then we have

$$\theta_1(r)=\int_0^1\frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{\gamma}}du>\int_0^1\frac{1}{1+u}du=\ln 2,\quad k_1(p)=B\left(\frac{1}{q},\frac{1}{p}\right)=\frac{\pi}{\sin(\pi/p)},$$

$$\theta_2(r)=\int_0^1\frac{1}{(1+u)^2}du=\frac{1}{2},\quad k_2(p)=B\left(\frac{p+2-2}{p},\frac{q+2-2}{q}\right)=B(1,1)=1,$$

$$\theta_3(r)=\int_0^1\frac{1}{(1+u)^3}\left(\frac{1}{u}\right)^{-\frac{1}{\gamma}}du>\int_0^1\frac{u}{(1+u)^3}du=\frac{1}{8},$$

$$k_3(p)=\frac{1}{2pq}B\left(\frac{1}{q},\frac{1}{p}\right)=\frac{(p-1)\pi}{2p^2\sin(\pi/p)}.$$

By Theorem 3.1, some corollaries are established as follows:

Corollary 3.3. *If* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda=1$, $\alpha<T\leqslant\infty$ *and* $f(t),g(t)>0$, $0<\int_\alpha^T f^p(t)dt<+\infty$ *and* $0<\int_\alpha^T g^q(t)dt<+\infty$ *, then we have*

$$\begin{aligned}\int_\alpha^T\int_\alpha^T \frac{f(x)g(y)}{x+y-2\alpha}dxdy \\ &< \left\{\int_\alpha^T\left(\frac{\pi}{\sin(\pi/p)}-\left(\frac{t-\alpha}{T-\alpha}\right)^{\frac{1}{q}}\cdot\ln 2\right)f^p(t)dt\right\}^{\frac{1}{p}}\\ &\times\left\{\int_\alpha^T\left(\frac{\pi}{\sin(\pi/p)}-\left(\frac{t-\alpha}{T-\alpha}\right)^{\frac{1}{p}}\cdot\ln 2\right)\cdot g^q(t)dt\right\}^{\frac{1}{q}}(1-r_1)^m,\quad \textit{for } T<\infty,\end{aligned}\tag{3.5}$$

and

$$\int_\alpha^\infty\int_\alpha^\infty\frac{f(x)g(y)}{x+y-2\alpha}dxdy<\frac{\pi}{\sin(\pi/p)}\left(\int_\alpha^\infty f^p(t)dt\right)^{\frac{1}{p}}\left(\int_\alpha^\infty g^q(t)dt\right)^{\frac{1}{q}}(1-\overline{r}_1)^m.\tag{3.6}$$

Remark 3.4. When $\alpha=0$ and $p=q=2$, the inequality (3.6) is reduced to a result which is equivalent to inequality (3.1) in [3] after simple computations. As a result, the inequalities (3.1), (3.2) and (3.5) – (3.6) are all extensions of (3.1) in [3].

Corollary 3.5. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha < T \leqslant \infty$ *and* $f(t), g(t) > 0$. *If*

$$0 < \int_\alpha^T \frac{1}{t-\alpha} f^p(t)\, dt < +\infty \quad \textit{and}$$
$$0 < \int_\alpha^T \frac{1}{t-\alpha} g^q(t)\, dt < +\infty,$$

then we obtain

$$\int_\alpha^T \int_\alpha^T \frac{f(x)g(y)}{(x+y-2\alpha)^2} dxdy < \left\{ \int_\alpha^T \left(1 - \frac{t-\alpha}{2(T-\alpha)}\right) \frac{1}{t-\alpha} \cdot f^p(t)dt \right\}^{\frac{1}{p}} \times \left\{ \int_\alpha^T \left(1 - \frac{t-\alpha}{2(T-\alpha)}\right) \frac{1}{t-\alpha} \cdot g^q(t)dt \right\}^{\frac{1}{q}} (1-r_2)^m, \quad \textit{for } T < \infty, \tag{3.7}$$

and

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x+y-2\alpha)^2} dxdy < \left(\int_\alpha^\infty \frac{1}{t-\alpha} f^p(t)dt \right)^{\frac{1}{p}} \left(\int_\alpha^\infty \frac{1}{t-\alpha} g^q(t)dt \right)^{\frac{1}{q}} (1-\overline{r}_2)^m. \tag{3.8}$$

Corollary 3.6. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda = 3$, $\alpha < T \leqslant \infty$ *and* $f(t), g(t) > 0$,

$$0 < \int_\alpha^T \frac{1}{(t-\alpha)^2} f^p(t)dt < +\infty,$$
$$0 < \int_\alpha^T \frac{1}{(t-\alpha)^2} g^q(t)dt < +\infty,$$

then we get

$$\int_\alpha^T \int_\alpha^T \frac{f(x)g(y)}{(x+y-2\alpha)^3} dxdy < \left\{ \int_\alpha^T \left(\frac{(p-1)\pi}{2p^2 \sin(\pi/p)} - \frac{1}{8} \left(\frac{t-\alpha}{T-\alpha}\right)^{1+\frac{1}{p}} \right) \frac{1}{(t-\alpha)^2} f^p(t)dt \right\}^{\frac{1}{p}} \times \left\{ \int_\alpha^T \left(\frac{(p-1)\pi}{2p^2 \sin(\pi/p)} - \frac{1}{8} \left(\frac{t-\alpha}{T-\alpha}\right)^{1+\frac{1}{q}} \right) \frac{1}{(t-\alpha)^2} g^q(t)dt \right\}^{\frac{1}{q}} (1-r_3)^m \quad T < \infty, \tag{3.9}$$

and

$$\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x+y-2\alpha)^3} dxdy < \frac{(p-1)\pi}{2p^2 \sin(\pi/p)} \left(\int_\alpha^\infty \frac{1}{(t-\alpha)^2} f^p(t)dt \right)^{\frac{1}{p}} \times \left(\int_\alpha^\infty \frac{1}{(t-\alpha)^2} g^q(t)dt \right)^{\frac{1}{q}} (1-\overline{r}_3)^m. \tag{3.10}$$

Since $k_\lambda(2) = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$, $\theta_\lambda(2) = \frac{1}{2} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$, and $\lambda > 2 - \min(2,2) = 0$, we also have

Corollary 3.7. *If* $p = q = 2$, $\lambda > 0$, $\alpha < T \leqslant \infty$ *and* $f(t), g(t) > 0$,

$$0 < \int_{\alpha}^{T} (t-\alpha)^{1-\lambda} f^2(t)\, dt < +\infty,$$

$$0 < \int_{\alpha}^{T} (t-\alpha)^{1-\lambda} g^2(t)\, dt < +\infty,$$

then we have

$$\int_{\alpha}^{T}\int_{\alpha}^{T} \frac{f(x)g(y)}{(x+y-2\alpha)^{\lambda}} dxdy \tag{3.11}$$

$$< B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\int_{\alpha}^{T}\left[1-\frac{1}{2}\left(\frac{t-\alpha}{T-\alpha}\right)^{\lambda/2}\right](t-\alpha)^{1-\lambda} f^2(t)dt\right\}^{\frac{1}{2}}$$

$$\times\left\{\int_{\alpha}^{T}\left[1-\frac{1}{2}\left(\frac{t-\alpha}{T-\alpha}\right)^{\lambda/2}\right](t-\alpha)^{1-\lambda} g^2(t)dt\right\}^{\frac{1}{2}}(1-\overline{R})^{m}, \text{ for } T<\infty$$

and

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty} \frac{f(x)g(y)}{(x+y-2\alpha)^{\lambda}} dxdy \tag{3.12}$$

$$< B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\int_{\alpha}^{\infty}(t-\alpha)^{1-\lambda} f^2(t)dt\right\}^{\frac{1}{2}}\left\{\int_{\alpha}^{\infty}(t-\alpha)^{1-\lambda} g^2(t)dt\right\}^{\frac{1}{2}}(1-\widetilde{R})^{m}.$$

Remark 3.8. The inequalities (3.11), (3.12) are new generalizations of (20) in [4] and improvements of the inequalities (4) and (12) in [6] respectively.

REFERENCES

[1] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge Univ. Press, Cambridge, UK, 1952.

[2] BICHENG YANG, On Hardy-Hilbert's integral inequality, *J. Math. Anal. Appl.*, **261** (2001), 295–306.

[3] MINGZHE GAO, LI TAN AND L. DEBNATH, Some improvements on Hilbert's integral inequality, *J. Math. Anal. Appl.*, **229** (1999), 682–689.

[4] MINGZHE GAO, On the Hilbert inequality, *Zeitschrift für Analysis und ihre Anwendungen*, **18**(4) (1999), 1117–1122.

[5] MINGZHE GAO, SHANG RONG WEI AND LEPING HE, On the Hilbert inequality with weights, *Zeitschrift für Analysis und ihre Anwendungen*, **21**(1) (2002), 257–263.

[6] BICHENG YANG, On Hilbert's integral inequality, *J. Math. Anal. Appl.*, **220** (1998), 778–785.