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ON SOME INEQUALITIES FOR THE SKEW LAPLACIAN ENERGY OF DIGRAPHS

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ABSTRACT. In this paper we introduce and investigate the skew Laplacian energy of a digraph. We establish upper and lower bounds for the skew Laplacian energy of a digraph.

Key words and phrases: Digraphs, skew energy, skew Laplacian energy.

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1. INTRODUCTION

In this paper we are concerned with simple directed graphs. A directed graph (or just digraph) G consists of a non-empty finite set $V(G) = \{v_1, v_2, \dots, v_n\}$ of elements called vertices and a finite set $\Gamma(G)$ of ordered pairs of distinct vertices called arcs. Two vertices are called adjacent if they are connected by an arc. The skew-adjacency matrix of G is the $n \times n$ matrix $S(G) = [a_{ij}]$ where $a_{ij} = 1$ whenever $(v_i, v_j) \in \Gamma(G)$, $a_{ij} = -1$ whenever $(v_j, v_i) \in \Gamma(G)$, $a_{ij} = 0$ otherwise. Hence $S(G)$ is a skew symmetric matrix of order n and all its eigenvalues are of the form $i\lambda$ where $i = \sqrt{-1}$ and $\lambda \in \mathbb{R}$. The skew energy of G is the sum of the absolute value of the eigenvalues of $S(G)$. For additional information on the skew energy of digraphs we refer to [1]. The degree of a vertex in a digraph G is the degree of the corresponding vertex of the underlying graph of G. Let $D(G) = \operatorname{diag}(d_1, d_2, \dots, d_n)$, the diagonal matrix with vertex degrees $d_1, d_2, \dots, d_n$ of $v_1, v_2, \dots, v_n$. Then $L(G) = D(G) - S(G)$ is called the Laplacian matrix of the digraph G. Let $\mu_1, \mu_2, \dots, \mu_n$ be the eigenvalues of $L(G)$. Then the set $\sigma_{SL}(G) = \{\mu_1, \mu_2, \dots, \mu_n\}$ is called the skew Laplacian spectrum of the digraph G. The Laplacian matrix of a simple, undirected (n, m) graph G_1 is $L(G_1) = D(G_1) - A(G_1)$, where $A(G_1)$ is the adjacency matrix of G_1. It is symmetric, singular, positive semi-definite and all its eigenvalues are real and non negative. It is well known that the smallest eigenvalue is zero and

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its multiplicity is equal to the number of connected components of G_1. The Laplacian spectrum of the graph G_1, consisting of the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ is the spectrum of its Laplacian matrix $L(G_1)$ [3, 4]. The spectrum of the graph G_1, consisting of the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the spectrum of its adjacency matrix $A(G_1)$. The ordinary and Laplacian eigenvalues obey the following well-known relations:

$$\sum_{i=1}^{n} \lambda_i = 0; \qquad \sum_{i=1}^{n} \lambda_i^2 = 2m, \tag{1.1}$$

$$\sum_{i=1}^{n} \alpha_i = 2m; \qquad \sum_{i=1}^{n} \alpha_i^2 = 2m + \sum_{i=1}^{n} d_i^2. \tag{1.2}$$

The energy of the graph G_1 is defined as

$$E(G_1) = \sum_{i=1}^{n} |\lambda_i|.$$

For a survey of the mathematical properties of the energy we refer to [5]. In order to define the Laplacian energy of G_1, Gutman and Zhou [6] introduced auxiliary "eigenvalues" β_i, $i = 1, 2, \ldots, n$, defined by

$$\beta_i = \alpha_i - \frac{2m}{n}.$$

Then it follows that

$$\sum_{i=1}^{n} \beta_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \beta_i^2 = 2M$$

where $M = m + \frac{1}{2}\sum_{i=1}^{n}(d_i - \frac{2m}{n})^2$.

If G_1 is an (n, m)-graph and its Laplacian eigenvalues are $\alpha_1, \alpha_2, \ldots, \alpha_n$, then the Laplacian energy of G_1 [6] is defined by

$$LE(G_1) = \sum_{i=1}^{n} |\beta_i| = \sum_{i=1}^{n} \left| \alpha_i - \frac{2m}{n} \right|.$$

Gutman and Zhou [6] have shown a great deal of analogy between the properties of $E(G_1)$ and $LE(G_1)$. Among others they proved the following two inequalities:

$$LE(G_1) \leq \sqrt{2Mn} \tag{1.3}$$

and

$$2\sqrt{M} \leq LE(G_1) \leq 2M. \tag{1.4}$$

Various bounds for the Laplacian energy of a graph can be found in [8, 9].

The main purpose of this paper is to introduce the concept of the skew Laplacian energy $SLE(G)$ of a simple, connected digraph G, and to establish upper and lower bounds for $SLE(G)$ which are similar to (1.3) and (1.4). We may mention here that the skew Laplacian energy of a digraph considered in [2] was actually the second spectral moment.

2. BOUNDS FOR THE SKEW LAPLACIAN ENERGY OF A DIGRAPH

We begin by giving the formal definition of the skew Laplacian energy of a digraph.

Definition 2.1. Let $S(G)$ be the skew adjacency matrix of a simple digraph G, possessing n vertices and m edges. Then the skew Laplacian energy of the digraph G is defined as

$$SLE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|,$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of the Laplacian matrix $L(G) = D(G) - S(G)$.

In analogy with (1.2), Adiga and Smitha [2] have proved that

$$\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} d_i = 2m \tag{2.1}$$

and

$$\sum_{i=1}^{n} \mu_i^2 = \sum_{i=1}^{n} d_i(d_i - 1). \tag{2.2}$$

We may observe that equations (2.1) and (2.2) are evident as (1.1) and (1.2), which follow from the trace equality.

Define $\gamma_i = \mu_i - \frac{2m}{n}$ for $i = 1, 2, \ldots, n$. On using (2.1) and (2.2) we see that

$$\sum_{i=1}^{n} \gamma_i = 0 \tag{2.3}$$

and

$$\sum_{i=1}^{n} \gamma_i^2 = 2M, \tag{2.4}$$

where

$$M = -m + \frac{1}{2} \sum_{i=1}^{n} \left(d_i - \frac{2m}{n} \right)^2.$$

Since $2m/n$ is the average vertex degree, we have $M + m = 0$ if and only if G is regular.

Theorem 2.1. *Let G be an (n, m)-digraph and let d_i be the degree of the i^{th} vertex of G, $i = 1, 2, \ldots, n$. If $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of the Laplacian matrix $L(G) = D(G) - S(G)$, where $D(G) = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ is the diagonal matrix and $S(G) = [a_{ij}]$ is the skew-adjacency matrix of G, then*

$$SLE(G) \leq \sqrt{2M_1 n}.$$

Here $M_1 = M + 2m = m + \frac{1}{2}\sum_{i=1}^{n}(d_i - \frac{2m}{n})^2$.

Proof. From (2.1) it is clear that

$$\sum_{i=1}^{n} \operatorname{Re}(\mu_i) = \sum_{i=1}^{n} d_i. \tag{2.5}$$

By Schur's unitary triangularization theorem, there is a unitary matrix U such that $U^* L(G) U = T = [t_{ij}]$, where T is an upper triangular matrix with diagonal entries $t_{ii} = \mu_i$, $i = 1, 2, \ldots, n$, i.e. $L(G) = [s_{ij}]$ and $T = [t_{ij}]$ are unitarily equivalent. That is,

$$\sum_{i,j=1}^{n} |s_{ij}|^2 = \sum_{i,j=1}^{n} |t_{ij}|^2 \geq \sum_{i=1}^{n} |t_{ii}|^2 = \sum_{i=1}^{n} |\mu_i|^2.$$

Thus

$$\sum_{i=1}^{n} d_i^2 + 2m \geq \sum_{i=1}^{n} |\mu_i|^2. \tag{2.6}$$

Let $\gamma_i = \mu_i - \frac{2m}{n}$, $i = 1, 2, \dots, n$. By the Cauchy-Schwarz inequality applied to the Euclidean vectors $(|\gamma_1|, |\gamma_2|, \dots, |\gamma_n|)$ and $(1, 1, \dots, 1)$, we have

$$SLE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| = \sum_{i=1}^{n} |\gamma_i| \leq \sqrt{\sum_{i=1}^{n} |\gamma_i|^2} \sqrt{n}. \tag{2.7}$$

Now by (2.5) and (2.6),

$$\begin{aligned} \sum_{i=1}^{n} |\gamma_i|^2 &= \sum_{i=1}^{n} \left(\left| \mu_i - \frac{2m}{n} \right| \right) \left(\left| \overline{\mu_i} - \frac{2m}{n} \right| \right) \\ &= \sum_{i=1}^{n} |\mu_i|^2 - \frac{2m}{n} \sum_{i=1}^{n} 2 \operatorname{Re} \mu_i + \frac{4m^2}{n} \\ &\leq 2m + \sum_{i=1}^{n} d_i^2 - \frac{4m}{n} \sum_{i=1}^{n} d_i + \frac{4m^2}{n} \\ &= 2M_1. \end{aligned} \tag{2.8}$$

Using (2.8) in (2.7), we conclude that

$$SLE(G) \leq \sqrt{2M_1 n}.$$

□

Second Proof. Consider the sum

$$S = \sum_{i=1}^{n} \sum_{j=1}^{n} (|\gamma_i| - |\gamma_j|)^2.$$

By direct calculation

$$S = 2n \sum_{i=1}^{n} |\gamma_i|^2 - 2 \left(\sum_{i=1}^{n} |\gamma_i| \sum_{j=1}^{n} |\gamma_j| \right).$$

It follows from (2.8) and the definition of $SLE(G)$ that

$$S \leq 4nM_1 - 2SLE(G)^2.$$

Since $S \geq 0$, we have $SLE(G) \leq \sqrt{2M_1 n}$. □

If $E(G_1)$ is the ordinary energy of a simple graph G_1 it is well-known [7] that

$$E(G_1) \leq \frac{2m}{n} + \sqrt{(n-1) \left[2m - \left(\frac{2m}{n} \right)^2 \right]}. \tag{2.9}$$

We prove an inequality similar to (2.9) involving the skew Laplacian energy of a digraph. Let G be an (n, m)-digraph. Suppose $\mu_1, \mu_2, \dots, \mu_n$ are the eigenvalues of the Laplacian matrix $L(G)$ with $|\gamma_1| \leq |\gamma_2| \leq \cdots \leq |\gamma_n| = k$, where $\gamma_i = \mu_i - \frac{2m}{n}$, $i = 1, 2, \dots, n$. Let $X =$

$(|\gamma_1| \leq |\gamma_2| \leq \cdots \leq |\gamma_{n-1}|)$ and $Y = (1, 1, \ldots, 1)$. By the Cauchy-Schwarz inequality we have

$$\left(\sum_{i=1}^{n-1} |\gamma_i| \right)^2 \leq (n-1) \sum_{i=1}^{n-1} |\gamma_i|^2.$$

That is,

$$(SLE(G) - |\gamma_n|)^2 \leq (n-1) \left(\sum_{i=1}^{n} |\gamma_i|^2 - |\gamma_n|^2 \right).$$

Using (2.8) in the above inequality we obtain

$$SLE(G) \leq k + \sqrt{(n-1)(2M_1 - k^2)},$$

where $k = |\gamma_n|$ and M_1 is as in Theorem 2.1.

Theorem 2.2. *We have*

$$2\sqrt{|M|} \leq SLE(G) \leq 2M_1.$$

Proof. Since $\sum_{i=1}^{n} \gamma_i = 0$, we have

$$\sum_{i=1}^{n} \gamma_i^2 + 2 \sum_{i<j}^{n} \gamma_i \gamma_j = 0.$$

Now, using (2.4) in the above equation we have

$$2M = -2 \sum_{i<j}^{n} \gamma_i \gamma_j.$$

This implies

$$2|M| = 2 \left| \sum_{i<j}^{n} \gamma_i \gamma_j \right| \leq 2 \sum_{i<j}^{n} |\gamma_i||\gamma_j|. \tag{2.10}$$

Now by (2.4),

$$SLE(G)^2 = \left(\sum_{i=1}^{n} |\gamma_i| \right)^2 = \sum_{i=1}^{n} |\gamma_i|^2 + 2 \sum_{i<j}^{n} |\gamma_i||\gamma_j|$$
$$\geq 2|M| + 2 \sum_{i<j}^{n} |\gamma_i||\gamma_j|,$$

which combined with (2.10) yields $SLE(G)^2 \geq 4|M|$. Thus

$$2\sqrt{|M|} \leq SLE(G).$$

To prove the right-hand inequality, note that for a graph with m edges and no isolated vertex, $n \leq 2m$. By Theorem 2.1, we have

$$SLE(G) \leq \sqrt{2M_1 n} \leq \sqrt{2M_1(2m)} = 2\sqrt{M_1 m}.$$

Since $M_1 \geq m$, we obtain $SLE(G) \leq 2M_1$. □

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