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A STUDY ON ALMOST INCREASING SEQUENCES

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Abstract

In this paper by using an almost increasing sequence a general theorem on $\varphi - \mid C, \alpha \mid_k$ summability factors, which generalizes some known results, has been proved under weaker conditions.

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Contents

1. Introduction

Let (φ_n) be a sequence of complex numbers and let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by σ_n^α and t_n^α the n-th Cesáro means of order α, with $\alpha > -1$, of the sequences (s_n) and (na_n), respectively, i.e.,

$$\sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \tag{1.1}$$

and

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1.2}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad \text{and} \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{1.3}$$

The series $\sum a_n$ is said to be $|C,\alpha|_k$ summable for $k \geq 1$ and $\alpha > -1$, if (see [5])

$$\sum_{n=1}^{\infty} n^{k-1} \left|\sigma_n^\alpha - \sigma_{n-1}^\alpha\right|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty. \tag{1.4}$$

and it is said to be $|C,\alpha;\beta|_k$ summable for $k \geq 1$, $\alpha > -1$ and $\beta \geq 0$, if (see [6])

$$\sum_{n=1}^{\infty} n^{\beta k+k-1} \left|\sigma_n^\alpha - \sigma_{n-1}^\alpha\right|^k = \sum_{n=1}^{\infty} n^{\beta k-1} |t_n^\alpha|^k < \infty. \tag{1.5}$$

The series $\sum a_n$ is said to be $\varphi-|C,\alpha|_k$ summable for $k \geq 1$ and $\alpha > -1$, if (see [2])

$$\sum_{n=1}^{\infty} n^{-k} \left|\varphi_n t_n^{\alpha}\right|^k < \infty. \tag{1.6}$$

In the special case when $\varphi_n = n^{1-\frac{1}{k}}$ (resp. $\varphi_n = n^{\beta+1-\frac{1}{k}}$) $\varphi-|C,\alpha|_k$ summability is the same as $|C,\alpha|_k$ (resp. $|C,\alpha;\beta|_k$) summability.

Bor [3] has proved the following theorem for $\varphi-|C,1|_k$ summability factors of infinite series.

Theorem 1.1. *Let (X_n) be a positive non-decreasing sequence and let (λ_n) be a sequence such that*

$$|\lambda_n|\, X_n = O(1) \quad \textit{as} \quad n \to \infty \tag{1.7}$$

and

$$\sum_{v=1}^{n} v X_v \left|\Delta^2 \lambda_v\right| = O(1) \quad \textit{as} \quad n \to \infty. \tag{1.8}$$

If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k}\,|\varphi_n|^k)$ is non-increasing and

$$\sum_{v=1}^{n} v^{-k} \left|\varphi_v t_v\right|^k = O(X_n) \quad \textit{as} \quad n \to \infty, \tag{1.9}$$

then the series $\sum a_n \lambda_n$ is $\varphi-|C,1|_k$ summable for $k \geq 1$.

The aim of this paper is to generalize Theorem 1.1 under weaker conditions for $\varphi-|C,\alpha|_k$ summability. For this we need the concept of almost increasing sequences. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence c_n and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is an almost increasing sequence but the converse need not be true as can be seen from the example $b_n = ne^{(-1)^n}$. So we are weakening the hypotheses of the theorem by replacing the increasing sequence with an almost increasing sequence.

2. Result

Now, we shall prove the following:

Theorem 2.1. *Let* (X_n) *be an almost increasing sequence and the sequence* (λ_n) *such that conditions (1.7) – (1.8) of Theorem 1.1 are satisfied. If there exists an* $\epsilon > 0$ *such that the sequence* $(n^{\epsilon-k}|\varphi_n|^k)$ *is non-increasing and if the sequence* (w_n^α)*, defined by (see [9])*

$$w_n^\alpha = \begin{cases} \left| \left| t_n^\alpha \right| \right|, & \alpha = 1 \\ \max_{1\leq v\leq n} \left| t_v^\alpha \right|, & 0 < \alpha < 1 \end{cases} \tag{2.1}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{-k}(w_n^\alpha |\varphi_n|)^k = O(X_m) \quad \text{as} \quad m \to \infty, \tag{2.2}$$

then the series $\sum a_n\lambda_n$ *is* $\varphi-|C,\alpha|_k$ *summable for* $k \geq 1,\ 0 < \alpha \leq 1$ *and* $k\alpha + \epsilon > 1$.

We need the following lemmas for the proof of our theorem.

Lemma 2.2. *([4]). If* $0 < \alpha \leq 1$ *and* $1 \leq v \leq n$*, then*

$$\left| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p \right| \leq \max_{1\leq m\leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p \right|. \tag{2.3}$$

Lemma 2.3. *([8]). If (X_n) is an almost increasing sequence and the conditions (1.7) and (1.8) of Theorem 1.1 are satisfied, then*

$$\sum_{n=1}^{m} X_n |\Delta\lambda_n| = O(1) \tag{2.4}$$

and

$$mX_m |\Delta\lambda_m| = O(1), \quad m \to \infty. \tag{2.5}$$

3. Proof of Theorem 2.1

Let (T_n^α) be the $n-$th (C,α), with $0<\alpha\le 1$, mean of the sequence $(na_n\lambda_n)$. Then, by (1.2), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v.$$

Using Abel's transformation, we get

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,$$

so that making use of Lemma 2.2, we have

$$\begin{aligned} |T_n^\alpha| &\le \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}|\Delta\lambda_v|\left|\sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p\right| + \frac{|\lambda_n|}{A_n^\alpha}\left|\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v\right| \\ &\le \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha |\Delta\lambda_v| + |\lambda_n| w_n^\alpha \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha, \quad \text{say.} \end{aligned}$$

Since

$$\left|T_{n,1}^\alpha + T_{n,2}^\alpha\right|^k \le 2^k\left(\left|T_{n,1}^\alpha\right|^k + \left|T_{n,2}^\alpha\right|^k\right),$$

to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k}\left|\varphi_n T_{n,r}^\alpha\right|^k < \infty \quad \text{for} \quad r=1,2, \quad \text{by} \quad (1.6).$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\sum_{n=2}^{m+1} n^{-k} \left| \varphi_n T_{n,1}^{\alpha} \right|^k$$

$$\leq \sum_{n=2}^{m+1} n^{-k} (A_n^{\alpha})^{-k} \left| \varphi_n \right|^k \left\{ \sum_{v=1}^{n-1} A_v^{\alpha} w_v^{\alpha} \left| \Delta \lambda_v \right| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} n^{-k} n^{-\alpha k} \left| \varphi_n \right|^k \left\{ \sum_{v=1}^{n-1} v^{\alpha k} (w_v^{\alpha})^k \left| \Delta \lambda_v \right| \right\} \left\{ \sum_{v=1}^{n-1} \left| \Delta \lambda_v \right| \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} v^{\alpha k} (w_v^{\alpha})^k \left| \Delta \lambda_v \right| \sum_{n=v+1}^{m+1} \frac{n^{-k} \left| \varphi_n \right|^k}{n^{\alpha k}}$$

$$= O(1) \sum_{v=1}^{m} v^{\alpha k} (w_v^{\alpha})^k \left| \Delta \lambda_v \right| \sum_{n=v+1}^{m+1} \frac{n^{\epsilon - k} \left| \varphi_n \right|^k}{n^{\alpha k + \epsilon}}$$

$$= O(1) \sum_{v=1}^{m} v^{\alpha k} (w_v^{\alpha})^k \left| \Delta \lambda_v \right| v^{\epsilon - k} \left| \varphi_v \right|^k \sum_{n=v+1}^{m+1} \frac{1}{n^{\alpha k + \epsilon}}$$

$$= O(1) \sum_{v=1}^{m} v^{\alpha k} (w_v^{\alpha})^k \left| \Delta \lambda_v \right| v^{\epsilon - k} \left| \varphi_v \right|^k \int_v^{\infty} \frac{dx}{x^{\alpha k + \epsilon}}$$

$$= O(1) \sum_{v=1}^{m} v \left| \Delta \lambda_v \right| v^{-k} (w_v^{\alpha} \left| \varphi_v \right|)^k$$

$$= O(1) \sum_{v=1}^{m-1} \Delta (v \left| \Delta \lambda_v \right|) \sum_{r=1}^{v} r^{-k} (w_r^{\alpha} \left| \varphi_r \right|)^k + O(1) m \left| \Delta \lambda_m \right| \sum_{v=1}^{m} v^{-k} (w_v^{\alpha} \left| \varphi_v \right|)^k$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1}\left|\Delta(v\left|\Delta\lambda_v\right|)\right|X_v + O(1)m\beta_m X_m\\
&= O(1)\sum_{v=1}^{m-1} v\left|\Delta^2\lambda_v\right|X_v + O(1)\sum_{v=1}^{m-1}\left|\Delta\lambda_{v+1}\right|X_{v+1} + O(1)m\left|\Delta\lambda_m\right|X_m\\
&= O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.3.
Again, since $|\lambda_n| = O(1/X_n) = O(1)$, by (1.7), we have that

$$
\begin{aligned}
&\sum_{n=1}^{m} n^{-k}\left|\varphi_n T_{n,2}^{\alpha}\right|^k\\
&\quad = \sum_{n=1}^{m}\left|\lambda_n\right|^{k-1}\left|\lambda_n\right| n^{-k}(w_n^{\alpha}\left|\varphi_n\right|)^k\\
&\quad = O(1)\sum_{n=1}^{m}\left|\lambda_n\right| n^{-k}(w_n^{\alpha}\left|\varphi_n\right|)^k\\
&\quad = O(1)\sum_{n=1}^{m-1}\Delta\left|\lambda_n\right|\sum_{v=1}^{n} v^{-k}(w_v^{\alpha}\left|\varphi_v\right|)^k + O(1)\left|\lambda_m\right|\sum_{n=1}^{m} n^{-k}(w_n^{\alpha}\left|\varphi_n\right|)^k\\
&\quad = O(1)\sum_{n=1}^{m-1}\left|\Delta\lambda_n\right|X_n + O(1)\left|\lambda_m\right|X_m = O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.3.

Therefore, we get that

$$\sum_{n=1}^{m} n^{-k} \left|\varphi_n T_{n,r}^{\alpha}\right|^k = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for} \quad r = 1, 2.$$

This completes the proof of the theorem.

4. Special Cases

1. If we take (X_n) as a positive non-decreasing sequence, $\alpha = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$ in Theorem 2.1, then we get Theorem 1.1.

2. If we take (X_n) as a positive non-decreasing sequence, $\alpha = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$ in Theorem 2.1, then we get a result due to Mazhar [7] for $|C,1|_k$ summability factors of infinite series.

3. If we take $\epsilon = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$ (resp. $\epsilon = 1$ and $\varphi_n = n^{\beta+1-\frac{1}{k}}$), then we get a new result related to $|C,\alpha|_k$ (resp. $|C,\alpha;\beta|_k$) summability factors.

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