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THE BEST CONSTANT FOR AN ALGEBRAIC INEQUALITY

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ABSTRACT. We determine the best constant λ for the inequality $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t} \geq \frac{\lambda}{1+16(\lambda-16)xyzt}$; where $x, y, z, t > 0$; $x+y+z+t = 1$. We also consider an analogous inequality with three variables. As a corollary we establish a refinement of Euler's inequality.

Key words and phrases: Best constant, Geometric inequality, Euler's inequality.

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1. INTRODUCTION

Recently the following inequality was proved [1, 2]:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{25}{1+48xyz}, \tag{1.1}$$

where $x, y, z > 0$; $x + y + z = 1$. This inequality is the special case of the inequality

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{\lambda}{1+3(\lambda-9)xyz}, \tag{1.2}$$

where $\lambda > 0$. Substituting in this inequality $x = y = \frac{1}{4}$, $z = \frac{1}{2}$ we obtain $0 < \lambda \leq 25$. So $\lambda = 25$ is the best constant for the inequality (1.2). As an immediate application one has the following geometric inequality [3]:

$$\frac{R}{r} \geq 2 + \lambda\frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a+b+c)^2}, \tag{1.3}$$

where R and r are respectively the circumradius and inradius, and a, b, c are sides of a triangle, and $\lambda \leq 8$. Substituting $a = b = 3$, $c = 2$ and the corresponding values $R = \frac{9\sqrt{2}}{8}$ and $r = \frac{1}{\sqrt{2}}$ in (1.3) we obtain $\lambda \leq 8$. So $\lambda = 8$ is the best constant for the inequality (1.3), which is a refinement of Euler's inequality.

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It is interesting to compare (1.3) with other known estimates of $\frac{R}{r}$. For example, it is well known that:

$$\frac{R}{r} \geq \frac{(a+b)(b+c)(c+a)}{4abc}. \tag{1.4}$$

For the triangle with sides $a = b = 3$, $c = 2$ inequality (1.3) is stronger than (1.4) even for $\lambda = 3$. But for $\lambda = 2$ inequality (1.4) is stronger than (1.3) for arbitrary triangles. This follows from the algebraic inequality:

$$\frac{(x+y)(y+z)(z+x)}{8xyz} \geq \frac{3(x^2+y^2+z^2)}{(x+y+z)^2}, \tag{1.5}$$

where $x, y, z > 0$, which is in turn equivalent to (1.1).

The main aim of the present article is to determine the best constant for the following analogue of the inequality (1.2):

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} \geq \frac{\lambda}{1 + 16(\lambda - 16)xyzt}, \tag{1.6}$$

where $x, y, z, t > 0$; $x + y + z + t = 1$.

It is known that the best constant for the inequality

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} \leq \lambda + \frac{16 - \lambda}{256xyzt}, \tag{1.7}$$

where $x, y, z, t > 0$; $x + y + z + t = 1$, is $\lambda = \frac{176}{27}$ (see e.g. [4, Corollary 2.13]). In [4] the problem on the determination of the best constants for inequalities similar to (1.7), with n variables was also completely studied:

$$\sum_{i=1}^{n} \frac{1}{x_i} \leq \lambda + \frac{n^2 - \lambda}{n^n \prod_{i=1}^{n} x_i},$$

where $x_1, x_2, \ldots, x_i > 0$, $\sum_{i=1}^{n} x_i = 1$. The best constant for this inequality is $\lambda = n^2 - \frac{n^n}{(n-1)^{n-1}}$. In particular if $n = 3$ then the strongest inequality is

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{9}{4} + \frac{1}{4xyz},$$

where $x, y, z > 0$, $x + y + z = 1$, which is in turn equivalent to the geometric inequality

$$p^2 \geq 16Rr - 5r^2,$$

where p is the semiperimeter of a triangle. But this inequality follows directly from the formula for the distance between the incenter I and the centroid G of a triangle:

$$|IG|^2 = \frac{1}{9}(p^2 + 5r^2 - 16Rr).$$

For some recent results see [4], [6] – [8] and especially [9].

2. PRELIMARY RESULTS

The results presented in this section aim to demonstrate the main ideas of the proof of Theorem 3.1 in a more simpler problem. Corollaries have an independent interest.

Theorem 2.1. *Let $x, y, z > 0$ and $x + y + z = 1$. Then the inequality (1.1) is true.*

Proof. We shall prove the equivalent inequality

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+48(xy+yz+zx)\geq 25.$$

Without loss of generality we may suppose that $x+y\leq\frac{1}{\sqrt[3]{3}}$. Indeed, if $x+y>\frac{1}{\sqrt[3]{3}}$, $y+z>\frac{1}{\sqrt[3]{3}}$, $z+x>\frac{1}{\sqrt[3]{3}}$ then by summing these inequalities we obtain $2>\frac{3}{\sqrt[3]{3}}$, which is false.

Let

$$f(x,y,z)=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+48(xy+yz+zx).$$

We shall prove that

$$f(x,y,z)\geq f\left(\frac{x+y}{2},\frac{x+y}{2},z\right)\geq 25.$$

The first inequality in this chain obtains, after simplifications, the form $\frac{1}{12}\geq xy(x+y)$, which is the consequence of $x+y\leq\frac{1}{\sqrt[3]{3}}$. Denoting $\frac{x+y}{2}=\ell$ $(z=1-2\ell)$ in the second inequality of the chain, after some simplification, we obtain

$$144\ell^4-168\ell^3+73\ell^2-14\ell+1\geq 0\Longleftrightarrow(3\ell-1)^2(4\ell-1)^2\geq 0.$$

□

Corollary 2.2. *Let $x,y,z>0$. Then inequality (1.5) holds true.*

Proof. Inequality (1.5) is homogeneous in its variables x,y,z. We may suppose, without loss of generality, that $x+y+z=1$, after which the inequality obtains the following form:

$$\frac{xy+yz+zx-xyz}{xyz}\geq 24(1-2(xy+yz+zx))$$

$$\Longleftrightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\geq 24-48(xy+yz+zx).$$

By Theorem 2.1 the last inequality is true. □

Corollary 2.3. *For an arbitrary triangle the following inequality is true:*

$$\frac{R}{r}\geq 2+8\frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a+b+c)^2}.$$

Proof. Using known formulas

$$S=\sqrt{p(p-a)(p-b)(p-c)},\quad R=\frac{abc}{4S},\quad r=\frac{S}{p},$$

where S and p are respectively, the area and semiperimeter of a triangle, we transform the inequality to

$$\frac{2abc}{(a+b-c)(b+c-a)(c+a-b)}\geq\frac{18(a^2+b^2+c^2)-12(ab+bc+ca)}{(a+b+c)^2}.$$

Using substitutions $a=x+y,\ b=y+z,\ c=z+x$, where x,y,z are positive numbers by the triangle inequality, we transform the last inequality to

$$\frac{(x+y)(y+z)(z+x)}{xyz}\geq\frac{24(x^2+y^2+z^2)}{(x+y+z)^2},$$

which follows from Corollary 2.2. □

3. MAIN RESULT

Theorem 3.1. *The greatest value of the parameter λ, for which the inequality*

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\geq\frac{\lambda}{1+16(\lambda-16)xyzt},$$

where $x,y,z,t>0$, $x+y+z+t=1$, is true is

$$\lambda=\frac{582\sqrt{97}-2054}{121}.$$

Proof. Substituting in the inequality (1.6) the values

$$x=y=z=\frac{5+\sqrt{97}}{72},\ t=\frac{19-\sqrt{97}}{24},$$

we obtain,

$$\lambda\leq\lambda_0=\frac{582\sqrt{97}-2054}{121}.$$

We shall prove that inequality (1.6) holds for $\lambda=\lambda_0$.

Without loss of generality we may suppose that $x\leq y\leq z\leq t$. We define sequences $\{x_n\},\ \{y_n\},\ \{z_n\}\ (n\geq 0)$ by the equalities

$$x_0=x,\quad y_0=y,\quad z_0=z,$$

$$x_{2k+1}=\frac{1}{2}(x_{2k}+y_{2k}),\quad y_{2k+1}=\frac{1}{2}(x_{2k}+y_{2k}),\quad z_{2k+1}=z_{2k},$$

and

$$x_{2k+2}=x_{2k+1},\quad y_{2k+2}=\frac{1}{2}(y_{2k+1}+z_{2k+1}),\quad z_{2k+2}=\frac{1}{2}(y_{2k+1}+z_{2k+1}),$$

where $k\geq 0$. From these equalities we obtain,

$$y_n=\frac{x+y+z}{3}+\frac{2z-x-y}{3}\left(-\frac{1}{2}\right)^n,$$

where $n\geq 1$. Then we have,

$$\lim y_n=\frac{x+y+z}{3}.$$

Since $x_{2k}=x_{2k-1}=y_{2k-1}$ and $z_{2k+1}=z_{2k}=y_{2k}$ for $k>0$, then we have also,

$$\lim x_n=\lim z_n=\lim y_n=\frac{x+y+z}{3}. \tag{3.1}$$

We note also that,

$$(x+y)^3(z+t)\leq\frac{1}{\lambda_0-16}. \tag{3.2}$$

Indeed, on the contrary we have,

$$(x+y)(z+t)^3\geq(x+y)^3(z+t)>\frac{1}{\lambda_0-16},$$

from which we obtain,

$$(x+y)^2(z+t)^2>\frac{1}{\lambda_0-16}.$$

Then

$$\frac{1}{16}=\left(\frac{(x+y)+(z+t)}{2}\right)^4\geq(x+y)^2(z+t)^2>\frac{1}{\lambda_0-16},$$

which is false, because $\lambda_0 < 32$. Therefore the inequality (3.2) is true. In the same manner, we can prove that

$$(x+z)^3(y+t) \leq \frac{1}{\lambda_0 - 16}. \tag{3.3}$$

Let

$$f(x,y,z,t) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} + 16(\lambda_0 - 16)(xyz + xyt + xzt + yzt).$$

Firstly we prove that

(3.4)

$$\begin{aligned} f(x,y,z,t) = f(x_0,y_0,z_0,t) &\geq f(x_1,y_1,z_1,t) \\ \iff \frac{1}{x} + \frac{1}{y} + 16(\lambda_0-16)xy(z+t) &\geq \frac{4}{x+y} + 16(\lambda_0 - 16)\left(\frac{x+y}{2}\right)^2 (z+t) \\ \iff \frac{1}{\lambda_0 - 16} &\geq 4xy(x+y)(z+t), \end{aligned}$$

which follows from (3.2).

Since for arbitrary $n \geq 0$ the inequality $x_n \leq y_n \leq z_n \leq t$ is true then in the same manner we can prove that

$$f(x_{2k}, y_{2k}, z_{2k}, t) \geq f(x_{2k+1}, y_{2k+1}, z_{2k+1}, t), \tag{3.5}$$

where $k > 0$.

We shall now prove that

$$f(x_{2k+1}, y_{2k+1}, z_{2k+1}, t) \geq f(x_{2k+2}, y_{2k+2}, z_{2k+2}, t), \tag{3.6}$$

where $k \geq 0$. Denote $x' = x_{2k+1},\ y' = y_{2k+1},\ z' = z_{2k+1},\ t' = t$. By analogy with (3.3) we may write,

$$(x'+z')^3(y'+t') \leq \frac{1}{\lambda_0 - 16}.$$

Since $x' = y'$ then we can write the last inequality in this form:

$$(y'+z')^3(x'+t') \leq \frac{1}{\lambda_0 - 16}. \tag{3.7}$$

Similar to (3.4), simplifying (3.6) we obtain,

$$\frac{1}{\lambda_0 - 16} \geq 4y'z'(y'+z')(x'+t'),$$

which follows from (3.7).

By (3.4) – (3.6) we have,

$$f(x,y,z,t) \geq f(x_n, y_n, z_n, t), \tag{3.8}$$

for $n \geq 0$. Denote $\ell = \frac{x+y+z}{3}$ then $t = 1 - 3\ell$. Since $f(x,y,z,t)$ is a continuous function for $x, y, z, t > 0$, then tending n to ∞ in (3.8), we obtain, by (3.1),

$$f(x,y,z,t) \geq \lim f(x_n, y_n, z_n, t) = f(\ell, \ell, \ell, 1 - 3\ell). \tag{3.9}$$

Thus it remains to show that

$$f(\ell, \ell, \ell, 1-3\ell) \geq \lambda_0. \tag{3.10}$$

After elementary but lengthy computations we transform (3.10) into

$$(4\ell - 1)^2 \left((\lambda_0 - 16)\ell(3\ell - 1)(8\ell + 1) + 3\right) \geq 0,$$

where $0 < \ell < \frac{1}{3}$. It suffices to show that

$$\lambda_0 - 16 \leq \frac{-3}{\ell(3\ell-1)(8\ell+1)} = g(\ell),$$

for $0 < \ell < \frac{1}{3}$. The function $g(\ell)$ obtains its minimum value at the point

$$\ell = \ell_0 = \frac{5+\sqrt{97}}{72} \in \left(0, \frac{1}{3}\right),$$

at which $g(\ell_0) = \lambda_0 - 16$. Consequently, the last inequality is true.

From (3.9) and (3.10) it follows that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} \geq \frac{\lambda_0}{1 + 16(\lambda_0 - 16)xyzt},$$

and the equality holds only for quadruples $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$, $(\ell_0, \ell_0, \ell_0, 1 - 3\ell_0)$ and 3 other permutations of the last.

The proof of Theorem 3.1 is complete. □

Remark 3.2. An interesting problem for further exploration would be to determine the best constant λ for the inequality

$$\sum_{i=1}^{n} \frac{1}{x_i} \geq \frac{\lambda}{1 + n^{n-2}(\lambda - n^2)\prod_{i=1}^{n} x_i},$$

where $x_1, x_2, \dots, x_n > 0$, $\sum_{i=1}^{n} x_i = 1$, for $n > 4$. It seems very likely that the number

$$\lambda = \frac{12933567 - 93093\sqrt{22535}}{4135801}\alpha + \frac{17887113 + 560211\sqrt{22535}}{996728041}\alpha^2 - \frac{288017}{17161},$$

where $\alpha = \sqrt[3]{8119 + 48\sqrt{22535}}$, is the best constant in the case $n = 5$. For greater values of n, it is reasonable to find an asymptotic formula of the best constant.

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