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DIFFERENTIAL SUBORDINATIONS AND SUPERORDINATIONS FOR ANALYTIC FUNCTIONS DEFINED BY THE DZIOK-SRIVASTAVA LINEAR OPERATOR

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ABSTRACT. In the present investigation, we obtain some subordination and superordination results involving Dziok-Srivastava linear operator $H_m^l[\alpha_1]$ for certain normalized analytic functions in the open unit disk. Our results extend corresponding previously known results.

Key words and phrases: Univalent functions, Starlike functions, Convex functions, Differential subordination, Convolution, Dziok-Srivastava linear operator.

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1. INTRODUCTION

Let $\mathcal{H}$ be the class of functions analytic in $\Delta := \{z : |z| < 1\}$ and $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = z + a_2 z^2 + \cdots$. Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z), \tag{1.1}$$

then p is a solution of the differential superordination (1.1). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be

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the best subordinant. Recently Miller and Mocanu [14] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

Using the results of Miller and Mocanu [14], Bulboacă [5] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [4]. Ali et al. [1] have used the results of Bulboacă [5] and obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$. Shanmugam et al. [19] obtained sufficient conditions for a normalized analytic function $f(z)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z) \quad \text{and} \quad q_1(z) \prec \frac{z^2f'(z)}{\{f(z)\}^2} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$.

In [2], for functions $f \in \mathcal{A}$ such that $\delta > 0$,

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\delta}\right\} > 0, \qquad z \in \Delta,$$

a class of Bazilevic type functions was considered and certain properties were studied. In this paper motivated by Liu [11], we define a class

$$B(\lambda, \delta, A, B) := \left\{f \in \mathcal{A} : (1-\lambda)\left(\frac{f(z)}{z}\right)^{\delta} + \lambda\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\delta} \prec \frac{1+Az}{1+Bz}\right\},$$

where $\delta > 0$, $\lambda \geq 0$, $-1 \leq B < A \leq 1$ and studied certain interesting properties based on subordination. Further we obtained a sandwich result for functions in the class $B(\lambda, \delta, A, B)$.

2. PRELIMINARIES

For our present investigation, we shall need the following definition and results.

Definition 2.1 ([14, Definition 2, p. 817])**.** Denote by Q, the set of all functions $f(z)$ that are analytic and injective on $\overline{\Delta} - E(f)$, where

$$E(f) = \left\{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Lemma 2.1 ([13, Theorem 3.4h, p. 132])**.** *Let $q(z)$ be univalent in the unit disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(1) *$Q(z)$ is starlike univalent in Δ, and*
(2) *$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in \Delta$.*

If

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2.2 ([19]). *Let q be a convex univalent function in Δ and $\psi, \gamma \in \mathbb{C}$ with*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\right\} > 0.$$

If $p(z)$ is analytic in Δ and

$$\psi p(z) + \gamma z p'(z) \prec \psi q(z) + \gamma z q'(z)$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2.3 ([5]). *Let $q(z)$ be convex univalent in the unit disk Δ and ϑ and φ be analytic in a domain D containing $q(\Delta)$. Suppose that*

(1) $\Re\left[\vartheta'(q(z))/\varphi(q(z))\right] > 0$ *for* $z \in \Delta$,
(2) $zq'(z)\varphi(q(z))$ *is starlike univalent in* Δ.

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\Delta) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in Δ, and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.1}$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

Lemma 2.4 ([14, Theorem 8, p. 822]). *Let q be convex univalent in Δ and $\gamma \in \mathbb{C}$. Further assume that $\Re\left[\overline{\gamma}\right] > 0$. If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, $p(z) + \gamma z p'(z)$ is univalent in Δ, then*

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z)$$

implies $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

For two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z).$$

For $\alpha_j \in \mathbb{C} \quad (j = 1, 2, \dots, l)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ $(j = 1, 2, \dots, m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!}$$
$$(l \le m + 1; l, m \in \mathbb{N}_0 := \{0, 1, 2, \dots\})$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)\cdots(a+n-1), & (n \in \mathbb{N} := \{1, 2, 3 \dots\}). \end{cases}$$

Corresponding to the function

$$h(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := z \; {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z),$$

the Dziok-Srivastava operator [7] (see also [8, 20]) $H_m^l(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)$ is defined by the Hadamard product

$$\begin{aligned} H_m^l(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) f(z) & \\ := h(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) * f(z) & \\ = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \cdots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1}} \frac{a_n z^n}{(n-1)!}. & \end{aligned} \tag{2.2}$$

For brevity, we write

$$H^l_m[\alpha_1]f(z) := H^l_m(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)f(z).$$

It is easy to verify from (2.3) that

$$z(H^l_m[\alpha_1]f(z))' = \alpha_1 H^l_m[\alpha_1+1]f(z) - (\alpha_1-1)H^l_m[\alpha_1]f(z). \tag{2.3}$$

Special cases of the Dziok-Srivastava linear operator include the Hohlov linear operator [9], the Carlson-Shaffer linear operator $L(a,c)$ [6], the Ruscheweyh derivative operator D^n [18], the generalized Bernardi-Libera-Livingston linear integral operator (*cf.* [3], [10], [12]) and the Srivastava-Owa fractional derivative operators (*cf.* [16], [17]).

The main object of the present paper is to find sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \left(\frac{H^l_m[\alpha_1]f(z)}{z}\right)^{\delta} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in Δ. Also, we obtain the number of known results as special cases.

3. MAIN RESULTS

We begin with the following:

Theorem 3.1. *Let $q(z)$ be univalent in Δ, $\lambda \in C$ and $\alpha_1 > 0$, $\delta > 0$. Suppose $q(z)$ satisfies*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)} + \frac{\lambda}{\delta}\right\} > 0. \tag{3.1}$$

If $f \in \mathcal{A}$ satisfies the subordination,

$$(1-\lambda\alpha_1)\left(\frac{H^l_m[\alpha_1]f(z)}{z}\right)^{\delta} + \lambda\alpha_1\left(\frac{H^l_m[\alpha_1]f(z)}{z}\right)^{\delta}\left(\frac{H^l_m[\alpha_1+1]f(z)}{H^l_m[\alpha_1]f(z)}\right) \prec q(z) + \frac{\lambda}{\delta}zq'(z), \tag{3.2}$$

then

$$\left(\frac{H^l_m[\alpha_1]f(z)}{z}\right)^{\delta} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \left(\frac{H^l_m[\alpha_1]f(z)}{z}\right)^{\delta}. \tag{3.3}$$

Then

$$\frac{zp'(z)}{\delta} := \alpha_1\left(\frac{H^l_m[\alpha_1]f(z)}{z}\right)^{\delta}\left(\frac{H^l_m[\alpha_1+1]f(z)}{H^l_m[\alpha_1]f(z)} - 1\right),$$

hence the hypothesis (3.2) of Theorem 3.1 yields the subordination:

$$p(z) + \frac{\lambda zp'(z)}{\delta} \prec q(z) + \frac{\lambda zq'(z)}{\delta}.$$

Now Theorem 3.1 follows by applying Lemma 2.2 with $\psi = 1$ and $\gamma = \frac{\lambda}{\delta}$. □

When $l = 2$, $m = 1$, $\alpha_1 = a$, $\alpha_2 = 1$, and $\beta_1 = c$ in Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let $q(z)$ be univalent in Δ, $\lambda \in C$ and $\alpha_1 > 0$, $\delta > 0$. Suppose $q(z)$ satisfies (3.1). If $f \in \mathcal{A}$ and satisfies the subordination,*

$$(1-\lambda a)\left(\frac{L(a,c)f(z)}{z}\right)^{\delta} + \lambda a\left(\frac{L(a,c)f(z)}{z}\right)^{\delta}\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right) \prec q(z) + \frac{\lambda}{\delta} zq'(z), \tag{3.4}$$

then

$$\left(\frac{L(a,c)f(z)}{z}\right)^{\delta} \prec q(z)$$

and $q(z)$ is the best dominant.

By taking $l = 1, m = 0$ and $\alpha_1 = 1$ in Theorem 3.1, we get the following corollary.

Corollary 3.3. *Let $q(z)$ be univalent in Δ, $\lambda \in C$ and $\alpha_1 > 0$, $\delta > 0$. Suppose $q(z)$ satisfies (3.1). If $f \in \mathcal{A}$ and satisfies the subordination,*

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\delta} + \lambda\left(\frac{f(z)}{z}\right)^{\delta}\left(\frac{zf'(z)}{f(z)}\right) \prec q(z) + \frac{\lambda}{\delta} zq'(z), \tag{3.5}$$

then

$$\left(\frac{f(z)}{z}\right)^{\delta} \prec q(z)$$

and $q(z)$ is the best dominant.

Corollary 3.4. *Let $-1 \leq B < A \leq 1$ and (3.1) hold. If $f \in \mathcal{A}$ and*

$$(1-\lambda\alpha_1)\left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^{\delta} + \lambda\alpha_1\left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^{\delta}\left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)}\right) \prec \frac{\lambda(A-B)z}{\delta(1+Bz)^2} + \frac{1+Az}{1+Bz},$$

then

$$\left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^{\delta} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Theorem 3.5. *Let $q(z)$ be univalent in Δ, $\lambda, \delta \in \mathbb{C}$. Suppose $q(z)$ satisfies*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0. \tag{3.6}$$

If $f \in \mathcal{A}$ satisfies the subordination:

$$1 + \gamma\delta\alpha_1\left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)} - 1\right) \prec 1 + \gamma\frac{zq'(z)}{q(z)}, \tag{3.7}$$

then

$$\left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^{\delta} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) = \left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^{\delta}.$$

It is clear that $p(0) = 1$ and $p(z)$ is analytic in Δ. By using the identity (2.3), from (3.3) we get,

$$\frac{zp'(z)}{p(z)} = \alpha_1\delta\left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)} - 1\right). \tag{3.8}$$

Using (3.8) in (3.7), we see that the subordination becomes

$$1 + \gamma\frac{zp'(z)}{p(z)} \prec 1 + \gamma\frac{zq'(z)}{q(z)}.$$

By setting

$$\theta(w) = 1 \quad \text{and} \quad \varphi(w) = \frac{\gamma}{w},$$

we observe that φ and θ are analytic in $\mathbb{C} \setminus \{0\}$. Also we see that

$$Q(z) := zq'(z)\varphi(q(z)) = \frac{\gamma zq'(z)}{q(z)},$$

and

$$h(z) := \vartheta(q(z)) + Q(z) = 1 + \gamma\frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in Δ and

$$\Re\frac{zh'(z)}{Q(z)} = \Re\left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right] \geq 0.$$

By the hypothesis of Theorem 3.5, the result now follows by an application of Lemma 2.1. □

Specializing the values of $l = 1$, $m = 0$, $\alpha_1 = 1$ and $q(z) = \frac{1}{(1-z)^{2b}} \quad (b \in C - \{0\})$, $\gamma = \frac{1}{b}$ and $\delta = 1$ in Theorem 3.5 above, we have the following corollary as stated in [21].

Corollary 3.6. *Let b be a non zero complex number. If $f \in \mathcal{A}$ and*

$$1 + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} - 1\right] \prec \frac{1+z}{1-z},$$

then

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{2b}}$$

and $\frac{1}{(1-z)^{2b}}$ is the best dominant.

Choosing the values of $l = 1$, $m = 0$, $\alpha_1 = 1$ and $q(z) = \frac{1}{(1-z)^{2ab}} \quad (b \in C - \{0\})$, $\gamma = \frac{1}{b}$ and $\delta = a \neq 0$ in Theorem 3.5 above, we have the following corollary as stated in [15].

Corollary 3.7. *Let b be a non zero complex number. If $f \in \mathcal{A}$ and*

$$1 + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} - 1\right] \prec \frac{1+z}{1-z},$$

then

$$\left(\frac{f(z)}{z}\right)^{a} \prec \frac{1}{(1-z)^{2ab}}$$

where $a \neq 0$ is a complex number and $\frac{1}{(1-z)^{2ab}}$ is the best dominant.

Similarly for $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $q(z) = \frac{1}{(1-z)^{2b}} \quad (b \in C - \{0\})$, $\gamma = \frac{1}{b}$ and $\delta = 1$ in Theorem 3.5 above, we get the following result as stated in [21].

Corollary 3.8. *Let b be a non zero complex number. If $f \in \mathcal{A}$ and*

$$1 + \frac{1}{b}\left[\frac{zf''(z)}{f'(z)} - 1\right] \prec \frac{1+z}{1-z},$$

then

$$f'(z) \prec \frac{1}{(1-z)^{2b}}$$

and $\frac{1}{(1-z)^{2b}}$ is the best dominant.

Next, applying Lemma 2.3, we have the following theorem.

Theorem 3.9. *Let $q(z)$ be convex univalent in Δ, $\lambda \in \mathcal{C}$ and $0 < \delta < 1$. Suppose $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left\{\frac{\delta}{\lambda}\right\} > 0 \tag{3.9}$$

and $\left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^\delta \in H[q(0), 1] \cap Q$. Let

$$(1 - \lambda\alpha_1)\left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^\delta + \lambda\alpha_1\left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^\delta\left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)}\right)$$

be univalent in Δ. If $f \in \mathcal{A}$ satisfies the superordination,

$$\begin{aligned} q(z) + \frac{\lambda}{\delta}zq'(z) \prec (1 - \lambda\alpha_1)&\left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^\delta \\ &+ \lambda\alpha_1\left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^\delta\left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)}\right)\end{aligned} \tag{3.10}$$

then

$$q(z) \prec \left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^\delta$$

and $q(z)$ is the best subordinant.

Proof. Define the function $p(z)$ by

$$p(z) := \left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^\delta. \tag{3.11}$$

Using (3.11), simple computation produces

$$\frac{zp'(z)}{\delta} := \alpha_1\left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^\delta\left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)} - 1\right),$$

then

$$q(z) + \frac{\lambda}{\delta}zq'(z) \prec p(z) + \frac{\lambda}{\delta}zp'(z).$$

By setting $\vartheta(w) = w$ and $\phi(w) = \frac{\lambda}{\delta}$, it is easily observed that $\vartheta(w)$ is analytic in C. Also, $\phi(w)$ is analytic in $C\backslash\{0\}$ and $\phi(w) \neq 0$, $(w \in C\backslash\{0\})$.

Since $q(z)$ is a convex univalent function, it follows that

$$\Re\left\{\frac{\vartheta'(q(z))}{\phi(q(z))}\right\} = \Re\left\{\frac{\delta}{\lambda}\right\} > 0, \quad z \in \Delta, \quad \delta, \lambda \in C, \delta, \lambda \neq 0.$$

Now Theorem 3.9 follows by applying Lemma 2.3. □

Concluding the results of differential subordination and superordination, we state the following sandwich result.

Theorem 3.10. *Let* q_1 *and* q_2 *be convex univalent in* Δ, $\lambda \in \mathbb{C}$ *and* $0 < \delta < 1$. *Suppose* q_2 *satisfies (3.1) and* q_1 *satisfies (3.9). If* $\left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^\delta \in \mathcal{H}[q(0), 1] \cap Q$,

$$(1-\lambda\alpha_1)\left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^\delta + \lambda\alpha_1\left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^\delta\left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)}\right)$$

is univalent in Δ. *If* $f \in \mathcal{A}$ *satisfies*

$$\begin{aligned} q_1(z) &+ \frac{\lambda}{\delta}zq_1'(z) \\ &\prec (1-\lambda\alpha_1)\left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^\delta + \lambda\alpha_1\left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^\delta\left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)}\right) \\ &\prec q_2(z) + \frac{\lambda}{\delta}zq_2'(z), \end{aligned} \tag{3.12}$$

then

$$q_1(z) \prec \left(\frac{H_m^l[\alpha_1]f(z)}{z}\right)^\delta \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

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