

# Journal of Inequalities in Pure and Applied Mathematics

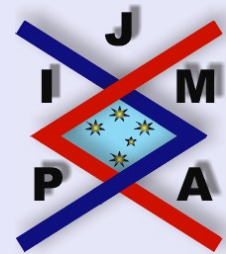
## A GENERAL OPTIMAL INEQUALITY FOR ARBITRARY RIEMANNIAN SUBMANIFOLDS

BANG-YEN CHEN

Department of Mathematics  
Michigan State University  
East Lansing, MI 48824-1027, USA

*EMail:* [bychen@math.msu.edu](mailto:bychen@math.msu.edu)

©2000 [Victoria University](#)  
ISSN (electronic): 1443-5756  
086-05



---

volume 6, issue 3, article 77,  
2005.

*Received 22 March, 2005;  
accepted 28 July, 2005.*

*Communicated by: W.S. Cheung*

---

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

## Abstract

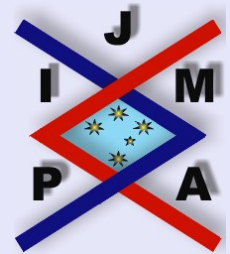
One of the most fundamental problems in submanifold theory is to establish simple relationships between intrinsic and extrinsic invariants of the submanifolds (cf. [6]). A general optimal inequality for submanifolds in Riemannian manifolds of constant sectional curvature was obtained in an earlier article [5]. In this article we extend this inequality to a general optimal inequality for arbitrary Riemannian submanifolds in an arbitrary Riemannian manifold. This new inequality involves only the  $\delta$ -invariants, the squared mean curvature of the submanifolds and the maximum sectional curvature of the ambient manifold. Several applications of this new general inequality are also presented.

*2000 Mathematics Subject Classification:* 53C40, 53C42, 53B25

*Key words:*  $\delta$ -invariants, Inequality, Riemannian submanifold, Squared mean curvature, Sectional curvature.

## Contents

1	Introduction .....	3
2	Preliminaries .....	5
3	A General Optimal Inequality .....	8
4	Some Applications .....	15
	References	



### A General Optimal Inequality for Arbitrary Riemannian Submanifolds

Bang-Yen Chen

Title Page

Contents



Go Back

Close

Quit

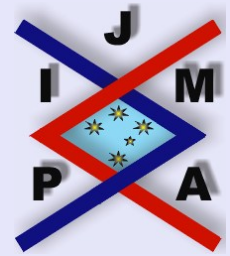
Page 2 of 23

# 1. Introduction

According to the celebrated embedding theorem of J.F. Nash [23], every Riemannian manifold can be isometrically embedded in some Euclidean spaces with sufficiently high codimension. The Nash theorem was established in the hope that if Riemannian manifolds could always be regarded as Riemannian submanifolds, this would then yield the opportunity to use extrinsic help. However, as observed by M. Gromov [18], this hope had not been materialized. The main reason for this is due to the lack of controls of the extrinsic properties of the submanifolds by the known intrinsic data.

In order to overcome the difficulty mentioned above, the author introduced in [4, 5] some new types of Riemannian invariants, denoted by  $\delta(n_1, \dots, n_k)$ . Moreover, he was able to establish in [5] an optimal general inequality for submanifolds in real space forms which involves his  $\delta$ -invariants and the main extrinsic invariant; namely, the squared mean curvature. Such inequality provides prima controls on the most important extrinsic curvature invariant by the initial intrinsic data of the Riemannian submanifolds in real space forms. As an application, he was able to discover new intrinsic spectral properties of homogeneous spaces via Nash's theorem. Such results extend a well-known theorem of Nagano [22]. Since then the  $\delta$ -invariant and the inequality established in [5] have been further investigated by many geometers (see for instance, [2, 8, 9, 10, 11, 14, 12, 13, 15, 17, 20, 21, 24, 25, 26, 27, 28, 29, 30]). Recently, the  $\delta$ -invariants have also been applied to general relativity theory as well as to affine geometry (see for instance, [7, 16, 19]).

In this article we use the same idea introduced in the earlier article [5] to extend the inequality mentioned above to a more general optimal inequality for



---

**A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds**

Bang-Yen Chen

---

Title Page

Contents



Go Back

Close

Quit

Page 3 of 23

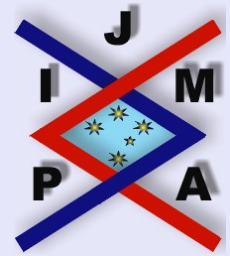
an arbitrary Riemannian submanifold in an arbitrary Riemannian manifold.

Our general inequality involves the  $\delta$ -invariant, the squared mean curvature of the Riemannian submanifold and the maximum of the sectional curvature function of the ambient Riemannian manifold (restricted to plane sections of the tangent space of the submanifold at a point on the submanifold). More precisely, we prove in Section 3 that, for any  $n$ -dimensional submanifold  $M$  in a Riemannian  $m$ -manifold  $\tilde{M}^m$ , we have the following general optimal inequality:

$$(1.1) \quad \delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k)H^2 + b(n_1, \dots, n_k) \max \tilde{K}$$

for any  $k$ -tuple  $(n_1, \dots, n_k) \in \mathcal{S}(n)$ , where  $\max \tilde{K}(p)$  denotes the maximum of the sectional curvature function of  $\tilde{M}^m$  restricted to 2-plane sections of the tangent space  $T_p M$  of  $M$  at  $p$ . (see Section 3 for details). (When  $k = 0$ , inequality (1.1) can be found in B. Suceavă's article [27]).

In the last section we provide several immediate applications of inequality (1.1). In particular, by applying our inequality we conclude that if  $M$  is a Riemannian  $n$ -manifold with  $\delta(n_1, \dots, n_k) > 0$  at some point in  $M$  for some  $k$ -tuple  $(n_1, \dots, n_k) \in \mathcal{S}(n)$ , then  $M$  admits no minimal isometric immersion into any Riemannian manifold with non-positive sectional curvature. In this section, we also apply inequality (1.1) to derive two inequalities for submanifolds in Sasakian space forms. In fact, many inequalities for submanifolds in various space forms obtained by various people can also be derived directly from inequality (1.1).




---

**A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds**

Bang-Yen Chen

---

Title Page

Contents



Go Back

Close

Quit

Page 4 of 23

## 2. Preliminaries

Let  $M$  be an  $n$ -dimensional submanifold of a Riemannian  $m$ -manifold  $\tilde{M}^m$ . We choose a local field of orthonormal frame

$$e_1, \dots, e_n, e_{n+1}, \dots, e_m$$

in  $\tilde{M}^m$  such that, restricted to  $M$ , the vectors  $e_1, \dots, e_n$  are tangent to  $M$  and hence  $e_{n+1}, \dots, e_m$  are normal to  $M$ . Let  $K(e_i \wedge e_j)$  and  $\tilde{K}(e_i \wedge e_j)$  denote respectively the sectional curvatures of  $M$  and  $\tilde{M}^m$  of the plane section spanned by  $e_i$  and  $e_j$ .

For the submanifold  $M$  in  $\tilde{M}^m$  we denote by  $\nabla$  and  $\tilde{\nabla}$  the Levi-Civita connections of  $M$  and  $\tilde{M}^m$ , respectively. The Gauss and Weingarten formulas are given respectively by (see, for instance, [3])

$$(2.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

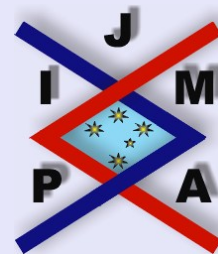
$$(2.2) \quad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$$

for any vector fields  $X, Y$  tangent to  $M$  and vector field  $\xi$  normal to  $M$ , where  $h$  denotes the second fundamental form,  $D$  the normal connection, and  $A$  the shape operator of the submanifold.

Let  $\{h_{ij}^r\}$ ,  $i, j = 1, \dots, n$ ;  $r = n + 1, \dots, m$ , denote the coefficients of the second fundamental form  $h$  with respect to  $e_1, \dots, e_n, e_{n+1}, \dots, e_m$ . Then we have

$$h_{ij}^r = \langle h(e_i, e_j), e_r \rangle = \langle A_{e_r} e_i, e_j \rangle,$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product.



**A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds**

Bang-Yen Chen

Title Page

Contents



Go Back

Close

Quit

Page 5 of 23

The mean curvature vector  $\vec{H}$  is defined by

$$(2.3) \quad \vec{H} = \frac{1}{n} \text{trace } h = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i),$$

where  $\{e_1, \dots, e_n\}$  is a local orthonormal frame of the tangent bundle  $TM$  of  $M$ . The squared mean curvature is then given by  $H^2 = \langle \vec{H}, \vec{H} \rangle$ . A submanifold  $M$  is called minimal in  $\tilde{M}^m$  if its mean curvature vector vanishes identically.

Denote by  $R$  and  $\tilde{R}$  the Riemann curvature tensors of  $M$  and  $\tilde{M}^m$ , respectively. Then the *equation of Gauss* is given by

$$(2.4) \quad R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) + \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle,$$

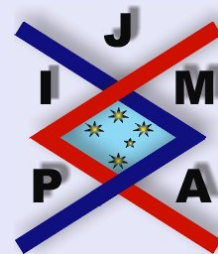
for vectors  $X, Y, Z, W$  tangent to  $M$ .

For any orthonormal basis  $e_1, \dots, e_n$  of the tangent space  $T_p M$ , the scalar curvature  $\tau$  of  $M$  at  $p$  is defined to be

$$(2.5) \quad \tau(p) = \sum_{i < j} K(e_i \wedge e_j),$$

where  $K(e_i \wedge e_j)$  denotes the sectional curvature of the plane section spanned by  $e_i$  and  $e_j$ .

Let  $L$  be a subspace of  $T_p M$  of dimension  $r \geq 1$  and  $\{e_1, \dots, e_r\}$  an orthonormal basis of  $L$ . The scalar curvature  $\tau(L)$  of the  $r$ -plane section  $L$  is



**A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds**

Bang-Yen Chen

Title Page

Contents



Go Back

Close

Quit

Page 6 of 23

defined by

$$(2.6) \quad \tau(L) = \sum_{\alpha < \beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.$$

When  $r = 1$ , we have  $\tau(L) = 0$ .

For integers  $k \geq 0$  and  $n \geq 2$ , let us denote by  $\mathcal{S}(n, k)$  the finite set consisting of unordered  $k$ -tuples  $(n_1, \dots, n_k)$  of integers  $\geq 2$  which satisfies

$$n_1 < n \quad \text{and} \quad n_1 + \dots + n_k \leq n.$$

Let  $\mathcal{S}(n)$  be the union  $\cup_{k \geq 0} \mathcal{S}(n, k)$ .

For any  $(n_1, \dots, n_k) \in \mathcal{S}(n)$ , the Riemannian invariants  $\delta(n_1, \dots, n_k)$  introduced in [5] are defined by

$$(2.7) \quad \delta(n_1, \dots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \dots + \tau(L_k)\},$$

where  $L_1, \dots, L_k$  run over all  $k$  mutually orthogonal subspaces of  $T_p M$  with  $\dim L_j = n_j$ ,  $j = 1, \dots, k$ .

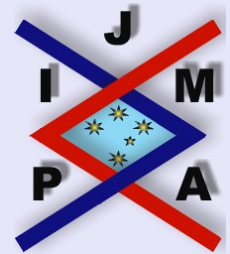
We recall the following general algebraic lemma from [4] for later use.

**Lemma 2.1.** *Let  $a_1, \dots, a_n, \eta$  be  $n + 1$  real numbers such that*

$$\left( \sum_{i=1}^n a_i \right)^2 = (n - 1) \left( \eta + \sum_{i=1}^n a_i^2 \right).$$

*Then  $2a_1 a_2 \geq \eta$ , with equality holding if and only if we have*

$$a_1 + a_2 = a_3 = \dots = a_n.$$



A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds

Bang-Yen Chen

Title Page

Contents



Go Back

Close

Quit

Page 7 of 23

### 3. A General Optimal Inequality

For each  $(n_1, \dots, n_k) \in \mathcal{S}(n)$ , let  $c(n_1, \dots, n_k)$  and  $b(n_1, \dots, n_k)$  be the positive numbers given by

$$(3.1) \quad c(n_1, \dots, n_k) = \frac{n^2(n+k-1 - \sum_{j=1}^k n_j)}{2(n+k - \sum_{j=1}^k n_j)},$$

$$(3.2) \quad b(n_1, \dots, n_k) = \frac{1}{2}(n(n-1) - \sum_{j=1}^k n_j(n_j-1)).$$

For an arbitrary Riemannian submanifold we have the following general optimal inequality.

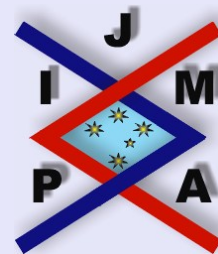
**Theorem 3.1.** *Let  $M$  be an  $n$ -dimensional submanifold of an arbitrary Riemannian  $m$ -manifold  $\tilde{M}^m$ . Then, for each point  $p \in M$  and for each  $k$ -tuple  $(n_1, \dots, n_k) \in \mathcal{S}(n)$ , we have the following inequality:*

$$(3.3) \quad \delta(n_1, \dots, n_k)(p) \leq c(n_1, \dots, n_k)H^2(p) + b(n_1, \dots, n_k) \max \tilde{K}(p),$$

where  $\max \tilde{K}(p)$  denotes the maximum of the sectional curvature function of  $\tilde{M}^m$  restricted to 2-plane sections of the tangent space  $T_p M$  of  $M$  at  $p$ .

The equality case of inequality (3.3) holds at a point  $p \in M$  if and only the following two conditions hold:

(a) There exists an orthonormal basis  $e_1, \dots, e_m$  at  $p$ , such that the shape



A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds

Bang-Yen Chen

Title Page

Contents



Go Back

Close

Quit

Page 8 of 23



operators of  $M$  in  $\tilde{M}^m$  at  $p$  take the following form :

$$(3.4) \quad A_{e_r} = \begin{bmatrix} A_1^r & \dots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & A_k^r & \\ & & 0 & \mu_r I \end{bmatrix}, \quad r = n + 1, \dots, m,$$

where  $I$  is an identity matrix and each  $A_j^r$  is a symmetric  $n_j \times n_j$  submatrix such that

$$(3.5) \quad \text{trace}(A_1^r) = \dots = \text{trace}(A_k^r) = \mu_r.$$

(b) For any  $k$  mutual orthogonal subspaces  $L_1, \dots, L_k$  of  $T_p M$  which satisfy

$$\delta(n_1, \dots, n_k) = \tau - \sum_{j=1}^k \tau(L_j)$$

at  $p$ , we have

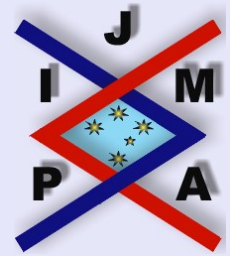
$$(3.6) \quad \tilde{K}(e_{\alpha_i}, e_{\alpha_j}) = \max \tilde{K}(p)$$

for any  $\alpha_i \in \Delta_i, \alpha_j \in \Delta_j$  with  $i \neq j$ , where

$$\Delta_1 = \{1, \dots, n_1\},$$

...

$$\Delta_k = \{n_1 + \dots + n_{k-1} + 1, \dots, n_1 + \dots + n_k\}.$$



**A General Optimal Inequality for Arbitrary Riemannian Submanifolds**

Bang-Yen Chen

Title Page

Contents



Go Back

Close

Quit

Page 9 of 23

*Proof.* Let  $M$  be an  $n$ -dimensional submanifold of a Riemannian  $m$ -manifold  $\tilde{M}^m$  and  $p$  be a point in  $M$ . Then the equation of Gauss implies that at  $p$  we have

$$(3.7) \quad 2\tau(p) = n^2H^2 - \|h\|^2 + 2\tilde{\tau}(T_pM),$$

where  $\|h\|^2$  is the squared norm of the second fundamental form  $h$  and  $\tilde{\tau}(T_pM)$  is the scalar curvature of the ambient space  $\tilde{M}^m$  corresponding to the subspace  $T_pM \subset T_p\tilde{M}^m$ , i.e.

$$\tilde{\tau}(T_pM) = \sum_{i < j} \tilde{K}(e_i, e_j)$$

for an orthonormal basis  $e_1, \dots, e_n$  of  $T_pM$ .

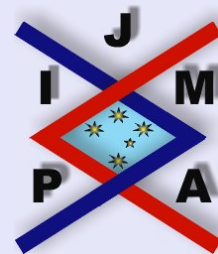
Let us put

$$(3.8) \quad \eta = 2\tau(p) - \frac{n^2(n+k-1-\sum n_j)}{n+k-\sum n_j}H^2 - 2\tilde{\tau}(T_pM).$$

Then we obtain from (3.7) and (3.8) that

$$(3.9) \quad n^2H^2 = \gamma(\eta + \|h\|^2), \quad \gamma = n+k - \sum n_j.$$

At  $p$ , let us choose an orthonormal basis  $e_1, \dots, e_m$  such that  $e_{\alpha_i} \in L_i$  for each  $\alpha_i \in \Delta_i$ . Moreover, we choose the normal vector  $e_{n+1}$  to be in the direction of the mean curvature vector at  $p$  (When the mean curvature vanishes at  $p$ ,  $e_{n+1}$



**A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds**

Bang-Yen Chen

Title Page

Contents



Go Back

Close

Quit

Page 10 of 23

can be chosen to be any unit normal vector at  $p$ ). Then (3.9) yields

$$(3.10) \quad \left( \sum_{A=1}^n a_A \right)^2 = \gamma \left[ \eta + \sum_{A \neq B} (h_{AB}^{n+1})^2 + \sum_{A=1}^n (a_A)^2 + \sum_{r=n+2}^m \sum_{A,B=1}^n (h_{AB}^r)^2 \right],$$

where  $a_A = h_{AA}^{n+1}$  with  $1 \leq A, B \leq n$ . Equation (3.10) is equivalent to

$$(3.11) \quad \left( \sum_{i=1}^{\gamma+1} \bar{a}_i \right)^2 = \gamma \left[ \eta + \sum_{i=1}^{\gamma+1} (\bar{a}_i)^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^m \sum_{i,j=1}^n (h_{ij}^r)^2 - \sum_{1 \leq \alpha_1 \neq \beta_1 \leq n_1} a_{\alpha_1} a_{\beta_1} - \sum_{\alpha_2 \neq \beta_2} a_{\alpha_2} a_{\beta_2} - \cdots - \sum_{\alpha_k \neq \beta_k} a_{\alpha_k} a_{\beta_k} \right],$$

where  $\alpha_2, \beta_2 \in \Delta_2, \dots, \alpha_k, \beta_k \in \Delta_k$  and

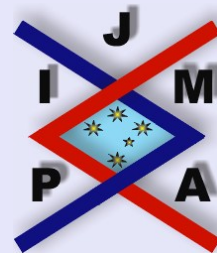
$$(3.12) \quad \bar{a}_1 = a_1, \quad \bar{a}_2 = a_2 + \cdots + a_{n_1},$$

$$(3.13) \quad \bar{a}_3 = a_{n_1+1} + \cdots + a_{n_1+n_2},$$

$$(3.14) \quad \dots$$

$$(3.15) \quad \bar{a}_{k+1} = a_{n_1+\dots+n_{k-1}+1} + \cdots + a_{n_1+\dots+n_k},$$

$$(3.15) \quad \bar{a}_{k+2} = a_{n_1+\dots+n_k+1}, \dots, \bar{a}_{\gamma+1} = a_n.$$



**A General Optimal Inequality for Arbitrary Riemannian Submanifolds**

Bang-Yen Chen

Title Page

Contents



Go Back

Close

Quit

Page 11 of 23

By applying Lemma 2.1 to (3.11) we obtain

$$(3.16) \quad \sum_{\alpha_1 < \beta_1} a_{\alpha_1} a_{\beta_1} + \sum_{\alpha_2 < \beta_2} a_{\alpha_2} a_{\beta_2} + \cdots + \sum_{\alpha_k < \beta_k} a_{\alpha_k} a_{\beta_k} \\ \geq \frac{\eta}{2} + \sum_{A < B} (h_{AB}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^m \sum_{A, B=1}^n (h_{AB}^r)^2,$$

where  $\alpha_i, \beta_i \in \Delta_i$ ,  $i = 1, \dots, k$ .

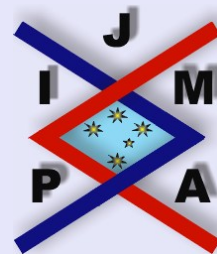
On the other hand, equation (2.6) and the equation of Gauss imply that, for each  $j \in \{1, \dots, k\}$ , we have

$$(3.17) \quad \tau(L_j) = \sum_{r=n+1}^m \sum_{\alpha_j < \beta_j} \left( h_{\alpha_j \alpha_j}^r h_{\beta_j \beta_j}^r - (h_{\alpha_j \beta_j}^r)^2 \right) + \tilde{\tau}(L_j), \\ \alpha_j, \beta_j \in \Delta_j.$$

where  $\tilde{\tau}(L_j)$  is the scalar curvature  $\tilde{M}^m$  associated with  $L_j \subset T_p$ .

Let us put  $\Delta = \Delta_1 \cup \cdots \cup \Delta_k$  and  $\Delta^2 = (\Delta_1 \times \Delta_1) \cup \cdots \cup (\Delta_k \times \Delta_k)$ . Then we obtain by combining (3.8), (3.16) and (3.17) that

$$(3.18) \quad \tau(L_1) + \cdots + \tau(L_k) \\ \geq \frac{\eta}{2} + \frac{1}{2} \sum_{r=n+1}^m \sum_{(\alpha, \beta) \notin \Delta^2} (h_{\alpha\beta}^r)^2 \\ + \frac{1}{2} \sum_{r=n+2}^m \sum_{j=1}^k \left( \sum_{\alpha_j \in \Delta_j} h_{\alpha_j \alpha_j}^r \right)^2 + \sum_{j=1}^k \tilde{\tau}(L_j)$$



A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds

Bang-Yen Chen

Title Page

Contents



Go Back

Close

Quit

Page 12 of 23

$$\begin{aligned} &\geq \frac{\eta}{2} + \sum_{j=1}^k \tilde{\tau}(L_j) \\ &= \tau - \frac{n^2(n+k-1 - \sum n_j)}{2(n+k - \sum n_j)} H^2 - \left( \tilde{\tau}(T_p M) - \sum_{j=1}^k \tilde{\tau}(L_j) \right). \end{aligned}$$

Therefore, by (2.7) and (3.18), we obtain

$$(3.19) \quad \tau - \sum_{j=1}^k \tau(L_j) \leq \frac{n^2(n+k-1 - \sum n_j)}{2(n+k - \sum n_j)} H^2 + \tilde{\tau}(T_p M) - \sum_{j=1}^k \tilde{\tau}(L_j),$$

which implies that

$$(3.20) \quad \delta(n_1, \dots, n_k) \leq \frac{n^2(n+k-1 - \sum n_j)}{2(n+k - \sum n_j)} H^2 + \tilde{\delta}^M(n_1, \dots, n_k),$$

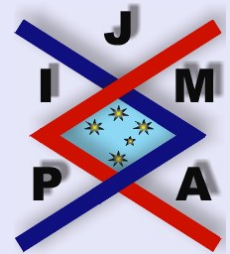
where

$$(3.21) \quad \tilde{\delta}^M(n_1, \dots, n_k) := \tilde{\tau}(T_p M) - \inf \{ \tilde{\tau}(\tilde{L}_1) + \dots + \tilde{\tau}(\tilde{L}_k) \}$$

with  $\tilde{L}_1, \dots, \tilde{L}_k$  run over all  $k$  mutually orthogonal subspaces of  $T_p M$  such that  $\dim \tilde{L}_j = n_j$ ;  $j = 1, \dots, k$ . Clearly, inequality (3.21) implies inequality (3.3).

It is easy to see that the equality case of (3.3) holds at the point  $p$  if and only if the following two conditions hold:

- (i) The inequalities in (3.16) and (3.18) are actually equalities;



A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds

Bang-Yen Chen

Title Page

Contents



Go Back

Close

Quit

Page 13 of 23

(ii) For any  $k$  mutual orthogonal subspaces  $L_1, \dots, L_k$  of  $T_p M$  which satisfy

$$(3.22) \quad \delta(n_1, \dots, n_k) = \tau - \sum_{j=1}^k \tau(L_j)$$

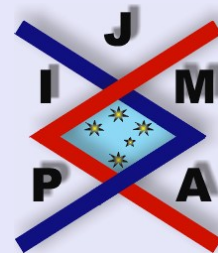
at  $p$ , we have

$$(3.23) \quad \tilde{K}(e_{\alpha_i}, e_{\alpha_j}) = \max \tilde{K}(p)$$

for any  $\alpha_i \in \Delta_i, \alpha_j \in \Delta_j$  with  $i \neq j$ .

It follows from Lemma 2.1, (3.16) and (3.18) that condition (i) holds if and only if there exists an orthonormal basis  $e_1, \dots, e_m$  at  $p$ , such that the shape operators of  $M$  in  $\tilde{M}^m$  at  $p$  satisfy conditions (3.4) and (3.5).

The converse can be easily verified. □




---

**A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds**

Bang-Yen Chen

---

Title Page

Contents



Go Back

Close

Quit

Page 14 of 23

## 4. Some Applications

The following results follow immediately from Theorem 3.1

**Theorem 4.1.** *Let  $M$  be an  $n$ -dimensional submanifold of the complex projective  $m$ -space  $CP^m(4\epsilon)$  of constant holomorphic sectional curvature  $4\epsilon$  (or the quaternionic projective  $m$ -space  $QP^m(4\epsilon)$  of quaternionic sectional curvature  $4\epsilon$ ). Then we have*

$$(4.1) \quad \delta(n_1, \dots, n_k)(p) \leq c(n_1, \dots, n_k)H^2(p) + 4b(n_1, \dots, n_k)\epsilon$$

for any  $k$ -tuple  $(n_1, \dots, n_k) \in \mathcal{S}(n)$ .

**Theorem 4.2.** *Let  $M$  be an  $n$ -dimensional submanifold of the complex hyperbolic  $m$ -space  $CH^m(4\epsilon)$  of constant holomorphic sectional curvature  $4\epsilon$  (or the quaternionic hyperbolic  $m$ -space  $QH^m(4\epsilon)$  of quaternionic sectional curvature  $4\epsilon$ ). Then we have*

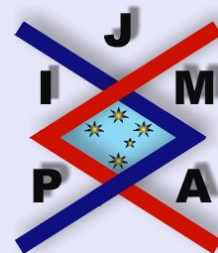
$$(4.2) \quad \delta(n_1, \dots, n_k)(p) \leq c(n_1, \dots, n_k)H^2(p) + b(n_1, \dots, n_k)\epsilon$$

for any  $k$ -tuple  $(n_1, \dots, n_k) \in \mathcal{S}(n)$ .

**Theorem 4.3.** *Let  $\tilde{M}^m$  be a Riemannian manifold whose sectional curvature function is bounded above by  $\epsilon$ . If  $M$  is a Riemannian  $n$ -manifold such that*

$$\delta(n_1, \dots, n_k)(p) > \frac{1}{2} \left( n(n-1) - \sum n_j(n_j-1) \right) \epsilon$$

for some  $k$ -tuple  $(n_1, \dots, n_k) \in \mathcal{S}(n)$  at some point  $p \in M$ , then  $M$  admits no minimal isometric immersion in  $\tilde{M}^m$ .



---

A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds

Bang-Yen Chen

---

Title Page

Contents



Go Back

Close

Quit

Page 15 of 23

In particular, we have the following non-existence result.

**Corollary 4.4.** *If  $M$  is a Riemannian  $n$ -manifold with*

$$\delta(n_1, \dots, n_k) > 0$$

*at some point in  $M$  for some  $k$ -tuple  $(n_1, \dots, n_k) \in \mathcal{S}(n)$ , then  $M$  admits no minimal isometric immersion into any Riemannian  $m$ -manifold  $\tilde{M}^m$  with non-positive sectional curvature, regardless of codimension.*

A  $(2m + 1)$ -dimensional manifold is called *almost contact* if it admits a tensor field  $\phi$  of type  $(1, 1)$ , a vector field  $\xi$  and a 1-form  $\eta$  satisfying

$$(4.3) \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

where  $I$  is the identity endomorphism. It is well-known that

$$\phi\xi = 0, \quad \eta \circ \phi = 0.$$

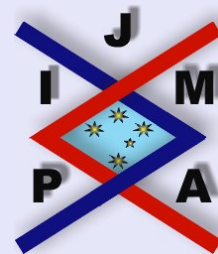
Moreover, the endomorphism  $\phi$  has rank  $2m$ .

An almost contact manifold  $(\tilde{M}, \phi, \xi, \eta)$  is called an *almost contact metric manifold* if it admits a Riemannian metric  $g$  such that

$$(4.4) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for vector fields  $X, Y$  tangent to  $\tilde{M}$ . Setting  $Y = \xi$  we have immediately that

$$\eta(X) = g(X, \xi).$$



**A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds**

Bang-Yen Chen

Title Page

Contents



Go Back

Close

Quit

Page 16 of 23



By a *contact manifold* we mean a  $(2m + 1)$ -manifold  $\tilde{M}$  together with a global 1-form  $\eta$  satisfying

$$\eta \wedge (d\eta)^m \neq 0$$

on  $M$ . If  $\eta$  of an almost contact metric manifold  $(\tilde{M}, \phi, \xi, \eta, g)$  is a contact form and if  $\eta$  satisfies

$$d\eta(X, Y) = g(X, \phi Y)$$

for all vectors  $X, Y$  tangent to  $\tilde{M}$ , then  $\tilde{M}$  is called a *contact metric manifold*.

A contact metric manifold is called *K-contact* if its characteristic vector field  $\xi$  is a Killing vector field. It is well-known that a *K-contact* metric  $(2n + 1)$ -manifold satisfies

$$(4.5) \quad \nabla_X \xi = -\phi X, \quad \tilde{K}(X, \xi) = 1$$

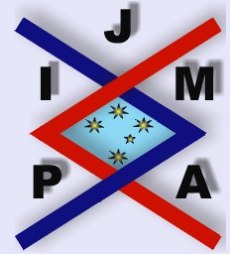
for  $X \in \ker \eta$ , where  $\tilde{K}$  denotes the sectional curvature on  $M$ .

A *K-contact* manifold is called *Sasakian* if we have

$$N_\phi + 2d\eta \otimes \xi = 0,$$

where  $N_\phi$  is the Nijenhuis tensor associated to  $\phi$ . A plane section  $\sigma$  in  $T_p \tilde{M}^{2m+1}$  of a Sasakian manifold  $\tilde{M}^{2m+1}$  is called  $\phi$ -section if it is spanned by  $X$  and  $\phi(X)$ , where  $X$  is a unit tangent vector orthogonal to  $\xi$ . The sectional curvature with respect to a  $\phi$ -section  $\sigma$  is called a  $\phi$ -sectional curvature. If a Sasakian manifold has constant  $\phi$ -sectional curvature, it is called a Sasakian space form.

An  $n$ -dimensional submanifold  $M^n$  of a Sasakian space form  $\tilde{M}^{2m+1}(c)$  is called a  $C$ -totally real submanifold of  $\tilde{M}^{2m+1}(c)$  if  $\xi$  is a normal vector field on



**A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds**

Bang-Yen Chen

Title Page

Contents



Go Back

Close

Quit

Page 17 of 23

$M^n$ . A direct consequence of this definition is that  $\phi(TM^n) \subset T^\perp M^n$ , which means that  $M^n$  is an anti-invariant submanifold of  $\tilde{M}^{2m+1}(c)$

It is well-known that the Riemannian curvature tensor of a Sasakian space form  $\tilde{M}^{2m+1}(\epsilon)$  of constant  $\phi$ -sectional curvature  $\epsilon$  is given by [1]:

$$(4.6) \quad \begin{aligned} \tilde{R}(X, Y)Z &= \frac{\epsilon + 3}{4}(\langle Y, Z \rangle X - \langle X, Z \rangle Y) \\ &+ \frac{\epsilon - 1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \langle X, Z \rangle \eta(Y)\xi \\ &- \langle Y, Z \rangle \eta(X)\xi + \langle \phi Y, Z \rangle \phi X - \langle \phi X, Z \rangle \phi Y - 2 \langle \phi X, Y \rangle \phi Z) \end{aligned}$$

for  $X, Y, Z$  tangent to  $\tilde{M}^{2m+1}(\epsilon)$ . Hence if  $\epsilon \geq 1$ , the sectional curvature function  $\tilde{K}$  of  $\tilde{M}^{2m+1}(\epsilon)$  satisfies

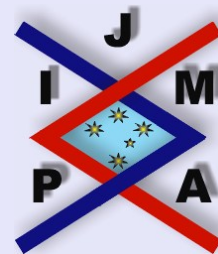
$$(4.7) \quad \frac{\epsilon + 3}{4} \leq \tilde{K}(X, Y) \leq \epsilon$$

for  $X, Y \in \ker \eta$ ; if  $\epsilon < 1$ , the inequalities are reversed.

From Theorem 3.1 and these sectional curvature properties (4.5) and (4.7) of Sasakian space forms, we obtain the following results for arbitrary Riemannian submanifolds in Sasakian space forms.

**Corollary 4.5.** *If  $M$  is an  $n$ -dimensional submanifold of a Sasakian space form  $\tilde{M}(\epsilon)$  of constant  $\phi$ -sectional curvature  $\epsilon \geq 1$ , then, for any  $k$ -tuple  $(n_1, \dots, n_k) \in \mathcal{S}(n)$ , we have*

$$(4.8) \quad \delta(n_1, \dots, n_k)(p) \leq c(n_1, \dots, n_k)H^2(p) + b(n_1, \dots, n_k)\epsilon.$$



A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds

Bang-Yen Chen

Title Page

Contents

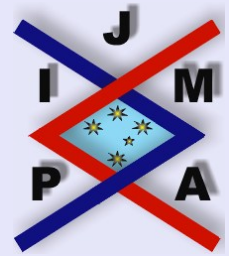


Go Back

Close

Quit

Page 18 of 23



**Corollary 4.6.** *If  $M$  is an  $n$ -dimensional submanifold of a Sasakian space form  $\tilde{M}(\epsilon)$  of constant  $\phi$ -sectional curvature  $\epsilon < 1$ , then, for any  $k$ -tuple  $(n_1, \dots, n_k) \in \mathcal{S}(n)$ , we have*

$$(4.9) \quad \delta(n_1, \dots, n_k)(p) \leq c(n_1, \dots, n_k)H^2(p) + b(n_1, \dots, n_k).$$

**Corollary 4.7.** *If  $M$  is an  $n$ -dimensional  $C$ -totally real submanifold of a Sasakian space form  $\tilde{M}(\epsilon)$  of constant  $\phi$ -sectional curvature  $\epsilon \leq 1$ , then, for any  $k$ -tuple  $(n_1, \dots, n_k) \in \mathcal{S}(n)$ , we have*

$$(4.10) \quad \delta(n_1, \dots, n_k)(p) \leq c(n_1, \dots, n_k)H^2(p) + b(n_1, \dots, n_k)\frac{\epsilon + 3}{4}.$$

Corollary 4.7 has been obtained in [13].

---

**A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds**

Bang-Yen Chen

---

Title Page

Contents



Go Back

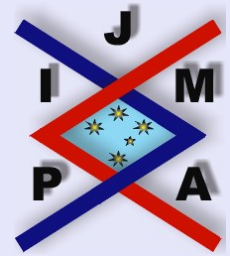
Close

Quit

Page 19 of 23

## References

- [1] D.E. BLAIR, *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhäuser, Boston, 2002.
- [2] D.E. BLAIR, F. DILLEN, L. VERSTRAELEN AND L. VRANCKEN, Calabi curves as holomorphic Legendre curves and Chen's inequality, *Kyungpook Math. J.*, **35** (1996), 407–416.
- [3] B.Y. CHEN, *Geometry of Submanifolds*, M. Dekker, New York, 1973.
- [4] B.Y. CHEN, Some pinching and classification theorems for minimal submanifolds, *Arch. Math.*, **60** (1993), 568–578.
- [5] B.Y. CHEN, Some new obstructions to minimal and Lagrangian isometric immersions, *Japan. J. Math.*, **26** (2000), 105–127.
- [6] B.Y. CHEN, *Riemannian Submanifolds*, in *Handbook of Differential Geometry, Volume I*, North Holland, (edited by F. Dillen and L. Verstraelen) 2000, pp. 187–418.
- [7] B.Y. CHEN, F. DILLEN AND L. VERSTRAELEN, Affine  $\delta$ -invariants and their applications to centroaffine geometry, *Differential Geom. Appl.*, **23** (2005), 341–354.
- [8] B.Y. CHEN AND I. MIHAI, Isometric immersions of contact Riemannian manifolds in real space forms, *Houston J. Math.*, **31** (2005), 743–764.
- [9] D. CIOROBOIU, B. Y. Chen inequalities for bi-slant submanifolds in Sasakian space forms, *Demonstratio Math.*, **36** (2003), 179–187.



---

A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds

Bang-Yen Chen

---

Title Page

Contents



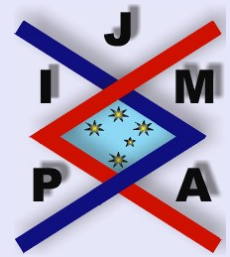
Go Back

Close

Quit

Page 20 of 23

- [10] D. CIOROBOIU, B. Y. Chen inequalities for semislant submanifolds in Sasakian space forms, *Int. J. Math. Math. Sci.*, **2003** no. 27, 1731–1738.
- [11] M. DAJCZER AND L.A. FLORIT, On Chen’s basic equality, *Illinois J. Math.*, **42** (1998), 97–106.
- [12] F. DEFEVER, I. MIHAI AND L. VERSTRAELEN, B. Y. Chen’s inequality for C-totally real submanifolds in Sasakian space forms, *Boll. Un. Mat. Ital., Ser. B*, **11** (1997), 365-374.
- [13] F. DEFEVER, I. MIHAI AND L. VERSTRAELEN, B.-Y. Chen’s inequalities for submanifolds of Sasakian space forms, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.*, (8) **4** (2001), 521–529.
- [14] F. DILLEN, M. PETROVIĆ AND L. VERSTRAELEN, Einstein, conformally flat and semi-symmetric submanifolds satisfying Chen’s equality, *Israel J. Math.*, **100** (1997), 163–169.
- [15] F. DILLEN AND L. VRANCKEN, Totally real submanifolds in  $S^6(1)$  satisfying Chen’s equality, *Trans. Amer. Math. Soc.*, **348** (1996), 1633–1646
- [16] F. DILLEN AND L. VRANCKEN, Improper affine spheres and  $\delta$ -invariants, Banach Center Publications, *Polish Acad. Sci.* (to appear).
- [17] S. FUNABASHI, Y.M. KIM AND J.S. PAK, On submanifolds of  $L \times_f F$  satisfying Chen’s basic equality, *Acta Math. Hungr.* **99** (2003), 189-201.
- [18] M. GROMOV, Isometric immersions of Riemannian manifolds, in: *Elie Cartan et les Mathématiques d’Aujourd’hui*, Astérisque 1985, pp. 129–133.




---

**A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds**

Bang-Yen Chen

---

Title Page

Contents



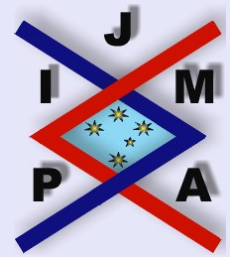
Go Back

Close

Quit

Page **21** of **23**

- [19] S. HAESSEN AND L. VERSTRAELEN, Ideally embedded space-times. *J. Math. Phys.*, **45** (2004), 1497–1510.
- [20] J.-K. KIM, Y.-M. SONG AND M. M. TRIPATHI, B.-Y. Chen inequalities for submanifolds in generalized complex space forms, *Bull. Korean Math. Soc.*, **40** (2003), 411–423.
- [21] Y. M. KIM, Chen’s basic equalities for submanifolds of Sasakian space form, *Kyungpook Math. J.*, **43** (2003), 63–71.
- [22] T. NAGANO, On the minimum eigenvalues of the Laplacians in Riemannian manifolds, *Sci. Papers College Gen. Edu. Univ. Tokyo*, **11** (1961), 177–182.
- [23] J.F. NASH, The imbedding problem for Riemannian manifolds, *Ann. of Math.*, **63** (1956), 20–63.
- [24] T. SASAHARA,  $CR$ -submanifolds in a complex hyperbolic space satisfying an equality of Chen, *Tsukuba J. Math.*, **23** (1999), 565–583.
- [25] T. SASAHARA, Chen invariant of  $CR$ -submanifolds, in: *Geometry of Submanifolds*, pp. 114–120, Kyoto, 2001.
- [26] T. SASAHARA, On Chen invariant of  $CR$ -submanifolds in a complex hyperbolic space, *Tsukuba J. Math.*, **26** (2002), 119–132.
- [27] B. SUCEAVĂ, Some remarks on B. Y. Chen’s inequality involving classical invariants, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)*, **45** (1999), 405–412.




---

**A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds**

Bang-Yen Chen

---

Title Page

Contents



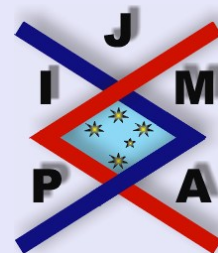
Go Back

Close

Quit

Page **22** of **23**

- [28] B. SUCEAVĂ, The Chen invariants of warped products of hyperbolic planes and their applications to immersibility problems, *Tsukuba J. Math.*, **25** (2001), 311–320.
- [29] M.M. TRIPATHI, J.S. KIM AND S.B. KIM, A note on Chen's basic equality for submanifolds in a Sasakian space form, *Int. J. Math. Math. Sci.*, **11** (2003), 711–716.
- [30] D.W. YOON, B.-Y. Chen's inequality for CR-submanifolds of locally conformal Kaehler space forms, *Demonstratio Math.*, **36** (2003), 189–198.




---

**A General Optimal Inequality for  
Arbitrary Riemannian  
Submanifolds**

Bang-Yen Chen

---

Title Page

Contents



Go Back

Close

Quit

Page 23 of 23